TRACIALLY QUASIDIAGONAL EXTENSIONS AND TOPOLOGICAL STABLE RANK

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Abstract. It is known that a unital simple $C^*$-algebra $A$ with tracial topological rank zero has real rank zero and topological stable rank one. In this note we construct unital $C^*$-algebras with tracial topological rank zero but topological stable rank two.

1. Introduction

The tracial topological rank of a unital $C^*$-algebra $A$, denoted by $\text{TR}(A)$, was introduced as a noncommutative analog of the covering dimension for a topological space $X$ ([Ln2] and [Ln3]; see also Definition 4.2 below). It plays an important role in the classification of amenable $C^*$-algebras (see [Ln3], [Ln5] and [Ln6]). As in the case of real rank (see [BP]), a unital commutative $C^*$-algebra $C(X)$ has tracial topological rank $k$ if and only if $\dim X = k$. It was shown in [HLX1] that if $\dim X = k$ and $\text{TR}(A) = m$ then $\text{TR}(C(X) \otimes A) \leq k + m$. (For the case of real rank see [NOP, Corollary 1.10].) At present, the most interesting cases are $C^*$-algebras with tracial topological rank at most one. If $A$ is a unital separable simple $C^*$-algebra with tracial topological rank zero, it was shown in [Ln4] that $A$ is quasidiagonal, has real rank zero, topological stable rank one and weakly unperforated $K_0(A)$.

We are also interested in $C^*$-algebras that are not simple. Let

\[0 \to J \to E \to A \to 0\]

be a short exact sequence with $\text{TR}(A) = 0$ and $\text{TR}(eJe) = 0$ for every projection $e \in A$. In [HLX2] it was shown that $\text{TR}(E) = 0$ if and only if the extension is tracially quasidiagonal. (For the definition of a tracially quasidiagonal extension see Definition 5.1.) Quasidiagonal extensions are tracially quasidiagonal. A natural question is whether there exist any tracially quasidiagonal extensions which are not quasidiagonal. Very recently, the first author showed that such extensions do indeed exist. The example given also shows that there are $C^*$-algebras with $\text{TR}(E) = 0$ such that real rank of $E$ is not

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zero. The specific extension \((\ast)\) given there has the property that both \(J\) and \(A\) (are simple and) have topological stable rank one. Moreover, the index map \(\partial_0 : K_0(A) \to K_1(J)\) is not trivial. It is known that, in this case, \(E\) has topological stable rank one if the index map \(\partial_1 : K_1(A) \to K_0(J)\) is zero (see [Ni, Lemma 3] and [LR, Proposition 4]). A natural question is whether there is a tracially quasidiagonal extension with \(\partial_1 \neq 0\). In this note we construct a tracially quasidiagonal extension \(0 \to J \to E \to A \to 0\) of \(C^*\)-algebras such that \(J\) is a simple AF-algebra, \(A\) is a simple AT-algebra with real rank zero and \(\partial_1 \neq 0\). That is, this extension is not a quasidiagonal extension. Moreover, \(E\) is a quasidiagonal \(C^*\)-algebra which has tracial topological rank zero and real rank zero, but topological stable rank two. To construct such an example we use the \(K_0\)-embedding property (see [BD]) and the generalized inductive limits in the sense of [BE].

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2. Preliminaries

Definition 2.1. Let \(A\) be a separable \(C^*\)-algebra. Then \(A\) is called quasidiagonal (QD) if there exists a faithful representation \(\pi : A \to B(H)\) and an increasing sequence of finite rank of projections, \(P_1 \leq P_2 \leq P_3 \leq \cdots\), such that \(\|P_n \pi(a) - \pi(a) P_n\| \to 0\) for all \(a \in A\) and \(P_n \to 1_H\) in the strong operator topology as \(n \to \infty\).

If \(A\) is a QD \(C^*\)-algebra, then there is an injective homomorphism which maps \(A\) into \(\prod_n M_{k(n)} / \oplus M_{k(n)}\) for some increasing sequence \(\{k(n)\}\).

Recently N. Brown and M. Dadarlat studied extensions of quasidiagonal \(C^*\)-algebras. Suppose that

\[
0 \to J \to E \to A \to 0
\]

is a short exact sequence of separable \(C^*\)-algebras such that \(A\) and \(J\) are QD \(C^*\)-algebras, and \(A\) is nuclear and satisfies the Universal Coefficient Theorem. Brown and Dadalart [BD] showed that if the indexes \(\partial_i : K_i(A) \to K_i(J)\) are both zero then \(E\) is a QD \(C^*\)-algebra. Moreover, they proved that for QD \(C^*\)-algebras \(J\) satisfying a certain property (namely, the \(K_0\)-embedding) the \(C^*\)-algebra \(E\) determined by the extension

\[
0 \to J \to E \to B \to 0
\]

with a QD \(C^*\)-algebra \(B\) is again QD if and only if \(E\) is stably finite. It follows from a result of J. Spielberg (see [Sp, Lemma 1.14]) that this is the case when \(\partial_1(K_1(B)) \cap K_0(J)_+ = \{0\}\). Spielberg [Sp, Lemma 1.14] also showed that all AF-algebras satisfy the \(\{K_0\text{-embedding}\}\) property.
Let $J$ be a stable (non-elementary) simple AF-algebra. Let $\rho : K_0(J) \rightarrow \text{Aff}(T(J))$, where $T(J)$ is the compact tracial space $\{\tau : \tau(p) = 1\}$ for some nonzero projection $p \in J$, be defined by $\rho([q]) = \tau(q)$ for $q \in K_0(J)$ and $\tau \in T(J)$. Since we have $K_1(M(J)/J) \cong K_0(J)$, there exists a unitary $u$ in $M(J)/J$ (or in $M_k(M(J)/J)$) such that $[u] \in \ker \rho$. This unitary $u$ together with $J$ give the essential extension

$$0 \rightarrow J \rightarrow E \rightarrow C(\mathbb{T}) \rightarrow 0,$$

so that $\partial_1(K_1(C(\mathbb{T})))$ is nonzero but generated by $[u]$, $[u] \in \ker \rho$. The important fact that we are using is that $\partial_1(K_1(C(\mathbb{T}))) \cap K_0(J)_+ = \{0\}$. Therefore, by the above remarks, $E$ is QD but the stable rank of $A$ is not one (by [Ni, Lemma 3]).

### 3. Construction

**Definition 3.1.** We now define a very special simple AF-algebra. Let $I_0$ be a unital separable simple AF $C^*$-algebra with $K_0(I_0) = \mathbb{Q} \oplus \mathbb{Z}$ and $K_0(I_0)_+ = \{(r, m) : r \in \mathbb{Q} \setminus \{0\}, m \in \mathbb{Z}\} \cup \{(0, 0)\}$. Let $B = C(\mathbb{T})$. Write $K_i(B) = \mathbb{Z}$ $(i = 0, 1)$. Take $u \in M(A)/A$ such that $[u] = (0, 1)$ in $\mathbb{Q} \oplus \mathbb{Z}$. Let $K$ be the $C^*$-algebra of all compact operators on a separable infinite dimensional Hilbert space. Set $I = I_0 \otimes K$. As in Section 2, we obtain an essential extension

$$0 \rightarrow I \rightarrow E_1 \xrightarrow{\pi} B \rightarrow 0,$$

such that $\partial_1([z]) = (0, 1)$ (in $\mathbb{Q} \oplus \mathbb{Z}$), where $z$ is the canonical generator for $B$. As in Section 2, $E_1$ is a QD $C^*$-algebra.

Let $\{e_{ij}\}$ be the matrix unit for $K$, and set $e_n = \sum_{i=1}^{n} e_{ii}$, $n = 1, 2, \ldots$. Here we identify $e_{11}$ with $1_{I_0}$.

**Definition 3.2.** Let $A$ be a $C^*$-algebra, $\mathcal{S} \subset A$ a finite subset of $A$ and $\varepsilon > 0$ a positive number. Recall that a positive linear map $L : A \rightarrow B$ (where $B$ is a $C^*$-algebra) is said to be $\mathcal{S}$-$\varepsilon$-multiplicative if

$$\|L(a)L(b) - L(ab)\| < \varepsilon,$$

for all $a, b \in \mathcal{S}$.

The following result is certainly known (cf. [Ln7, Proposition 2.3]).

**Proposition 3.3.** Let $A = M_n$. For any $\varepsilon > 0$ there is $\delta > 0$ such that if $L : A \rightarrow B$, where $B$ is a unital $C^*$-algebra, is a $\mathcal{S}$-$\delta$-multiplicative contractive completely positive linear map, where $\mathcal{S}$ contains the matrix unit $\{e_{ij}\}_{i,j=1}^{n}$, then there is a homomorphism $h : A \rightarrow B$ such that

$$\|L - h\| < \varepsilon.$$

**Definition 3.4.** We denote by $\delta_n$ the value of $\delta$ corresponding to $\varepsilon = 1/2^n$ in Proposition 3.3. We may assume that $0 < \delta_{n+1} < \delta_n < 1$. 
3.3 such that the following holds:

Let \( \{ \xi_1, \xi_2, \ldots \} \) be a dense sequence of \( \mathbb{T} \), where each point is repeated infinitely many times. Let \( \{ a_1, a_2, \ldots \} \) be a dense sequence in the unit ball of \( E_1 \). Let \( S_n = \{ 0, a_1, \ldots, a_n \} , n = 1, 2, \ldots \), and let \( \mathcal{F}_1 = S_1 \). Write \( I = \bigcup_{n=1}^{\infty} S_n \), where the \( S_n \) are finite dimensional \( C^* \)-algebras and \( S_n \subset S_{n+1} \). Let \( S_1 \) be a finite subset of \( S_1 \) and let \( \eta_1 > 0 \) be chosen according to Proposition 3.3 such that the following holds:

For any unital \( C^* \)-algebra \( B' \) there is a homomorphism \( \bar{h}_1 : S_1 \to B' \) such that

\[
\| L'|_{S_1} - \bar{h}_1 \| < 1/2,
\]

for any \( \delta_1-\eta_1 \)-multiplicative contractive completely positive linear map \( L' : E \to B' \).

Let \( \mathcal{F}_1 = S_1 \cup S_1 \). Since \( E_1 \) is quasidiagonal, there is a (unital) contractive completely positive linear map \( \psi_1 : E_1 \to M_{k(1)} \), which is \( \mathcal{F}_1 \)-\( 1/2 \)-\( 1/2 \)-\( \eta_1/2^2 \)-multiplicative with \( \| \psi_1(a) \| \geq (1/2)\| a \| \) for all \( a \in \mathcal{F}_1 \). Let \( p_1 = 1_{1_0} \otimes \varepsilon_{k(1)} \), where \( \varepsilon_{k(1)} \) is a rank \( k(1) \) projection in \( \mathbb{K} \). So \( p_1 \in I \) and \( \psi_1 \) may be viewed as a map from \( E_1 \) to \( p_1(\mathcal{C} \cdot 1_{1_0} \otimes \mathbb{K}) p_1 \cong M_{k(1)} \). Put \( \phi_1(a) = \pi(a)(\xi_1) \cdot (1_{E_1} - p_1) \) for \( a \in E_1 \). Let \( C_1 = \phi_1(E) \oplus p_1(\mathcal{C} \cdot 1_{1_0} \otimes \mathbb{K}) p_1 \) and \( C_1' = p_1(\mathcal{C} \cdot 1_{1_0} \otimes \mathbb{K}) p_1 \).

Define \( L_1 : E_1 \to E_2 = M_2(E_1) \) by

\[
L_1(a) = \text{diag}(a, \phi_1(a), \psi_1(a)),
\]

for \( a \in E_1 \). Note that \( L_1(S_1) \) is contained in \( S_1 \oplus C_1' \). Moreover, \( S_2 \otimes M_2 + C_1' \) is contained in a finite dimensional \( C^* \)-subalgebra of \( M_2(I) \). Let \( C_1'' \) be the finite dimensional \( C^* \)-subalgebra generated by \( S_2 \otimes M_2 + C_1' \). Since \( S_1 \subset S_2 \), we have \( L_1(S_1) \subset C_1'' \). Let \( S_2 \) be a finite subset of \( C_1'' \) and let \( 1 > \eta_2 > 0 \) be chosen according to Proposition 3.3 such that the following holds:

For any unital \( C^* \)-algebra \( B' \) there exists a homomorphism \( \bar{h}_2 : C_1'' \to B' \) such that

\[
\| L'|_{C_1''} - \bar{h}_2 \| < 1/2^2,
\]

for any \( \delta_2-\eta_2 \)-multiplicative contractive completely positive linear map \( L' : E_2 \to B' \).

Set \( I_2 = M_2(I) \). Let \( \mathcal{F}_2 \) be a finite subset of \( E_2 \) containing \( L_1(\mathcal{F}_1) \), \( \{ (a_{ij})_{i,j=1}^{k(1)} : a_{ij} = 0, a_1 \text{ or } a_2 \} \), \( I_{C_2} \), the standard generators of \( C_1' \), \( S_2 \), and another matrix unit \( \{ u_{ij} \}_{i,j=1}^{2} \), where \( u_{11} \) and \( u_{22} \) are identified with \( \text{diag}(1_{E_1}, 0), \text{diag}(0, 1_{E_1}) \). Since \( E_1 \) (unital) is quasidiagonal, there is a \( \mathcal{F}_2 \)-\( 1/3 \)-\( 1/2^2 \)-\( \eta_2 \)-\( \delta_{k(1)/2^2} \)-multiplicative contractive completely positive linear map \( \psi_2 : E_2 \to M_{k(2)} \) such that \( \psi_2(M_2(\mathcal{C} \cdot 1_E)) \) is a homomorphism and \( \| \psi_2(a) \| \geq (1 - 1/4)\| a \| \) for all \( a \in \mathcal{F}_2 \), and such that there is homomorphism \( \bar{h}_2 : C_1 \to M_{k(2)} \) such that

\[
\| \psi_2(a) \| < 1/4.
\]
Note that such a map $h_2$ exists by Proposition 3.3. We may assume that $k(2) > k(1) > 2$.

Let $E_3 = M_{2+1}(E_2) = M_3(E_1)$ and $I_3 = M_{2+1}(I_2)$. Let $p'_2 = 1_{I_0} \otimes e_{k(2)}$ and $p_2 = \text{diag}(p'_2, p'_2) \in I_2$. Define $\phi_1^{(2)}(a) = \pi_2(a)(\xi_1) \cdot 1_{E_1}$ for $a \in E_2$, where the image of $\phi_1^{(2)}$ is identified with $M_2(\mathbb{C} \cdot 1_{E_2})$. Define $\phi_2(a) = \pi_2(a)(\xi_2) \cdot (1_{E_1} - p_2)$ for $a \in E_2$, where the image of $\phi_2$ is identified with $M_2(\mathbb{C} \cdot (1 - p_2'))$. Let $\Psi_2(a) = \text{diag}(\psi_2(a), \psi_2(a))$ for $a \in E_2$. We now view $\Psi_2$ as a map $\Psi_2 : E_2 \rightarrow p_2(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K})p_2 \subset p_2I_2p_2$. In particular, $\Psi_2(1_{E_2}) = p_2$. Define

$$L_2(a) = \text{diag}(a, \phi_1^{(2)}(a), \phi_2(a), \Psi_2(a)).$$

It should be noted that the part $\text{diag}(\phi_2(a), \Psi_2(a))$ is in $E_2$ and $L_2$ is unital. Let

$$C_2 = \phi_1^{(2)}(E_1) \oplus \phi_2(E_2) \oplus p_2M_2(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K})p_2,$$

and $C'_2 = p_2(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K})p_2$. Note that $L_2(C''_1)$ is contained in $C''_1 \oplus C'_2$ and $S_3 \otimes M_3 + C'_2$ is contained in a finite dimensional $C^*$-subalgebra of $I_3$. Let $C''_2$ be the finite dimensional $C^*$-subalgebra generated by $S_3 \otimes M_3 + C'_2$. Then, since $S_2 \subset S_3$ and $C'_1 \subset C'_2$, we have $L_2(C''_1) \subset C''_2$.

Let $S_3$ be a finite subset of $C''_2$ and let $\eta_3 > 0$ be chosen according to Proposition 3.3 such that the following holds:

For any unital $C^*$-algebra $B'$ there exists a homomorphism $\widetilde{h}_3 : C''_2 \rightarrow B'$ such that

$$\|L'_1| C''_2 - \widetilde{h}_3]\| < 1/2^3,$$

for any $S_3$-multiplicative contractive completely positive linear map $L' : E_3 \rightarrow B'$.

Let $E_4 = M_4(E_3)$ and $I_4 = M_4(I_3)$. Let $D_2$ be a finite subset of $C_2$ containing $1_{C_2}$ and the standard generators of $C_2$. Let $\mathcal{F}_3$ be a finite subset of $E_3$ containing $L_2(\mathcal{F}_2)$, $\{(a_{ij})_{i,j=1}^{3 \times 2} : a_{ij} = 0, a_2, or a_3\}$, $D_2$, $S_3$ and another matrix unit $\{u_{ij}\}_{i,j=1}^{3 \times 1}$, where $u_{ii}$ is identified with a diagonal element with $1_{E_2}$ in the $i$th position and zero elsewhere.

Since $E_3$ is quasidiagonal and $E_3 = M_3(E_1)$, there is a $\mathcal{F}_3$-$1/4 \cdot 1/2^3 \cdot \eta_3 \cdot \delta_{\text{dim}C_2/2}$- multiplicative contractive completely positive linear map $\psi_3 : E_3 \rightarrow M_{k(3)}$ (where $k(3) > k(1) + k(2)$) such that $(\psi_3)|_{M_{k(3)}(\mathbb{C} \cdot 1_E)}$ is a homomorphism, and there is a homomorphism $h_3 : C_2 \rightarrow M_{k(3)}$ such that

$$\|\psi_3|_{C_2} - h_3\| < 1/2^3.$$

Define $\phi_1^{(3)}(a) = \pi_3(a)(\xi_1)$ for $a \in E_3$, where the image of $\phi_1^{(3)}$ is identified with $M_3(\mathbb{C} \cdot 1_{E_1})$, $i = 1, 2$. Let $p'_3 = 1_{I_0} \otimes e_{k(3)}$ and $p_3 = \text{diag}(p'_3, \ldots, p'_3)$, where $p'_3$ is repeated $3!$ times.

Thus, $p_3 \in I_3$. Let $\Psi_3(a) = \text{diag}(\psi_3(a), \ldots, \psi_3(a))$ for $a \in E_3$, where $\psi_3(a)$ is repeated $3!$ times. We view $\Psi_3$ as a map $\Psi_3 : E_3 \rightarrow p_3M_3(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K})p_3$. 
Define $\phi_3(a) = \pi_3(a)(\xi_3)$, where the image of $\phi_3$ is identified with $M_{3}(\mathbb{C} \cdot (1_{E_1} - p_3^3))$. Define $L_3 : E_3 \rightarrow E_4$ by (for any $a \in E_3$) 

$$L_3(a) = \text{diag}(a, \phi_1^{(1)}(a), \phi_2^{(3)}(a), \phi_3(a), \Psi_3(a)).$$

Note that $\text{diag}(\phi_3(a), \Psi_3(a)) \in E_3$. Put 

$$C_3 = \bigoplus_{i=1}^{2} \phi_i^{(3)}(E_3) \oplus \phi_3(E_3) \oplus p_3 M_{3}(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K}) p_3,$$

$$C_3 = p_3 M_{3}(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K}) p_3.$$

Let $C''_n$ be the finite dimensional $C^*$-algebra generated by $S_4 \otimes M_{41} + C'_3$. Then, since $L_2(C''_n)$ is contained in $C''_n \oplus C''_3$, $S_3 \subset S_4$ and $C_2 \subset C_3$, we have $L_2(C''_n) \subset C''_3$.

We continue the construction in this fashion. With $C_n = \bigoplus_{i=1}^{n-1} \phi_i^{(n)}(E_n) \oplus \phi_n(E_n) \oplus p_n M_n(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K}) p_n$, let $E_{n+1} = M_{n+1}(E_n)$, $I_{n+1} = M_{n+1}(I_n)$ and $C' = p_n M_n(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K}) p_n$, and let $C''_n$ be the finite dimensional $C^*$-subalgebra generated by $S_{n+1} \otimes M_{(n+1)} + C''_n$. Let $S_{n+1}$ be a finite set in $C''_n$ and let $1 > \eta_{n+1} > 0$ be chosen according to Proposition 3.3 such that the following holds:

For any unital $C^*$-algebra $B'$ there exists a homomorphism $h_{n+1} : C''_n \rightarrow B'$ such that $\|L'| C''_n - h_{n+1}\| < 1/2^{n+1}$ for any $S_{n+1} \cdot \eta_{n+1}$-multiplicative contractive completely positive map $L' : E_{n+1} \rightarrow B'$.

Let $D_n$ be a finite subset of $C_n$ containing $1_{C_n}$ and the standard generators of $C_n$, and let $F_{n+1}$ be a finite subset of $E_{n+1}$ containing $L_n(\mathcal{F}_n)$, $S_{n+1}$, $(a)_{i,j}^{(n+1)} : a_{i,j} = 0, a_1, \ldots, a_{n+1}$, $D_n$ and a matrix unit $\{u_{i,j}\}_{i,j=1}^{1+n}$, where $u_{i,j}$ is identified with $\text{diag}(0, \ldots, 0, 1_{E_n}, 0, \ldots, 0)$ (with $1_{E_n}$ in the $i$th position). Since $E_1$ is quasidiagonal and $E_{n+1} = M_{(n+1)}(E_1)$, there is a $\mathcal{F}_{n+1}$-bijective, $1/(n+2) \cdot 1/2^{n+1}$, $\eta_{n+1} \cdot \delta_{\text{dim} C_n/2^{n+1}}$-multiplicative contractive completely positive linear map $\psi_{n+1} : E_{n+1} \rightarrow M_{k(n+1)}$ such that $(\psi_{n+1})_{M_{(n+1)}}(\mathbb{C} \cdot 1_{E_1})$ is a homomorphism, and there is a homomorphism, $h_{n+1} : C_n \rightarrow M_{k(n+1)}$ such that 

$$\| (\psi_{n+1})_{|C_n} - h_{n+1} \| < 1/2^{n+1}.$$

Define $\phi_i^{(n+1)}(a) = \pi_{(n+1)}(a)(\xi_i)$ for $a \in E_{n+1}$ and identify the image of $\phi_i^{(n+1)}$ with $M_{(n+1)}(\mathbb{C} \cdot 1_{E_1})$, $i = 1, 2, \ldots, n$. Let $p_{n+1} = 1_{I_0} \otimes e_{k(n+1)}$ and $p_{n+1} = \text{diag}(p_{n+1}, \ldots, p_{n+1})$, where $p_{n+1}$ is repeated $(n+1)!$ times. Put $\Psi_{n+1}(a) = \text{diag}(\psi_{n+1}(a), \ldots, \psi_{n+1}(a))$, where $\psi_{n+1}(a)$ is repeated $(n+1)!$ times. Thus the image of $\psi_{n+1}$ can be identified with $p_{n+1} M_{(n+1)}(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K}) p_{n+1}$. Note that $\Psi_{n+1}(1_{E_{n+1}}) = p_{n+1}$. Define $\phi_{n+1}(a) = \pi_{(n+1)}(a)(\xi_i)$, with its image identified with $M_{(n+1)}(\mathbb{C} \cdot (1_{E_1} - p_{n+1}^n))$. Note that the unit of $M_{(n+1)}(\mathbb{C} \cdot (1_{E_1} - p_{n+1}^n))$ is $1_{E_{n+1}} - p_{n+1}$. Define 

$$L_{n+1}(a) = \text{diag}(a, \phi_1^{(n+1)}(a), \phi_2^{(n+1)}(a), \ldots, \phi_{n+1}(a), \phi_{n+1}(a), \Psi_{n+1}(a), \phi_{n+1}(a), \Psi_{n+1}(a)).$$
where $a \in E_{n+1}$. Let
\[
C_{(n+1)} = \bigoplus_{i=1}^{n+1} \phi_i^{(n+1)}(E_{n+1}) \oplus \phi_{n+1}(E_{n+1}) \\
\oplus p_{n+1}M_{(n+1)!}(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K})p_{n+1}
\]
and
\[
C'_n = p_{n+1}M_{(n+1)!}(\mathbb{C} \cdot 1_{I_0} \otimes \mathbb{K})p_{n+1}.
\]
Let $C''_{n+1}$ be the finite dimensional $C^*$-algebra generated by $S_{n+2} \otimes M_{(n+2)!} + C''_{n+1}$. Then, since $L_{n+1}(C''_n)$ is contained in $C''_n \oplus C'_{n+1}$, $S_{n+1} \subset S_{n+2}$, and $C'_n \subset C''_{n+1}$, we have $L_{n+1}(C''_n) \subset C''_{n+1}$. Let $S_{n+2}$ be a finite set in $C''_{n+1}$ and let $1 > \eta_{n+2} > 0$ be chosen according to Proposition 3.3 such that the following holds:

For any unital $C^*$-algebra $B'$ there exists a homomorphism $\tilde{h}_{n+1} : C''_{n+1} \to B'$ such that $||L'_i|C''_{n+1} - \tilde{h}_{n+1}|| < 1/2^{n+2}$ for any $S_{n+2} - \eta_{n+2}$-multiplicative completely contractive positive map $L' : E_{n+2} \to B'$.

It is easy to verify that $(E_n, L_n)$ forms a general inductive limit in the sense of [BE]. Denote by $E$ the $C^*$-algebra defined by this inductive limit. We will write $L_{n+k} : E_n \to E_{n+k}$ for the decomposition $L_{n+k-1} \circ \cdots \circ L_n$ and $L_{n,\infty} : E_n \to E$ for the map induced by the inductive limit. We will also use the fact that $||L_n(a)|| = ||a|| = ||L_{n,\infty}(a)||$ for all $a \in E_n$, $n = 1, 2, \ldots$.

Let $I_1 = 1, I_{n+1} = M_{(n+1)!}(I)$. Then $I_n \cong I_0 \otimes \mathbb{K}$ and $I_n$ is an ideal of $E_n$.

Set $J_0 = \bigcup_{n=1}^\infty L_{n,\infty}(I_n)$ and $J = J_0$.

**Proposition 3.6.** $J$ is an ideal of $E$.

**Proof.** Let $a \in E$ and $b \in J$. We want to show that $ab, ba \in J$. For any $\varepsilon > 0$, there are $a' \in \bigcup_{n=1}^\infty L_{n,\infty}(E_n)$ and $b' \in J_0$ such that $||a - a'|| < \varepsilon$ and $||b - b'|| < \varepsilon$. It suffices to show that $a'b', b'a' \in J$. To simplify notation, without loss of generality, we may assume that $a \in \bigcup_{n=1}^\infty L_{n,\infty}(E_n)$ and $b \in J_0$. Therefore, there is an integer $n > 0$ such that $a = L_{n,\infty}(a_1)$ and $b = L_{n,\infty}(b_1)$, where $a_1 \in E_n$ and $b_1 \in I_n$. Moreover, there is an integer $N > n$ such that
\[
||L_{n,N+k} \circ L_{n,N}(a_1)L_{n,N+k} \circ L_{n,N}(b_1) - L_{n,N+k}(L_{n,N}(a_1)L_{n,N}(b_1))|| < \varepsilon
\]
for all $k > 0$. By the definition, $L_{n,N}(b_1) \in I_N$. Therefore $L_{n,N+k}(L_{n,N}(a_1)L_{n,N}(b_1)) \in I_{N+k}$. This implies that
\[
dist(ab, J) < \varepsilon
\]
for all $\varepsilon > 0$. Hence $ab \in J$. Similarly $ba \in J$. \hfill $\Box$

**Definition 3.7.** Let $B_1 = C(T)$ and $B_{n+1} = M_{(n+1)!}(C(T))$, $n = 1, 2, \ldots$ Define $h_n : B_n \to B_{n+1}$ by $h_n(b) = \text{diag}(b, b(\xi_1), \ldots, b(\xi_n))$, $n = 1, 2, \ldots$. Let $B_\infty = \lim_n(B_n, h_n)$. Then $B_\infty$ is a unital simple $C^*$-algebra with $\text{TR}(B_\infty) = 0$. 

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(see Definition 4.2), $K_1(B_\infty) = \mathbb{Z}$ and $K_0(B_\infty) = \mathbb{Z} \oplus \mathbb{Q}$ with $K_0(B_\infty)_+ = \{(n, r) : n > 0, r \in \mathbb{Q}_+ \setminus \{0\} \cup \{0\}\}.$

**Proposition 3.8.** Let $\pi_\infty : E \to E/J$ be the quotient map. Then $\pi_\infty(E) \cong B_\infty.$

**Proof.** We first show that, for each $n$, $L_{n,\infty}(E_n) \cap J = L_{n,\infty}(I_n).$ Let $a \in E_n \setminus I_n.$ Then, by the construction, for all $m > 0,$

\[ \text{dist}(L_{n,m}(a), I_{n+m}) \geq \|\pi_n!(a)\|, \]

where $\pi_n! : E_n \to E_n/I_n$ is the quotient map. This implies that

\[ \text{dist}(L_{n,\infty}(a), J) \geq \|\pi_n!(a)\|. \]

Therefore $L_{n,\infty}(E_n) \cap J = L_{n,\infty}(I_n).$

Now we have

\[ L_{n,\infty}(E_n)/J \cong B_n. \]

From the construction there is an isomorphism from $L_n(E_n)/I_{n+1}$ to $L_{n,\infty}(E_n)/J.$ Denote by $j_n : L_{n,\infty}(E_n)/J \to L_{n+1,\infty}(E_{n+1})/J$ the map induced by $L_n$ and by $\gamma_n$ the isomorphism from $L_{n,\infty}(E_n)/J$ onto $B_n.$ We obtain the following intertwining:

\[ \begin{array}{ccc}
L_{n,\infty}(E_n)/J & \xrightarrow{j_n} & L_{n+1,\infty}(E_{n+1})/J \\
\downarrow{\gamma_n} & & \downarrow{\gamma_{n+1}} \\
B_n & \xrightarrow{h_{n,\infty}} & B_{n+1}
\end{array} \]

This implies that $B_\infty \cong E/J.$ \qed

### 4. Tracial topological rank of $E$

Through the rest of paper, we will write $f_{\delta_2}^{\delta_1}$ (where $0 < \delta_2 < \delta_1 < 1$) for the following non-negative continuous function on $[0, \infty):$

\[ f_{\delta_2}^{\delta_1}(t) = \begin{cases} 1, & t \geq \delta_1, \\ (t - \delta_2)/(\delta_1 - \delta_2), & \delta_2 < t < \delta_1, \\ 0, & t \leq \delta_2. \end{cases} \]

**Definition 4.1.** Let $a$ and $b$ be two positive elements in a $C^*$-algebra $A.$ We write $[a] \leq [b]$ if there exists $x \in A$ such that $a = x^*x$ and $xx^* \in b\mathcal{A}b,$ and $[a] = [b]$ if $a = x^*x$ and $b = xx^*.$ For more information on this relation, see [Cu1], [Cu2] and [HLX1].

**Definition 4.2** ([Ln4] and [HLX1]). Recall that a unital $C^*$-algebra $A$ is said to have tracial topological rank zero if the following holds: For any $\varepsilon > 0,$ any finite subset $\mathcal{F} \subset A$ containing a nonzero element $a \in A_+,$ and any real
numbers $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, there is a projection $p \in A$ and a finite dimensional $C^*$-subalgebra $B$ of $A$ with $1_B = p$ such that

1. $\|xp - px\| < \varepsilon$ for all $x \in \mathcal{F}$,
2. $p \sigma p \in \mathcal{L}$ for all $x \in \mathcal{F}$, and
3. $[f^\varepsilon_n((1-p)a(1-p))] \leq [f^\varepsilon_n(pap)]$.

If $A$ has tracial topological rank zero, we will write $\text{TR}(A) = 0$. If $A$ is non-unital, we say that $A$ has tracial topological rank zero if $\text{TR}(\tilde{A}) = 0$.

We will show that the $C^*$-algebra $E$ constructed in the previous section has tracial topological rank zero. The proof is similar to that in [Ln7]. We will use the following two lemmas:

**Lemma 4.3** ([HLX1, Lemm 1.8]). Let $0 < \sigma_4 < \sigma_3 < 1$. There is $\delta = \delta(\sigma_3, \sigma_4) > 0$ such that for any $C^*$-algebra $A$, any $a, b \in A_+$ with $\|a\| \leq 1$, $\|b\| \leq 1$, and any $\sigma_1, \sigma_2$ with $\sigma_3 < \sigma_2 < \sigma_1 < 1$, $\|a - b\| < \delta$ implies $[f^\varepsilon_n(a)] \leq [f^\varepsilon_n(b)]$.

**Lemma 4.4** ([Ln7, Lemma 3.4]). Let $0 < \sigma_4 < \sigma_3 < 1$. There is $\delta_1 = \delta(\sigma_3, \sigma_4) > 0$ such that for any $C^*$-algebra $A$, any $a, b \in A_+$ and $x \in A$ with $\|x\| \leq 1$, $\|a\| \leq 1$, $\|b\| \leq 1$ and any $\sigma_1, \sigma_2$ with $\sigma_3 < \sigma_2 < \sigma_1 < 1$, then $\|x^*x - a\| < \delta_1$ and $\|xx^* - b\| < \delta_1$ imply $[f^\varepsilon_n(a)] \leq [f^\varepsilon_n(b)]$.

**Lemma 4.5.** $\text{TR}(E) = 0$.

**Proof.** By Definition 1.11 (see also Proposition 1.17) in [HLX1], it suffices to show the following:

For any $\varepsilon > 0$, any $0 < \sigma_2 < \sigma_1 < 1$, any finite subset $\mathcal{F}$ of $E$ and a nonzero element $a \in E_+$, there is a projection $p \in E$ and a finite dimensional $C^*$-subalgebra $C \subset E$ with $1_C = p$ such that

1. $\|xp - px\| < \varepsilon$ for all $x \in \mathcal{F}$,
2. $\text{dist}(x, C) < \varepsilon$ for all $x \in \mathcal{F}$, and
3. $[f^\varepsilon_n((1-p)a(1-p))] \leq [f^\varepsilon_n(pap)]$ for some $0 < \sigma_4 < \sigma_3 < \sigma_2$.

Without loss generality, we may assume that $\|a\| = 1$. Fix $0 < d_2 < d_1 < \min \{1/8, \sigma_2\}$. Let $\delta(d_1, d_2) > 0$ be as in Lemma 4.4. There is an integer $n$ such that $1/n < \varepsilon/4$, and a finite subset $S \subset E_n$ such that $\mathcal{F} \cup \{a\} \subset L_n(\infty)(S)$. Suppose that $L_n(\infty)(b) = a$, where $0 \leq b \leq 1$ is in $E_n$ and $\|b\| = 1$. We may assume that $L_n(\infty)(S') \subset L_{l, \infty}(\mathcal{F})$, where $S' = \{cd : c, d \in S\}$ and $\mathcal{F}_l$ is as in Definition 3.5.

Choose a large integer $l > (n+1)^2$ such that $\max \{1/2^{l-2}, 1/l\} < \delta(d_1, d_2)/2$ and $\|\psi_l(L_{n, l-1}(b))\| \geq (1/2)\|b\|$. (Note that $1/l < \varepsilon/16$.) For $s \in S$ write (in $E_l$)

$$L_n,l(s) = \text{diag}(s, L(s)), \quad \text{with} \quad L_n,l(1_{E_n}) = \text{diag}(1_{E_n}, L(1_{E_n})),$$
where \( L(s) \in C_l \). (See the construction of \( E \).) Since \( L_{t,\infty} \) is \( \mathcal{F}_{l-1/(l+1)2^l} \cdot \delta_{\dim C_l/2^l} \)-multiplicative, by Proposition 3.3 there is a homomorphism \( h : C_l \to E \) such that

\[
\| L_{t,\infty} | C_l - h \| < 1/2^{l-1}.
\]

Let \( p' = \text{diag}(0, L(1_{E_1})) \). Then \( p' \in C_l \). Hence there is a projection \( p \in h(C_l) \) such that \( \| L_{t,\infty}(p') - p \| \leq 1/2^{l-1} \). Since \( L_{t,\infty} \) is \( \mathcal{F}_{l-1/(l+1)2^l} \cdot \delta_{\dim C_l/2^l} \)-multiplicative, we have

1. \( \| px - xp \| < \varepsilon \) for \( x \in \mathcal{F} \), and
2. \( pxp \in \varepsilon \cdot h(C_l) \) for \( x \in \mathcal{F} \).

To show (3) we consider two cases:

**Case (i):** \( b \in (I_n)_+ \). We may assume that

\[
\| e_l b - b \| < \min\{\delta(d_1, d_2)/4, \varepsilon/4\}.
\]

Let \( b_1 = e_l b e_l \) and \( b'_1 = L_{n,l-1}(b_1) \). Thus, \( \psi(b'_1) \neq 0 \). We have

\[
L_{n,l}(b_1) = \text{diag}(b_1, \Phi_n(b_1), \psi_l(b'_1), \ldots, \psi_l(b'_1)),
\]

where \( \Phi_n : I_n \to I_l \) is a contractive completely positive linear map and \( \Phi_n(I_n) \) is contained in \( C_l \) and \( \psi_l \) is repeated \( l \) times. Note that \( \| \psi(b'_1) \| > 1/4 \). So, \( \text{diag}(\psi_l(b'_1), \ldots, \psi_l(b'_1)) \) has an eigenvalue \( \lambda \) with \( \lambda \geq 1/4 \) and its rank in \( C'_l \) (see the construction of \( E \)) is at least \( l \). We have

\[
[b_1] \leq [e_l] \quad \text{and} \quad (1/4)[e_l] \leq [\text{diag}(\psi_l(b'_1), \ldots, \psi_l(b'_1))],
\]

where \( \psi_l(b'_1) \) is repeated \( l \) times.

Put \( c = \text{diag}(0, \Phi_n(b_1), \psi_l(b'_1), \ldots, \psi_l(b'_1)) \) and \( b' = \text{diag}(b_1, 0, \ldots, 0) \). Since \( \{u_{ij}\}_{i=1} \in \mathcal{F}_l \), there is \( x \in \mathcal{F}_l \) such that

\[
x^* x = b' \quad \text{and} \quad x x^* \in C',
\]

where \( C' = e_l C'_l e_l \). Moreover, \( c \) contains an eigenvalue \( \lambda \) with \( \lambda \geq 1/4 \) and the corresponding spectral projection \( e \) larger than a projection in \( C'_l \) with rank \( l \). Therefore, there exists \( v \in C_l \) such that

\[
v^* v = e_l \quad \text{and} \quad vv^* \leq e.
\]

Note that \( f_{1/8}^{1/4}(c) \geq c \). This implies that there is \( z \in C_l \) such that

\[
z^* z = x x^* \quad \text{and} \quad z z^* f_{1/8}^{1/4}(c) = z z^*.
\]

Let \( y = L_{t,\infty}(x) \) and \( b'' = \rho L_{t,\infty}(b')p \). Since \( L_{t,\infty} \) is \( \mathcal{F}_{l-1/(l+1)2^l} \cdot \delta_{\dim C_l/2^l} \)-multiplicative and \( \| L_{t,\infty} | C_l - h \| < 1/2^{l-1} \), we have

\[
\| y^* y - b'' \| < 1/2^{l-2} \quad \text{and} \quad \| y y^* - h(x x^*) \| < 1/2^{l-2}.
\]

We also have

\[
\| b'' - (1 - p)a(1 - p) \| < 1/2^{l-2} \quad \text{and} \quad \| h(c) - p a p \| < 1/2^{l-2}.
\]
Moreover,
\[ h(z^*z) = h(xx^*) \quad \text{and} \quad h(zz^*)h(f_{1/8}^{1/4}(c)) = h(zz^*). \]

Therefore, by Lemma 4.4,
\[ [f_{\sigma_2}^{\sigma_2^*}((1 - p)a(1 - p))] \leq [f_{d_2}^{\sigma_4}(h(xx^*)]]. \]

We also have that \([f_{d_2}^{\sigma_4}(h(xx^*))] = [f_{d_2}^{\sigma_4}(h(zz^*))]\). Therefore
\[ [f_{d_2}^{\sigma_4}(h(zz^*))] \leq [h(zz^*)] \leq [h(f_{1/8}^{1/4}(c))]. \]

It then follows from the proof of Lemma 4.4 that there are \(0 < \sigma_4 < \sigma_3 < d_2\) such that
\[ [h(f_{1/8}^{1/4}(c))] \leq [f_{\sigma_4}^{\sigma_2}(pap)], \]

because \(||h(c) - pap|| < 1/2^{l-2}\). Hence
\[ [f_{\sigma_4}^{\sigma_2}((1 - p)a(1 - p))] \leq [f_{\sigma_4}^{\sigma_2}(pap)]. \]

**Case (ii)**: \(b \in (E_n)_+ \setminus I_n\). This part of the proof is just a slight modification of the proof in Case (i). Take \(0 < \sigma_{10} < \sigma_9 \quad \text{etc.} \quad \sigma_4 < \sigma_3 < d_2\). We note that \(\phi_i^{(n+1)} \circ L_n(a)\) has the form
\[ \text{diag}(\pi_n(a)(\xi_1), \phi_1^{(n)}(a), \ldots, \phi_{n-1}^{(n)}(a), \pi_n(a)(\xi_n)) \]

for \(0 < i < n\) and \(a \in E_n\). Since \(\{\xi_n\}\) is dense in \(\mathbb{T}\), without loss of generality we may assume that \(\pi_n(b)(\xi_m) \neq 0\) and \(n < m < m! < l\). By the construction we may write
\[ L_n,t(b) = \text{diag}(b, L'(b), \pi_n(b)(\xi_m), \ldots, \pi_n(b)(\xi_m), L''(b)), \]

where \(\pi_n(b)(\xi_m)\) is repeated \(m!\) times and \(L'(b), L''(b) \in C_l\). Note that
\[ \text{diag}(0, L'(b), \pi_n(b)(\xi_m), \ldots, \pi_n(b)(\xi_m), L''(b)) \]
\[ \geq \text{diag}(0, 0, \ldots, 0, \pi_n(b)(\xi_m), \ldots, \pi_n(b)(\xi_m), 0). \]

Since \(\{u_{ij}\} \subset F_1\), there is \(z_k \in F_1\) such that
\[ z_k^*z_k = \text{diag}(b, 0, 0, \ldots, 0) \quad \text{and} \quad z_k z_k^* = \text{diag}(0, \ldots, 0, b, 0), \]

where \(b\) is in the \((k + 1)\)st position. We note that there is \(c \in M_{l!\cdot n!}(\mathbb{C} \cdot 1_{E_n})\) such that
\[ cc^* = \text{diag}(0, 1_{E_n}, 0, \ldots, 0) \]

and
\[ cc^* \leq \text{diag}(0, \ldots, 0, \pi_n(b)(\xi_m), \ldots, \pi_n(b)(\xi_m), 0). \]

Indeed, since \(\pi_n(b)(\xi_m) \neq 0\), \(\pi_n(b)(\xi_m)\) is unitarily equivalent to \(\text{diag}(\alpha_1, \ldots, \alpha_{n!})\) for some \(\alpha_1, \ldots, \alpha_{n!} \in \mathbb{C}\) with \(\alpha_1 \neq 0\). Since \(t\pi_n(b)(\xi_m)\) is repeated \(m!\) times and \(m > n\), we have
\[ [\text{diag}(0, 1_{E_n}, 0, \ldots, 0)] \leq [\text{diag}(0, 0, \ldots, 0, \pi_n(b)(\xi_m), \ldots, \pi_n(b)(\xi_m), 0)]. \]
Note also $\text{diag}(0, b, 0, \ldots, 0) \leq \text{diag}(0, 1_{E_n}, 0, \delta, 0)$ and $(L_{t, \infty})|_{M_{0}(C_{1} E_{1})}$ is a homomorphism. Therefore we have
\[
[f_{\sigma_{2}}((1 - p)a(1 - p))] \leq [f_{d_{2}}(L_{t, \infty}(z_{1}^{*} z_{1}))] \\
= [f_{d_{2}}(L_{t, \infty}(z_{1}^{*} z_{1}^{*}))] \\
\leq [f_{\sigma_{4}}(L_{t, \infty}(c^{*} c))] \\
\leq [f_{\sigma_{4}}(L_{t, \infty}(c^{*} c^{*}))] \\
\leq [f_{\sigma_{6}}(L_{t, \infty}(\text{diag}(0, \pi_{n!}(b)(\xi_{n}), \ldots, \pi_{n!}(b)(\xi_{n}), 0)))] \\
\leq [f_{\sigma_{6}}(\text{diag}(0, L'(b), \pi_{n!}(b)(\xi_{n}), \ldots, \pi_{n!}(b)(\xi_{n}), L''(b)))] \\
\leq [f_{\sigma_{10}}(pap)].
\]
and hence
\[
[f_{\sigma_{2}}((1 - p)a(1 - p))] \leq [f_{\sigma_{10}}(pap)].
\]
This shows that $\text{TR}(E) = 0$. \hfill \qed

**Corollary 4.6.**  $J$ is an AF algebra.

**Proof.** Given a finite subset $\mathcal{F} \subset J$, we may assume that there is a finite subset $\mathcal{G} \subset I_{n}$ such that $L_{n, \infty}(\mathcal{G}) = \mathcal{F}$. Furthermore, we may assume that $\mathcal{G} \subset S_{k(n)} \otimes M_{n!}$ with $k(n) > n$. Since $C''_{l} \oplus C'_{l+1} \subset C''_{l+1}$ for $1 \leq l$ and
\[
L_{n, k(n)}(S_{k(n)} \otimes M_{n!}) \subset S_{k(n)} \otimes M_{n!} \oplus C'_{n} \oplus C'_{n+1} \oplus \cdots \oplus C'_{k(n)-1},
\]
it follows that $L_{n, k(n)}(S_{k(n)} \otimes M_{n!}) \subset C''_{k(n)}$. By the choice of $S_{k(n)}$ and $\eta_{k(n)}$, we see that there is a homomorphism $\tilde{h}_{k(n)} : C''_{k(n)} \rightarrow E$ such that
\[
\|L_{k(n), \infty}(C''_{k(n)} - \tilde{h}_{k(n)})\| < 1/2^{k(n)}.
\]
Let $B_{0} = \tilde{h}_{k(n)}(C''_{k(n)})$. Then $B_{0}$ is a finite dimensional $C^{*}$-algebra and for any $x \in \mathcal{G}$ we have
\[
\|L_{n, \infty}(x) - \tilde{h}_{k(n)}(L_{n, k(n)}(x))\| \\
\leq \|L_{n, \infty}(x) - L_{k(n), \infty}(L_{n, k(n)}(x))\| \\
\quad + \|L_{k(n), \infty}(L_{n, k(n)}(x)) - \tilde{h}_{k(n)}(L_{n, k(n)}(x))\| \\
< 1/2^{k(n)}.
\]
This implies that
\[
\text{dist}(\mathcal{F}, B_{0}) < 1/2^{k(n)}.
\]
From this one sees that $J$ is an AF-algebra. \hfill \qed
5. Tracially quasidiagonal extensions

**Definition 5.1.** Let
\[ 0 \to I \to E \to A \to 0, \]
be a short exact sequence of $C^*$-algebras. In [HLX2], we say that $(E, I)$ is **tracially quasidiagonal** if, for any $\varepsilon > 0$, any nonzero $a \in E_+$, any finite subset $\mathcal{F} \subset E$ and any $0 < \sigma_4 < \sigma_3 < \sigma_2 < \sigma_1 < 1$, there exists a $C^*$-subalgebra $D \subset E$ with $1_D = p$ such that

1. $\|px - xp\| < \varepsilon$ for all $x \in \mathcal{F}$,
2. $pxp \in \varepsilon D$ for all $x \in \mathcal{F}$,
3. $D \cap I = pIp$ and $(D, D \cap I)$ is quasidiagonal, and
4. $[f^{\sigma_2}_{\sigma_1}((1 - p)a(1 - p))] \leq [f^{\sigma_4}_{\sigma_3}(pap)]$.

In [HLX2] we showed that if $\text{TR}(I) = 0 = \text{TR}(A)$ then $\text{TR}(E) = 0$ if $(E, I)$ is tracially quasidiagonal. Moreover, if $\text{TR}(eJe) = 0$ for every projection $e \in E$, then $\text{TR}(E) = 0$ also implies that the extension is tracially quasidiagonal.

It is clear that if $(E, I)$ is quasidiagonal, then $(E, I)$ is tracially quasidiagonal. (Take $p = 1$.) On the other hand, Corollary 5.5 below says that there are tracially quasidiagonal extensions that are not quasidiagonal.

**Theorem 5.2.** The extension
\[ 0 \to J \to E \to B_\infty \to 0, \]
is tracially quasidiagonal.

**Proof.** Since $\text{TR}(B_\infty) = \text{TR}(E) = \text{TR}(J) = 0$, and $\text{TR}(eJe) = 0$ for every projection $e \in E$ by Lemma 4.5 and Corollary 4.6, it follows from [HLX2] that the extension is tracially quasidiagonal.

**Lemma 5.3.** Let $A_n$ be a sequence of unital $C^*$-algebras and $A = \lim_{n \to \infty} (A_n, \varphi_{n,m})$ be a generalized inductive limit in the sense of [BE]. Suppose that each $\varphi_{n,m} : A_n \to A_m$ is unital with $\|\varphi_n(a)\| = \|\varphi_{n,m}(a)\|$ for $a \in A_n$ and $m > n$. Let $u \in A$ be a unitary. Then for any $\varepsilon > 0$ there is $n > 0$ and a unitary $v \in A_n$ such that
\[ \|\varphi_n(v) - u\| < \varepsilon, \]
where $\varphi_n$ is an induced map by the inductive limit from $A_n$ into $A$.

**Proof.** By definition there is a sequence $\{\varphi_{n_k}(a_k)\}$, where $a_k \in A_{n_k}$ that converges to $u$. Therefore, we may assume that
\[ \|\varphi_{n_k}(a_k) - u\| < 1/2^{k+2}. \]
and
\[ \|\varphi_{n_k}(a_k)\varphi_{n_k}(a_k) - 1\| < 1/2^{k+2}, \quad \|\varphi_{n_k}(a_k)\varphi_{n_k}(a_k) - 1\| < 1/2^{k+2}. \]
From the definition of a generalized inductive limit it follows that there is \( m_k > n_k \) such that for any \( x, y \in A_{n_k} \)

\[
\| \varphi_{m_k}(\varphi_{n_k,m_k}(x) + \varphi_{n_k,m_k}(y)) - (\varphi_{n_k}(x) + \varphi_{n_k}(y)) \| < 1/2^{k+2},
\]

\[
\| \varphi_{m_k}(\varphi_{n_k,m_k}(x)^*) - \varphi_{n_k}(x)^* \| < 1/2^{k+2},
\]

\[
\| \varphi_{m_k}(\varphi_{n_k,m_k}(x)\varphi_{n_k,m_k}(y)) - \varphi_{n_k}(x)\varphi_{n_k}(y) \| < 1/2^{k+2}.
\]

Set \( b_k = \varphi_{n_k,m_k}(a_k) \in A_{m_k} \). Since

\[
\| \varphi_{n_k}(a_k^*a_k) - 1 \| = \| \varphi_{n_k}(a_k^*a_k) - \varphi_{m_k}(\varphi_{n_k,m_k}(a_k^*)a_k) \|
\]

\[
+ \| \varphi_{m_k}(\varphi_{n_k,m_k}(a_k^*)a_k) - 1 \|
\]

\[
\leq \| \varphi_{n_k}(a_k^*a_k) - \varphi_{m_k}(\varphi_{n_k,m_k}(a_k^*)a_k) \|
\]

\[
+ \| \varphi_{m_k}(\varphi_{n_k,m_k}(a_k^*)a_k) - 1 \|
\]

\[
\leq 1/2^{k+2} + \| \varphi_{m_k}(\varphi_{n_k,m_k}(a_k^*)a_k) - 1 \|
\]

\[
\leq 3/2^{k+2},
\]

we have

\[
\| \varphi_{m_k}(b_k^*b_k - 1) \| \leq \| \varphi_{m_k}(b_k^*b_k - 1) - (\varphi_{n_k}(a_k^*a_k) - 1) \|
\]

\[
+ \| \varphi_{n_k}(a_k^*a_k) - 1 \|
\]

\[
< 1/2^{k+2} + 3/2^{k+2} = 1/2^k,
\]

\[
\| \varphi_{m_k}(b_k^*b_k - 1) \| < 1/2^k
\]

and

\[
\| \varphi_k(b_k) - u \| = \| \varphi_{m_k}(\varphi_{n_k,m_k}(a_k)) - u \|
\]

\[
\leq \| \varphi_{m_k}(\varphi_{n_k,m_k}(a_k)) - \varphi_{n_k}(a_k) \| + \| \varphi_{n_k}(a_k) - u \|
\]

\[
< 1/2^{k+1}.
\]

Hence \( b_k \) is an invertible element in \( A_{m_k} \). Set \( v = b_k|b_k|^{-1} \). Then \( v \) is a unitary, and the distance between \( v \) and \( b_k \) is small (depending on \( k \)). Hence, taking a sufficient large \( k \), we have a unitary \( v \in A_{m_k} \) such that

\[
\| \varphi_{m_k}(v) - u \| < \varepsilon. \tag*{□}
\]

**Theorem 5.4.** A C*-algebra \( E \) has topological stable rank two.

**Proof.** Take a unitary \( u \in B_\infty \) such that \( 0 \neq [u]_1 \in K_1(B_\infty) \). Suppose that \( u \) can be lifted to a unitary \( \tilde{u} \) in \( E \). We will get a contradiction.

Since a system \( (E_n, L_{n,m}) \) is a generalized inductive limit, by Lemma 5.3 there exists \( n \in \mathbb{N} \) and a unitary \( v \in E_n \) such that

\[
\| L_{n,\infty}(v) - \tilde{u} \| < 1.
\]
From the commutative diagram

\[
\begin{array}{ccc}
E_n & \xrightarrow{L_n,\infty} & E \\
\downarrow \pi_n & & \downarrow \pi \\
B_n & \xrightarrow{h_n,\infty} & B_{\infty}
\end{array}
\]

we conclude that \(w_n = \pi_n!(v)\) is a unitary in \(B_n\) and \(h_n,\infty(w_n) = \pi_\infty \circ L_n,\infty(v)\). Set \(w = h_n,\infty(w_n)\). Then \(w \in B_{\infty}\) is a unitary, and

\[
\|w - u\| = \|\pi_\infty \circ L_n,\infty(v) - \pi_\infty(\tilde{u})\| \leq \|L_n,\infty(v) - \tilde{u}\| < 1.
\]

Hence, \([w] = [u]_1\) in \(K_1(B_{\infty})\). Therefore, \([w_n]_1 \neq 0\) in \(K_1(B_n)\), because the induced map \((h_n,\infty)_* : K_1(B_n) \to K_1(B_{\infty})\) is an identity (see Definition 3.7).

On the other hand, from the construction we have \(0 \neq \partial_1([w_n]_1) \in K_0(I_n)\). But this gives a contradiction, for

\[
\partial_1([w_n]_1) = \partial_1 \circ \pi_*([v]_1) = 0.
\]

So by [Ni, Lemma 3] or [LR, Proposition 4] \(E\) has topological stable rank more than one. Hence by [Rf, Corollary 4.2] we conclude that \(E\) has topological stable rank two. \(\square\)

**Corollary 5.5.** The extension

\[
0 \rightarrow J \rightarrow E \rightarrow B_{\infty} \rightarrow 0
\]

is not quasidiagonal.

**Proof.** From the previous theorem it follows that the index map \(\partial_1 : K_1(B_{\infty}) \to K_0(J)\) is non-zero. Hence, by [BrD, Theorem 8] the extension is not quasidiagonal. \(\square\)

**Remark 5.6.**

1. The \(C^*\)-algebra \(E\) is QD in [HLX3, Theorem 4.6].

2. A \(C^*\)-algebra \(E\) in (1) is not an AH-algebra, that is, it can not be written as the inductive limit of direct sums of homogeneous \(C^*\)-algebras. Indeed, since \(J\) and \(B_{\infty}\) have real rank zero (see [G, Theorem 9]), if \(E\) is an AH-algebra, the extension has to be quasidiagonal by [BrD, Proposition 11]. This is a contradiction to Corollary 5.5.

3. In [La7] the first author constructed an example of a unital \(C^*\)-algebra \(A\) which has tracial topological rank zero, but real rank greater than zero. This also gives an extension \(0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0\) with a non-zero index map \(\partial_0 : K_0(B) \to K_1(I)\). In the present note we have constructed another tracially quasidiagonal extension in Corollary 5.5 with a non-zero index map \(\partial_1 : K_1(B) \to K_0(I)\). This implies that a more complicated index is needed to characterize tracially quasidiagonal extensions.
References


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