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SOME REMARKS ABOUT REINHARDT DOMAINS IN $\mathbf{C}^n$

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ABSTRACT. We show that, given a bounded Reinhardt domain D in $\mathbb{C}^n$, there exists a hyperconvex domain Ω such that Ω contains D and every holomorphic function on a neighborhood of $\overline{D}$ extends to a neighborhood of $\overline{\Omega}$. As a consequence of this result, we recover an earlier result stating that every bounded fat Reinhardt domain having a Stein neighbourhoods basis must be hyperconvex. We also study the connection between the Caratheodory hyperbolicity of a Reinhardt domain and that of its envelope of holomorphy. We give an example of a Caratheodory hyperbolic Reinhardt domain in $\mathbf{C}^3$, for which the envelope of holomorphy is not Caratheodory hyperbolic, and we show that no such example exists in $\mathbf{C}^2$.

1. Introduction

Let D be a domain in $\mathbf{C}^n$. We say that D is *Reinhardt* if D is invariant under some action (for a precise definition see Section 2). Reinhardt domains are important objects in complex analysis, and characterizations of properties such as pseudoconvexity, hyperconvexity, and various kinds of hyperbolicity, have been given for such domains in, e.g., [CCW], [Zw1], [Zw2].

The aim of this note is to establish further properties of Reinhardt domains. Our main result, stated in Section 3, is an analogue of the well known fact that every holomorphic function on a neighbourhood of the closure of the Hartogs triangle extends holomorphically to a neighbourhood of the closure of the unit bidisk. In particular, we show that, given a bounded Reinhardt domain D in $\mathbf{C}^n$, there exists a hyperconvex domain Ω such that Ω contains D and every holomorphic function on a neighbourhood of $\overline{D}$ extends to a neighbourhood of $\overline{\Omega}$. As a consequence of this result, we recover an earlier result given in [LNN] which states that every bounded fat Reinhardt domain having a Stein neighbourhoods basis must be hyperconvex. In Section 4, we study the connection between the Caratheodory hyperbolicity of a Reinhardt domain and that of its envelope of holomorphy. It is easy to give an example of a Caratheodory hyperbolic Reinhardt domain in $\mathbf{C}^3$ such that its envelope

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of holomorphy contains a complex line and hence is not Caratheodory hyperbolic. However, we show in Proposition 4.1 that such an example cannot exist in $\mathbf{C}^2$. We conclude the paper with an explicit description of a Stein neighbourhoods basis for the closure of a hyperconvex Reinhardt domain.

2. Preliminaries

A subsect D of $\mathbf{C}^n$ is said to be *Reinhardt* if for every $(\theta_1, \dots, \theta_n) \in \mathbf{R}^n$ we have

$$(z_1, \dots, z_n) \in D \Rightarrow (e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n) \in D.$$

Given a Reinhardt subset D in $\mathbf{C}^n$, we denote by $\log D_*$ its *logarithmic image*, i.e,

$$\log D_* = \{(\log|z_1|, \dots, \log|z_n|) : (z_1, \dots, z_n) \in D_*\},$$

where $D_* = \{(z_1, \dots, z_n) \in D : z_1 \dots z_n \neq 0\}$. In a slight abuse of notation we write $\log D$ instead of $\log D_*$ whenever $D = D_*$.

If D is a domain in $\mathbf{C}^n$ we write $\widehat{D}$ for the envelope of holomorphy of D. In case D is a Reinhardt domain, by a result in [Ca] $\widehat{D}$ is pseudoconvex Reinhardt.

Next, we set

$$V_j = \{z \in \mathbf{C}^n : z_j = 0\}, \quad 1 \leq j \leq n, \quad V = \bigcup_{1 \leq j \leq n} V_j.$$

The following useful criterion for pseudoconvexity of a Reinhardt domain can be found in [Zw1].

Lemma 2.1. *Let D be a Reinhardt domain in $\mathbf{C}^n$. Then the following conditions are equivalent.*

(i) *D is pseudoconvex.*

(ii) *$\log D_*$ is convex, and if $D \cap V_j \neq \emptyset$ for some $1 \leq j \leq n$, then*

$$(z_1, \dots, z_{j-1}, z_j, \dots, z_n) \in D \Rightarrow (z_1, \dots, z_{j-1}, \lambda z_j, \dots, z_n) \in D, \quad |\lambda| < 1.$$

As the referee pointed out to us, in case D is a Reinhardt domain containing 0, the above-mentioned results in [Ca] and [Zw1] are well known and go back to a 1906 paper by Hartogs.

3. Extending holomorphic functions near Reinhardt compact sets

It is well known that every holomorphic function on a neighbourhood of the (closed) Hartogs triangle $\{(z, w) : |z| \leq |w| \leq 1\}$ extends holomorphically to a neighbourhood of the closed unit bidisk. Notice that the unit bidisk is a hyperconvex (in fact, convex) domain, while the Hartogs triangle is *not* hyperconvex. In this section we are interested in finding an extension of this phenomenon to a certain class of compact Reinhardt sets. We first recall that a bounded domain D in $\mathbf{C}^n$ is said to be *hyperconvex* if there is a negative exhaustive continuous plurisubharmonic function for D. It is a remarkable fact that for a bounded domain D to be hyperconvex it is enough to have a weak plurisubharmonic barrier at every point $\xi \in \partial D$, i.e, there exists a non-constant negative plurisubharmonic function ψ on D such that

$$\lim_{z \to \xi} \psi(z) = 0.$$

This fact is perhaps most clearly explained in [Bl]. If the domain in question is pseudoconvex Reinhardt, then we have the following simpler criterion.

LEMMA 3.1. *Let D be a bounded pseudoconvex Reinhardt domain in $\mathbf{C}^n$. Then we have:*

(i) *At every point $\xi \in (\partial D) \backslash V$ there exists a weak plurisubharmonic barrier which extends to a plurisubharmonic function in a neighbourhood of ξ in $\mathbf{C}^n$.*
(ii) *D is hyperconvex if and only if there exists a weak plurisubharmonic barrier at every point $\xi \in (\partial D) \cap V$.*

Proof. (i) The proof is implicitly contained in that of Theorem 2.14 in [CCW]. We therefore omit the details.

(ii) This part follows immediately from Lemma 3.1(i) and Theorem 1.6 in [Bl]. □

We also need a piece of terminology: A compact set K in $\mathbf{C}^n$ is called *admissible* if K has a neighbourhood basis of connected open sets.

THEOREM 3.2. *Let K be an admissible compact Reinhardt domain in $\mathbf{C}^n$. Assume that $K_* \neq \emptyset$ and that either $\log K_*$ is not contained in a hyperplane or $0 \in K$. Then there exists an admissible compact set $\widetilde{K}$ having the following properties:*

(a) *$K \subset \widetilde{K}$ and $\mathrm{Int}(\widetilde{K})$ is a bounded hyperconvex Reinhardt domain.*
(b) *Every holomorphic function on a neighbourhood of K extends to a holomorphic function on a neighbourhood of $\widetilde{K}$.*

The following lemma is the key ingredient in the proof of Theorem 3.2.

LEMMA 3.3. *Let $\{U_k\}_{k \geq 1}$ be a decreasing sequence of bounded pseudoconvex Reinhardt domains in $\mathbf{C}^n$ satisfying $\overline{U_{k+1}} \subset U_k$. Let*

$$G = \bigcap_{k \geq 1} U_k.$$

Assume that $\mathrm{Int}(G)$ is not empty. Then $\mathrm{Int}(G)$ is a hyperconvex Reinhardt domain.

Proof of Lemma 3.3. Set $\Omega = \mathrm{Int}(G)$. We first show that Ω is connected. For this, it suffices to check that $\Omega_* = \Omega \setminus V$ is connected. Indeed, take two arbitrary points $a, b \in \Omega_*$ and set

$$a' = (\log |a_1|, \dots, \log |a_n|) \in \log \Omega_*,$$
$$b' = (\log |b_1|, \dots, \log |b_n|) \in \log \Omega_*.$$

There exist two small balls S_1 and S_2 centered at a' and b' such that $S_1 \cup S_2 \subset \log \Omega_*$. Then

$$S_1 \cup S_2 \subset \log(U_n)_*, \quad n \geq 1.$$

Since $\log(U_n)_*$ is a convex domain in $\mathbf{R}^n$, we infer that

$$\mathrm{conv}(S_1 \cup S_2) \subset \log(U_n)_*, \quad n \geq 1.$$

Thus $\mathrm{conv}(S_1 \cup S_2) \subset \log G_*$, and consequently

$$\mathrm{conv}(S_1 \cup S_2) \subset \log \Omega_*.$$

This implies that Ω_* is connected. Notice that $\widehat{\Omega} \subset U_k$ for all k. Thus $\widehat{\Omega} = \Omega$, and hence Ω is a pseudoconvex Reinhardt domain.

It remains to check that Ω is hyperconvex. By Lemma 3.1(i) it is enough to show the existence of a weak plurisubharmonic barrier at every point $a \in (\partial\Omega) \cap V$. To see this, let $a \in (\partial\Omega) \cap V$. We first show that a cannot be 0.

Assume $a = 0$. Take an arbitrary point $p = (p_1, \dots, p_n)$ in Ω_*. Fix $k \geq 1$. Since $\overline{\Omega} \subset \overline{U_{k+1}} \subset U_k$, we can find $\varepsilon > 0$ such that

$$\{z \in \mathbf{C}^n : |z_1| < \varepsilon, \dots, |z_n| < \varepsilon\} \cup \{p\} \subset U_k.$$

Since $\log(U_k)_*$ is convex we have

$$\begin{aligned} (1) \qquad \log(U_k)_* &\supset \mathrm{conv}\left((\log|p_1|, \dots, \log|p_n|) \cup \{(x_1, \dots, x_n) : x_j < \log \varepsilon\}\right) \\ &\supset \{(x_1, \dots, x_n) : x_1 < \log|p_1|, \dots, x_n < \log|p_n|\}. \end{aligned}$$

It follows that

$$(2) \qquad \{z \in \mathbf{C}^n : |z_i| < |p_i|, 1 \leq i \leq n\} \subset \overline{U_k}, \quad k \geq 1.$$

Therefore, $0 \in \mathrm{Int}(G) = \Omega$, which is a contradiction. This proves the claim that $a \neq 0$.

We may therefore assume that a is of the form $a = (0, 0, \dots, 0, a_{k+1}, \dots, a_n)$, where $1 \leq k < n$ and $a_j \neq 0$ for all $k+1 \leq j \leq n$. Let π denote the projection $(z_1, \dots, z_n) \mapsto (z_{k+1}, \dots, z_n)$. By Lemma 2.1 $\pi(\Omega)$ is a pseudoconvex Reinhardt domain in $\mathbf{C}^{n-k}$. We claim that $\pi(a) \notin \pi(\Omega)$. Otherwise we can find $\alpha = (c_1, c_2, \dots, c_k, a_{k+1}, \dots, a_n) \in \Omega_*$. Set

$$\widetilde{\alpha} = (\log|c_1|, \dots, \log|c_k|, \log|a_{k+1}|, \dots, \log|a_n|) \in \log \Omega_*.$$

Then there exists a small ball $\widetilde{S}$ centered at $\widetilde{\alpha}$ such that $\widetilde{S} \subset \log \Omega_*$. This implies that there exists a Reinhardt neighbourhood (in $\mathbf{C}^n$) S of α such that

$$\log S = \widetilde{S} \subset \log \Omega_*.$$

We deduce that there exists $\delta > 0$ such that

$$\min(|z_1|, \dots, |z_n|) > \delta, \quad (z_1, \dots, z_n) \in S.$$

Fix $m \geq 1$. We have $U_m \cap V_j \neq \emptyset$ for $j = 1, \dots, k$. By Lemma 2.1 we get

$$(z_1, \dots, z_n) \in S \Rightarrow (0, \dots, 0, z_{k+1}, \dots, z_n) \in U_m.$$

Now, let $\widetilde{\pi}$ be the projection $(x_1, \dots, x_n) \mapsto (x_{k+1}, \dots, x_n)$ and fix $(z_1^0, \dots, z_n^0) \in S$. We can find $\varepsilon > 0$ small enough such that

$$\left\{(z_1, \dots, z_k, z_{k+1}^0, \dots, z_n^0) : |z_j| < \varepsilon, \ j = 1, \dots, k\right\} \subset U_m.$$

This implies that

$$\log(U_m)_* \supset (\log|z_1^0|, \dots, \log|z_n^0|) \cup \Big\{\{(x_1, \dots, x_k) : x_j < \log\varepsilon, \ j = 1, \dots, k\} \times (\log|z_{k+1}^0|, \dots, \log|z_n^0|)\Big\}.$$

and hence

$$\begin{aligned} \log(U_m)_* \supset \operatorname{conv}\Big((\log|z_1^0|, \dots, \log|z_n^0|) \\ \cup \left\{(x_1, \dots, x_k, \log|z_{k+1}^0|, \dots, \log|z_n^0|) : x_j < \log\varepsilon, \ j = 1, \dots, k\right\}\Big) \\ \supset \left\{(x_1, \dots, x_k, \log|z_{k+1}^0|, \dots, \log|z_n^0|) : x_j < \log|z_j^0|, \ j = 1, \dots, k\right\}. \end{aligned}$$

It follows that

$$\log(U_m)_* \supset \{(x_1, \dots, x_k) : x_j < \log\delta\} \times \widetilde{\pi}(\log S).$$

This implies that G contains a neighbourhood of a. In other words, $a \in \operatorname{Int}(G) = \Omega$, which is absurd. Thus $\pi(a) \notin \pi(D)$ and therefore $\pi(a) \in \partial\pi(D)$. Hence we can find a weak plurisubharmonic barrier u at $\pi(a)$ in $\pi(D)$. It follows that $u \circ \pi$ is a weak plurisubharmonic barrier for a in D. Therefore Ω is hyperconvex. □

Proof of Theorem 3.2. For each $k \geq 1$ we define

$$D_k = \left\{z \in \mathbf{C}^n : \operatorname{dist}(z, K) < \frac{1}{k}\right\}.$$

As K is admissible, D_k is a Reinhardt domain for all $k \geq 1$. Set

$$U_k = \widehat{D_k}, \quad \widetilde{K} = \bigcap_{k=1}^{\infty} U_k, \quad \Omega = \operatorname{Int}(\widetilde{K}).$$

First, note that for all k, U_k is a bounded pseudoconvex Reinhardt domain. Next, we claim that $\overline{U_{k+1}} \subset U_k$ for all k. To see this, we let W_k be a pseudoconvex domain which is relatively compact in U_k and satisfies $D_{k+1} \subset W_k$. Such a domain exists since $\overline{D_{k+1}} \subset D_k$. Thus $\overline{U_{k+1}} \subset U_k$, and the claim follows. Hence we have

$$\overline{\widetilde{K}} \subset \overline{U_{k+1}} \subset U_k, \quad k \geq 1.$$

We infer that $\overline{\widetilde{K}} = \widetilde{K}$, so $\widetilde{K}$ is compact, and obviously $K \subset \widetilde{K}$. It remains to prove that Ω is hyperconvex. By Lemma 3.3, this will follow if we can show that $\Omega \neq \emptyset$. There are two cases to be considered:

(a) If $\log K_*$ is not contained in any hyperplane (in $\mathbf{R}^n$), then $\operatorname{conv}(\log K_*)$ has non-empty interior. Since $\log(U_k)_*$ is a convex set containing $\operatorname{conv}(\log K_*)$, we infer that $\Omega \neq \emptyset$.

(b) If $0 \in K$, then we take a point $a = (a_1, \ldots, a_n) \in K_*$. Using (1) and (2) from the proof of Lemma 3.3 we see that

$$\{(z_1, \ldots, z_n) : |z_j| < |a_j|\} \subset D_k, \quad k \geq 1.$$

Thus $0 \in \Omega$.

The proof is thereby finished. □

REMARKS. (1) It follows easily from the construction of $\widetilde{K}$ that $\widetilde{K}$ is the *largest* compact set satisfying (b) in Theorem 3.2.

(2) If $K = \{(z, w) : |z| \leq |w| \leq 1\}$ is the closed Hartogs triangle, then (1) and (2) imply that $\widetilde{K} = \{(z, w) : \max(|z|, |w|) \leq 1\}$ is the closed unit bidisk. Thus we recover the well known fact mentioned at the beginning of the section.

(3) The condition $K_* \neq \emptyset$ is absolutely necessary for $\operatorname{Int}(\widetilde{K}) \neq \emptyset$ to hold. Indeed, assume that $K_* = \emptyset$ and $\operatorname{Int}(\widetilde{K}) \neq \emptyset$. Then we can find $\delta \neq 0$ such that the hypersurface $z_1 \ldots z_n = \delta$ meets $\operatorname{Int}(\widetilde{K})$. It follows that the function

$$f(z) = \frac{1}{z_1 \ldots z_n - \delta}$$

is holomorphic on a neighbourhood of K but is nonextendible to any neighbourhood of $\widetilde{K}$. This contradicts the choice of $\widetilde{K}$.

(4) If $\log K_*$ is contained in a hyperplane H of $\mathbf{R}^n$ and $K \cap V = \emptyset$, then $\operatorname{Int}(\widetilde{K}) = \emptyset$. Indeed, notice that $\operatorname{conv}(\log K_*)$ is a convex compact subset of H. Thus there exists a decreasing sequence of convex domains L_k in $\mathbf{R}^n$ such that $\bigcap_{k \geq 1} L_k = \operatorname{conv}(\log K_*)$. Let $\widetilde{L}_k$ be the pseudoconvex Reinhardt domain in $\mathbf{C}^n$ defining by $\log \widetilde{L}_k = L_k$. Clearly, $K \subset \widetilde{L}_k$ for all k, and $\bigcap_{k \geq 1} \widetilde{L}_k$ has empty interior. As $K \cap V = \emptyset$, we see that $K \subset \widetilde{L}_k$ for all k. It follows that $\operatorname{Int}(\widetilde{K}) = \emptyset$.

From Theorem 3.2 we deduce immediately the following result of [LNN].

COROLLARY 3.4. *Let D be a bounded Reinhardt domain in $\mathbf{C}^n$. Assume that D is fat, i.e,* $\operatorname{Int}(\overline{D}) = D$, *and that $\overline{D}$ has a Stein neighbourhoods basis. Then D is hyperconvex.*

As a converse to this statement, at the end of this paper we will give a concrete construction of a Stein neighbourhoods basis for the closure of a hyperconvex Reinhardt domain.

4. Caratheodory hyperbolicity of Reinhardt domains

In this section we are concerned with the relation between Caratheodory hyperbolicity of a Reinhardt domain D and that of its envelope of holomorphy $\widehat{D}$. Before formulating the main result of this section, we introduce some notation and terminology.

Let D be an arbitrary domain in $\mathbf{C}^n$ and Δ the unit disk in $\mathbf{C}$. We denote by $H(D,\Delta)$ the set of holomorphic maps from D into Δ. The Caratheodory distance on D is then defined by

$$c_D(z,w) = \sup\{\rho(f(z), f(w)) : f \in H(D,\Delta)\}$$

for $z, w \in D$. Here ρ is the Poincaré distance on Δ. The domain D is called Caratheodory hyperbolic if $c_D(z,w) > 0$ for $z \neq w$. In other words, D is Caratheodory hyperbolic if and only if the space $H^\infty(D)$ of bounded holomorphic functions on D separates the points of D.

The following simple example shows that the Caratheodory hyperbolicity of D does not imply that of $\widehat{D}$. Consider the following two domains:

$$D = \{(z_1,z_2,z_3) \in \mathbf{C}^3 : \max(|z_1z_2|, |z_1z_3|, |z_2|, |z_3|) < 1\},$$
$$D' = D\backslash\{(z_1,0,0)\}.$$

Clearly D is not Caratheodory hyperbolic (as it contains a complex line), while D' is so (since the four functions z_1z_2, z_1z_3, z_2, z_3 separate points of D'). However $\widehat{D'} = D$ since the line $\{(z_1,0,0)\}$ is of codimension 2 in $\mathbf{C}^3$.

We now show that in $\mathbf{C}^2$ this implication does hold.

Proposition 4.1. *Let D be a Caratheodory hyperbolic domain in $\mathbf{C}^2$. Then $\widehat{D}$ is Caratheodory hyperbolic in $\mathbf{C}^2$.*

To prove this result we need the following lemma.

Lemma 4.2. *Let D be a Reinhardt domain in $\mathbf{C}^n$ satisfying $D \cap V = \emptyset$. Then we have $\widehat{D} \cap V = \emptyset$ and*

$$\log \widehat{D} = \operatorname{conv}(\log D).$$

Proof of Lemma 4.2. The result is undoubtedly well known, but due to a lack of a suitable reference we offer a short proof. As $D \cap V = \emptyset$, we have $\widehat{D} \cap V = \emptyset$. Since $\log \widehat{D}$ is a convex domain containing $\log D$, we see that $\operatorname{conv}(\log D) \subset \log \widehat{D}$. To obtain the reverse inclusion, we let D' be the Reinhardt domain in $\mathbf{C}^n_*$ satisfying $\log D' = \operatorname{conv}(\log D)$. It is clear that $D \subset D'$ and D' is pseudoconvex in $\mathbf{C}^n$. The desired conclusion now follows. □

Proof of Proposition 4.1. In view of Theorem 2.5.1 (iv) in [Zw2] we need to show that $\log(\widehat{D})_*$ contains no affine line and that $\widehat{D} \cap V_j$ is either empty or Caratheodory hyperbolic (viewed as subset of $\mathbf{C}$) for $j = 1, 2$. To this end, we proceed in three steps.

Step 1: $\widehat{D_*}$ is Caratheodory hyperbolic. Indeed, since $\widehat{D_*} \cap V = \emptyset$, by Theorem 2.5.1 (iv) in [Zw2] and Lemma 4.2 it suffices to show that $\operatorname{conv}(\log D_*)$ contains no affine line. Suppose there is an affine line $l \subset \operatorname{conv}(\log D_*)$. Let x_0 be any point in $\log(D_*)\backslash l$. By considering the convex hull of $x_0 \cup l$ we see that there exists an affine line l' parametrized by

$$l' = \{(a_1 x + b_1, a_2 x + b_2) : x \in \mathbf{R}\}, \quad a_1^2 + a_2^2 \neq 0,$$

such that $l' \cap \log D_* \neq \emptyset$ and $l' \subset \operatorname{conv}(\log D_*)$. It follows that $\widehat{D_*}$ contains the analytic curve $\varphi(\mathbf{C})$, where

$$\varphi(z) = (e^{a_1 z + b_1}, e^{a_2 z + b_2}).$$

Since each function in $\mathcal{H}^\infty(D)$ extends to an element of $\mathcal{H}^\infty(\widehat{D})$, by the Liouville theorem we deduce that every function in $\mathcal{H}^\infty(D)$ is constant on $\varphi(\mathbf{C})$. Since D is Caratheodory hyperbolic, this implies that $\varphi(\mathbf{C}) \cap D = \emptyset$. Thus $l' \cap \log D_* = \emptyset$, which is a contradiction. Hence $\widehat{D_*}$ is Caratheodory hyperbolic.

Step 2: $\log(\widehat{D})_*$ contains no affine line. If not, then by the same reasoning as in the first step we get a holomorphic map $\psi : \mathbf{C} \to \mathbf{C}_*^2$ such that $\psi(\mathbf{C}) \subset (\widehat{D})_*$. On the other hand, by the result obtained in Step 1, $\widehat{D_*}$ is Caratheodory hyperbolic. According to Theorem 2.5.1 (vi) in [Zw2] there exists a biholomorphism mapping Φ from $\mathbf{C}_*^2$ onto $\mathbf{C}_*^2$ which is bounded on $\widehat{D_*}$. The Riemann extension theorem implies that Φ extends to a bounded holomorphic mapping on D, which we also denote by Φ. Now, applying the Liouville theorem to the composite map $\Phi \circ \psi$, we deduce that Φ is constant on $\psi(\mathbf{C})$. This is absurd and hence proves the claim of Step 2.

Step 3: $\widehat{D} \cap V_j$ is either empty or Caratheodory hyperbolic (viewed as subset of $\mathbf{C}$) for $j = 1, 2$. Suppose this is not the case. We may assume, without loss of generality, that $D' = \widehat{D} \cap V_2 \neq \emptyset$ and that D' is not Caratheodory hyperbolic. Since each component of D' is a Reinhardt domain in $\mathbf{C}$, there are only two cases to be considered:

(a) If $D' = \mathbf{C}$, then obviously $\widehat{D}$ contains the analytic curve $\varphi(z) = (z, 0)$. By the Liouville theorem we have $\varphi(\mathbf{C}) \cap D = \emptyset$. Thus $\widehat{D} \backslash \varphi(\mathbf{C})$ is a pseudoconvex domain containing D, but strictly contained in $\widehat{D}$, which is absurd.

(b) If $D' = \mathbf{C}_*$, then by considering the analytic curve $\varphi(z) = (e^z, 0)$ and using an argument similar to that of the previous case, we also arrive at a contradiction.

The proof of Proposition 4.1 is thus complete. □

To conclude this section, we present an example showing that for Kobayashi hyperbolicity the conclusion of Proposition 4.1 fails even in the case $\mathbf{C}^2$.

EXAMPLE 4.3. Let φ be an upper semicontinuous function defined on $[0,1)$ with values in $(-\infty,0)$. Assume that $\lim_{t\to 1-0}\varphi(t)=-\infty$. Then the Hartogs domain

$$\Omega_\varphi(\Delta)=\{(z_1,z_2)\in\Delta\times\mathbf{C}:|z_2|<e^{-\varphi(|z|)}\}$$

is Kobayashi hyperbolic, but its envelope of holomorphy is not.

Proof. Since φ is locally bounded from below, the domain $\Omega_\varphi(\Delta)$ is Kobayashi hyperbolic (see [TT]). To compute the envelope of holomorphy of $\Omega_\varphi(\Delta)$, we let $\widetilde{\varphi}$ be the largest subharmonic minorant of φ on Δ. It is easy to see that $\widetilde{\varphi}(z)$ goes to $-\infty$ when z tends to $\partial\Delta$. By applying the maximum principle we obtain $\widetilde{\varphi}\equiv-\infty$. This implies that the envelope of holomorphy of $\Omega_\varphi(\Delta)$ equals $\Delta\times\mathbf{C}$, which is obviously not Kobayashi hyperbolic. This completes the proof. □

5. Appendix

In this section we will give a concrete description of a Stein neighbourhoods basis for the closure of a hyperconvex Reinhardt domain.

We first fix some notations. For $\alpha>0$ we define the following (possibly multivalued) function

$$z^\alpha=\begin{cases}e^{\alpha(\log|z|+i\arg z)}, & z\neq 0,\\ 0, & z=0.\end{cases}$$

Given a subset D in $\mathbf{C}^n$, we set

$$D^\alpha=\{(z_1^\alpha,\dots,z_n^\alpha):(z_1,\dots,z_n)\in D\}.$$

PROPOSITION 5.1. *Let D be a bounded hyperconvex Reinhardt domain in $\mathbf{C}^n$. Then the domains $\{D^\alpha\}_{\alpha>1}$ form a Stein neighbourhoods basis of $\overline{D}$.*

We require the following result due to Zwonek ([Zw2, Corollary 2.6.11])

LEMMA 5.2. *A bounded pseudoconvex Reinhardt domain D in $\mathbf{C}^n$ is hyperconvex if and only if $D\cap V_j\neq\emptyset$ for every $j\in\{1,\dots,n\}$ satisfying $\overline{D}\cap V_j\neq\emptyset$. In particular, $0\notin\partial D$.*

Proof of Proposition 5.1. We split the proof into three steps.

Step 1: D^α is a pseudoconvex Reinhardt domain for any $\alpha>0$. First it is immediate that D^α is a Reinhardt domain. It is also easy to check that $\log(D^\alpha)_*=\alpha(\log D_*)$. Since D is a pseudoconvex Reinhardt domain, by Lemma 2.1 so is D^α.

Step 2: $\overline{D^\beta}\subset D^\alpha$ for all $0<\beta<\alpha$. Without loss of generality we may assume that $\beta=1$. Let $a\in\overline{D}$. As $\log D_*$ is convex we have

$$\overline{\log D_*}\subset\alpha(\log D_*)=\log(D^\alpha)_*.$$

Thus, if $a \notin V$, then $a \subset D^\alpha$. If $a = 0$, then by Lemma 5.2 we have $a \in D$, and thus $a \in D^\alpha$. It remains to consider the case $a \neq 0$. We may assume that $a = (0, \ldots, 0, a_{k+1}, \ldots, a_n)$, where $a_j \neq 0$ for $k+1 \leq j \leq n$. Let π be the projection $(z_1, \ldots, z_n) \mapsto (z_{k+1}, \ldots, z_n)$. From the above reasoning we see that $\pi(a) \in \pi(D^\alpha)$. Now by Lemma 5.2 we have $D \cap V_j \neq \emptyset$ for all $1 \leq j \leq k$. Thus Lemma 2.1 implies that $a \in D^\alpha$.

Step 3: $\overline{D} = \bigcap_{\alpha>1} D^\alpha$. Let $a = (a_1, \ldots, a_n) \in \bigcap_{\alpha>1} D^\alpha$. We have to show that $a \in \overline{D}$. If $a \notin V$, then since $\log D_*$ is convex we have

$$(\log |a_1|, \ldots, \log |a_n|) \in \bigcap_{\alpha>1} \alpha(\log D_*) = \overline{\log D_*}.$$

It follows that $a \in \overline{D}$. If $a \in V$, then we may assume that $a = (0, \ldots, 0, a_{k+1}, \ldots, a_n)$, where $a_j \neq 0$ for $k+1 \leq j \leq n$. It follows from the above argument that $\pi(a) \in \overline{\pi(D)}$, where π is the projection $(z_1, \ldots, z_n) \mapsto (z_{k+1}, \ldots, z_n)$. On the other hand, we notice easily that $\overline{D} \cap V_j \neq \emptyset$ for all $1 \leq j \leq k$. Thus Lemma 5.2 implies $D \cap V_j \neq \emptyset$ for all $1 \leq j \leq k$. By Lemma 2.1 we get $a \in \overline{D}$. □

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