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ERRATUM TO "THE LOOP SPACE OF THE Q-CONSTRUCTION"

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ABSTRACT. We correct an error in [1], and provide a new shorter proof of Theorem B′.

Introduction

The purpose of this note is to point out that Lemma 1.5 of [1] is not true, but that a slightly weaker version is true and suffices for the applications in the paper.

Lemma 1.5 is an ingredient in the proof of our analogues for simplicial sets of Theorems A and B of Quillen [3], which we called Theorems A′ and B′. Our proof followed the same general lines as Quillen's, making use of his technique of constructing a quasifibration cell by cell. The problem with the proof in the paper is a cavalier application of the method of proof used in the lemma on page 90 of [3]; we ignored the degenerate cells in the skeletal filtration of the base space, or more precisely, the cells in the total space lying over them.

Lemma 1.5′ below is a weakened form of Lemma 1.5 of [1] that provides a homotopy cartesian square rather than a quasifibration. This is enough for the proof of Theorem B′, which we reprove below using the new lemma. The proof of Lemma 1.5′ given here is completely combinatorial, and we thank Rick Jardine for dramatically shortening our original proof.

A counterexample to the statement of Lemma 1.5 is provided by letting Y be the constant simplicial set with one point, and letting Z be some bisimplicial set such that all the face (and degeneracy) maps $Z(A,\cdot) \to Z(A',\cdot)$ arising from maps $A' \to A$ in Δ are homotopy equivalences, and the map $|Z([0],\cdot)| \to |Z|$ is not a homeomorphism. Taking $\rho \in Y([0])$ we see that $|Z_\rho| = |Z([0],\cdot)|$ and $F^{-1}(\rho) = |Z|$, and these two spaces are not homeomorphic, contrary to the claim in Lemma 1.5.

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Let Δ be the category of finite nonempty ordered sets. For $m \in \Delta$, let Δ^m denote the simplicial set represented by m. For $m \in \mathbb{N}$ let $[m] \in \Delta$ be the ordered set $\{0 < 1 < \cdots < m\}$, and let $\Delta^m = \Delta^{[m]}$. Similarly, for a simplicial set X and $m \in \Delta$ we'll let $X_m = X(m)$, and for $m \in \mathbb{N}$ we'll let $X_m = X([m])$. Given simplicial sets X and Y, the external product $X \boxtimes Y$ is the bisimplicial set defined by $(X \boxtimes Y)_{m,n} = X_m \times Y_n$. Given a simplicial set Y define a bisimplicial set YL by $YL(m,n) = Y(m)$, and define a bisimplicial set YR by $YL(m,n) = Y(n)$. Observe that $YL \cong Y \boxtimes \Delta^0$ and $YR \cong \Delta^0 \boxtimes Y$.

LEMMA 1.5$'$. *Suppose Z is a bisimplicial set, Y is a simplicial set, and $F : Z \to YL$ is a map. For $m \in \Delta$ and $y \in Y_m$ define a simplicial set Z_y by $Z_y(n) = \{z \in Z_{m,n} \mid Fz = y\}$. Suppose that for all $m \in \Delta$, all $y \in Y_m$, and all $g : p \to m$ in Δ, the natural map $Z_y \to Z_{g^*y}$ defined by $z \mapsto (g \boxtimes 1)^* z$ is a homotopy equivalence. There is a map $\Delta^m \boxtimes Z_y \to Z$ defined by $(g, z) \mapsto (g \boxtimes 1)^* z$, and it fits into the following commutative square of bisimplicial sets.*

$$\begin{tikzcd} \Delta^m \boxtimes Z_y \arrow[r] \arrow[d] & Z \arrow[d, "F"] \\ \Delta^m L \arrow[r, "y"] & YL \end{tikzcd}$$

The square is homotopy cartesian.

Proof. The square is not cartesian, so we introduce the pullback W_y, forming the following cartesian square.

$$\begin{tikzcd} W_y \arrow[r] \arrow[d] & Z \arrow[d, "F"] \\ \Delta^m L \arrow[r, "y"] & YL \end{tikzcd} \tag{1}$$

Consider the induced map $\Delta^m \boxtimes Z_y \to W_y$. It is a homotopy equivalence, for using the realization lemma [4, 5.1] we may fix the first argument of the bisimplicial sets and fix a simplex $g : p \to m$ of Δ^m. That reduces us to considering the map $Z_y \to Z_{g^*y}$, which is a homotopy equivalence by assumption.

It is enough to show that (1) is homotopy cartesian. The commutativity of the diagram

$$\begin{tikzcd} \Delta^p \boxtimes Z_{g^*y} \arrow[d, "\sim"'] & \Delta^p \boxtimes Z_y \arrow[l, "\sim"'] \arrow[r, "\sim"] & \Delta^m \boxtimes Z_y \arrow[d, "\sim"] \\ W_{g^*y} \arrow[rr] & & W_y \end{tikzcd}$$

and the contractibility of the simplices shows that the map $W_{g^*y} \to W_y$ is always a homotopy equivalence, hence we can diagonalize our bisimplicial sets and apply [5, Lemma 1.4.B][1] to conclude that (1) is homotopy cartesian. □

Theorem B′. *Suppose $F : X \to Y$ is a map of simplicial sets. Suppose for any $m \in \Delta$, any $y \in Y_m$, and any $f : p \to m$ that the map $y|F \to f^*y|F$ induced by f is a homotopy equivalence. Then the square*

(2)
$$\begin{tikzcd} y|F \arrow[r] \arrow[d] & X \arrow[d, "F"] \\ y|Y \arrow[r] & Y \end{tikzcd}$$

is homotopy cartesian.[2]

Proof. We proceed as in the proof of [3, Theorem B, p. 99]. Let $Z = Y|F$, let $F : Z \to YL$ be the evident projection, and for a simplex y of Y we observe that $Z_y = y|F$. The lemma tells us that the large left hand rectangle of the diagram

(3)
$$\begin{tikzcd} \Delta^m \boxtimes (y|F) \arrow[r] \arrow[d] & Y|F \arrow[r, "\sim"] \arrow[d] & XR \arrow[d, "F"] \\ \Delta^m \boxtimes (y|Y) \arrow[r] \arrow[d, "\sim"'] & Y|Y \arrow[r, "\sim"] \arrow[d, "\sim"] & YR \\ \Delta^m L \arrow[r, "y"] & YL & \end{tikzcd}$$

is homotopy cartesian, and the indicated maps are homotopy equivalences by [1, Lemmas 1.3-4].[3] Hence the upper left hand square is also homotopy cartesian, and hence, so is the large upper rectangle. The long horizontal composite maps are independent of the left hand variable in Δ^m, so diagonalizing yields the result. □

References

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[1] A modern combinatorial proof of Lemma 1.4.B can be constructed from [2, IV, 5.2 and 5.7].

[2] Since $y|Y$ is contractible [1, 1.4] the theorem identifies the homotopy fiber of F.

[3] The bisimplicial set $Y|Y$ is the 2-fold edgewise subdivision of Y, defined by $Y_{m,n} = Y(m * n)$ for $m, n \in \Delta$, where $m * n$ is the disjoint union of m and n ordered so the elements of m are all less than the elements of n. The maps in the diagram are the evident projections. The bisimplicial set $Y|F$ is defined as the pullback making the right hand square of (3) cartesian, and one may define $y|F = Z_y$.

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