Illinois Journal of Mathematics
Volume 47, Number 3, Fall 2003, Pages 633–648
S 0019-2082

LOWER BOUNDS FOR GENERALIZED UPCROSSINGS OF ERGODIC AVERAGES

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Abstract. New lower bound inequalities are obtained for generalized upcrossings of ergodic averages. The results and techniques are compared with those of E. Bishop on upper bounds. Moreover, a connection between these results and spatial oscillations is established.

1. Introduction

Upcrossing inequalities (u.i.) are a basic tool in studying ergodic averages. In particular, they imply the ergodic theorem. E. Bishop [2][3] established upper bounds for (generalized) upcrossings by using two different techniques and in a more general setting than that of ergodic averages. We complement these results by giving lower bounds for generalized upcrossings in the setting of measure preserving transformations and Cesaro averages. The main motivation for studying lower bounds is that they give information on the number of spatial oscillations for the ergodic averages. Lower bounds, in the form of reverse inequalities, have also been studied in [5], but from a different perspective.

We now describe the main result in our paper which establishes a strikingly tight inequality. (Precise definitions are given later in the paper.) Let $w_{\eta,\alpha,n}(x)$ denote the number of generalized upcrossings up to time n with respect to a function f and a transformation τ. Setting $w_{\eta,\alpha}(x) = \sup_n w_{\eta,\alpha,n}(x)$, a constructive result of Bishop implies (using classical arguments)

$$\int \eta\ w_{\eta,\alpha}(x) d\mu(x) \leq \int (f-\alpha)_+ d\mu(x).$$

Under appropriate conditions, our Theorem 2 shows that

$$\int (f-\alpha-\eta)_+ d\mu(x) \leq \int \eta\ w_{\eta,\alpha}(x) d\mu(x).$$

Received June 9, 2002; received in final form October 29, 2002.
2000 *Mathematics Subject Classification.* Primary 28D99. Secondary 47A35.
Research partially supported by NSERC.

We note the fact that our result, in contrast to Bishop's theorem, requires using the function $w_{\eta,\alpha}(x)$; it is not possible to obtain a similar result using the finite time quantity $w_{\eta,\alpha,n}(x)$.

The paper is organized as follows. In Section 2 we introduce the main definitions and proceed to prove the basic counting inequalities. We present new results on lower bounds along with Bishop's results on upper bounds (as presented in [3]). This can be done with little extra effort and shows the similarities and the differences between our arguments and those of Bishop. Section 3 introduces the concepts and intermediate results needed to integrate the pointwise inequalities from Section 2; our main result, Theorem 2, is then proved. Section 4 draws connections between generalized upcrossings and other measures of spatial oscillations. Furthermore, Proposition 2 gives information on the pointwise asymptotics of generalized upcrossings. For completeness, the brief Section 5 states the dual results for downcrossings. Finally, the Appendix states, for the reader's convenience, a known result needed in the main body of the paper.

2. Pointwise inequalities for generalized upcrossings

We adopt the convention that pointwise inequalities not containing explicit quantifiers referring to a point x are valid for all values of x for which the quantities involved are defined. In our setting, this means almost everywhere (a.e.) on the measure space.

Definition 1. Given an integer $n \geq 0$, a sequence $P = (s_1, t_1, \ldots, s_m, t_m)$ is called *n-admissible* if $-1 \leq s_1 < t_1 \leq s_2 < t_2 \leq \cdots \leq s_m < t_m \leq n$. We let $|P| = m$ denote the *size* of P. The finite set of all n-admissible sequences is denoted by $\mathcal{P}^n$. We allow the empty sequence $P = \emptyset$ and define $|P| = 0$ in this case.

Definition 2. Let $a_{-1}, a_0, \ldots, a_n$ and $b_{-1}, b_0, \ldots, b_n$ be given real numbers. A sequence $P = (u_1, v_1, \ldots, u_N, v_N)$ is called an *n-crossing sequence* if it is an n-admissible sequence which satisfies

$$\begin{aligned} a_{u_i} &\leq b_{v_i}, \ i = 1, \ldots, N, \\ a_{u_{i+1}} &\leq b_{v_i}, \ i = 1, \ldots, N-1. \end{aligned} \tag{1}$$

Thus, a crossing sequence is a special kind of admissible sequence. The finite set of all n-crossing sequences is denoted by $\mathcal{P}_0^n$.

For a nonempty admissible sequence $P = (s_1, t_1, \ldots, s_m, t_m)$ we define

$$S(P) = \sum_{i=1}^{|P|} (b_{t_i} - a_{s_i}). \tag{2}$$

If P is empty we define $S(P) = 0$. We let $\mathcal{P}_1^n$ be the set of n-admissible sequences P_1 with $S(P_1)$ maximal in $\mathcal{P}^n$ (i.e., the maximum is taken over $\mathcal{P}^n$), and we let $\mathcal{P}_2^n$ be the set of sequences P_2 in $\mathcal{P}_1^n$ with $|P_2|$ maximal in $\mathcal{P}_1^n$.

The next lemma is essentially contained in Lemma 6 of [3, pp. 234–235]. We have added material needed to prove lower bounds and we rewrote the lemma to serve our needs.

Lemma 1. *The following two statements hold for any $n \geq 0$:*

(i) $\mathcal{P}_1^n \subseteq \mathcal{P}_0^n$.

(ii) *If $P_2 \in \mathcal{P}_2^n$, then*

$$|P_2| \geq |P| \text{ for all } P \in \mathcal{P}_0^n.$$

Proof. (i) Let $P_1 = (s_1, t_1, \dots, s_m, t_m)$ belong to $\mathcal{P}_1^n$. If $P_1 \notin \mathcal{P}_0^n$, there would exist i, $1 \leq i \leq m$, such that $a_{s_i} > b_{t_i}$, or $1 \leq i \leq m-1$ such that $a_{s_{i+1}} > b_{t_i}$. Then, deleting (s_i, t_i) or (t_i, s_{i+1}) from P_1 we would obtain an n-admissible sequence Q with $S(Q) > S(P_1)$, contradicting the maximality of $S(P_1)$. Thus $\mathcal{P}_1^n \subseteq \mathcal{P}_0^n$.

(ii) Let $P_2 = (s_1, t_1, \dots, s_m, t_m)$ belong to $\mathcal{P}_2^n$ and let $P = (u_1, v_1, \dots, u_N, v_N) \in \mathcal{P}_0^n$. It is not possible that there exists $i \in \{1, \dots, N-1\}$ and $j \in \{1, \dots, m\}$ such that

$$s_j < v_i \leq u_{i+1} < t_j, \tag{3}$$

for otherwise (i.e., if (3) holds) $P' = (s_1, t_1, \dots, s_j, v_i, u_{i+1}, t_j, \dots, s_m, t_m)$ would be n-admissible with $|P'| = |P_2| + 1$ and

$$\begin{aligned} S(P') &= \sum_{k=1}^{j-1} \left(b_{t_j} - a_{s_j}\right) + b_{v_i} - a_{s_j} + b_{t_j} - a_{u_{i+1}} + \sum_{k=j+1}^{m} \left(b_{t_j} - a_{s_j}\right) \\ &= S(P) + b_{v_i} - a_{u_{i+1}} \geq S(P), \end{aligned}$$

which contradicts the fact that $P_2 \in \mathcal{P}_2^n$.

For convenience set $t_0 = -1$ and $s_{m+1} = n$. By a similar argument, we see that it is also not possible that, for some $i = 1, \dots, N$ and some $j = 0, \dots, m$,

$$t_j \leq u_i < v_i \leq s_{j+1}. \tag{4}$$

Let us note that $u_i < t_m$ for all $i = 1, \dots, N$. Thus for each $i = 1, \dots, N$ there exists $j = j(i) = 1, \dots, m$ such that

$$u_i \in [t_{j-1}, t_j).$$

Now if $i_1 < i_2$, we have $t_{j(i_1)-1} \leq u_{i_1}$, and keeping in mind that (3) and (4) do not hold, we conclude

$$s_{j(i_1)} < v_{i_1}, \text{ and thus } t_{j(i_1)} \leq u_{i_2}.$$

This shows that the map $i \to j(i)$ is injective, so $N \leq m$. $\square$

We specialize the general setting just introduced to the following situation: For a given real-valued and measurable function $f(x)$ and a measurable point transformation τ on a measure space $(X, \mathcal{F}, \mu)$ we let $f_j(x) = T^j f(x) = f(\tau^j x)$ (so that $f_0 = f$). We define $\mathcal{I} = \{A \in \mathcal{F} : \tau^{-1}(A) = A\}$ and call a set $A \in \mathcal{I}$ an invariant subset. We use the notation $A_t f(x) = 1/(t+1) \sum_{j=0}^{t} f(\tau^j x)$ and set $A_{-1} f(x) = 0$ for all x. For real numbers α and η ($\eta > 0$) and given x we specialize the fixed finite sequences $\{a_i\}$ and $\{b_i\}$ in Definition 2 as follows:

$$b_t = b_t(x) = b_{t,\eta,\alpha}(x) = \sum_{j=0}^{t} \left(f_j(x) - \alpha - \eta\right),$$

$$a_s = a_s(x) = a_{s,\alpha}(x) = \sum_{j=0}^{s} \left(f_j(x) - \alpha\right).$$

Of course, in the expressions above, a sum over an empty set has the value 0. We extend the notations introduced earlier in a natural way by making explicit reference to the point x and possibly to other parameters. For example, if $P = (s_1, t_1, \ldots, s_m, t_m) \in \mathcal{P}^n$ we define $S(P)(x) = \sum_{j=1}^{m} \left(b_{t_j}(x) - a_{s_j}(x)\right)$. Then $\mathcal{P}_1^n(x, \eta, \alpha)$ (or, more compactly, $\mathcal{P}_1^n(x)$) is the set of elements $P \in \mathcal{P}^n$ such that $S(P)(x)$ is maximal. In particular, $\mathcal{P}_0^n$ specializes to $\mathcal{P}_0^n(x, \eta, \alpha)$. An element P of this set is called an *n-generalized upcrossing sequence at x*. The reason for using the term *generalized upcrossing* instead of *crossing* is given by Proposition 1 (see also the remarks preceding Corollary 1) and our use of crossings in Section 5. We will omit some of the parameters (mainly α and η) if this causes no confusion. We use the notation

$$\lambda_{\eta,\alpha,n}(x) = \max_{P \in \mathcal{P}^n} S(P)(x) = S(P_1)(x), \tag{5}$$

where P_1 is any element in $\mathcal{P}_1^n(x)$.

In the remainder of this paper, except for Section 5, the above conventions and assumptions will be used freely without explicit mention.

Definition 3. For a given integer $n \geq 0$, we define the (maximal) *number of n-generalized upcrossings* at x by

$$w_{\eta,\alpha,n}(x) = \max\{|P| : P \text{ is an } n\text{-generalized upcrossing sequence at } x\}.$$

Also, we define the number of generalized upcrossings at x by

$$w_{\eta,\alpha}(x) = \lim_{n \to \infty} w_{\eta,\alpha,n}(x).$$

Without further restrictions nothing prevents $w_{\eta,\alpha}(x)$ from being infinite at this moment. We use the word *generalized* to distinguish $w_{\eta,\alpha,n}(x)$ from the usual (geometric) upcrossings defined in Definition 4.

Lemma 1 implies the following corollary.

Corollary 1.

$$|P_2| = w_{\eta,\alpha,n}(x) \text{ and } \lambda_{\eta,\alpha,n}(x) = S(P_2)(x), \tag{6}$$

where P_2 *is any sequence in* $\mathcal{P}_2^n(x)$.

The following lemma is key to the proof of the lower bounds in Theorem 2.

Lemma 2. *For all* $n \geq 1$ *the following holds:*

$$\lambda_{\eta,\alpha,n-1}(\tau x) - \lambda_{\eta,\alpha,n}(x) \leq \eta\ w_{\eta,\alpha,n-1}(\tau x) - (f(x) - \alpha - \eta)_+. \tag{7}$$

Proof. Suppose $P' = (s'_1, t'_1, \ldots, s'_{m'}, t'_{m'}) \in \mathcal{P}^{n-1}$ is nonempty, and let $s_i = s'_i + 1$, $t_i = t'_i + 1$ for $i = 1, \ldots, m'$. This defines $P = (s_1, t_1, \ldots, s_{m'}, t_{m'}) \in \mathcal{P}^n$ with $s_1 \geq 0$. Then

$$S(P')(\tau x) - \eta\ m' = S(P)(x) \leq \lambda_{\eta,\alpha,n}(x) - (f(x) - \alpha - \eta)_+. \tag{8}$$

The equality in (8) can be checked directly. To prove the inequality in (8) notice first that if $(f(x) - \alpha - \eta) \leq 0$ the inequality holds. On the other hand, if $(f(x) - \alpha - \eta) \geq 0$, define $Q = (s_0 = -1, t_0 = 0, s_1, t_1, \ldots, s_{m'}, t_{m'}) \in \mathcal{P}^n$ and notice that $S(P)(x) = S(Q)(x) - (f(x) - \alpha - \eta) \leq \lambda_{\eta,\alpha,n}(x) - (f(x) - \alpha - \eta)_+$. In the case when $P' = \emptyset$, equation (8) still holds since $\lambda_{\eta,\alpha,n}(x) \geq (f(x) - \alpha - \eta)_+$. Therefore, since P' is an arbitrary element of $\mathcal{P}^{n-1}$, evaluating (8) at $P' \in \mathcal{P}_2^{n-1}(\tau x)$ and using (6) we obtain

$$\lambda_{\eta,\alpha,n-1}(\tau x) - \lambda_{\eta,\alpha,n}(x) \leq \eta\ w_{\eta,\alpha,n-1}(\tau x) - (f(x) - \alpha - \eta)_+, \tag{9}$$

which is valid for any $n \geq 1$. □

The next result complements the previous lemma; it can be found, in a more general setting, in [3].

Lemma 3.

$$\eta\ w_{\eta,\alpha,n}(x) \leq \lambda_{\eta,\alpha,n-1}(\tau x) - \lambda_{\eta,\alpha,n}(x) + (f(x) - \alpha)_+. \tag{10}$$

Proof. Let $P = (s_1, t_1, \ldots, s_m, t_m) \in \mathcal{P}^n$ and define $P' = (s'_1, t'_1, \ldots, s'_{m'}, t'_{m'}) \in \mathcal{P}^{n-1}$ as follows: If $t_1 > 0$ and $m \geq 1$, let $t'_i = t_i - 1$, $s'_i = s_i - 1$ for $i = 1, \ldots, m$ and $m' = m$ with the understanding that $s'_1 = -1$ if $s_1 = -1$. If $t_1 = 0$ and $m \geq 2$, take $m' = m - 1$ and $t'_i = t_{i+1} - 1$, $s'_i = s_{i+1} - 1$ for $i = 1, \ldots, m'$. In the case $t_1 = 0$ and $m = 1$, take $P' = \emptyset$. We then obtain

$$\begin{aligned} S(P)(x) &\leq S(P')(\tau x) + (f(x) - \alpha)_+ - \eta\ m \\ &\leq \lambda_{\eta,\alpha,n-1}(\tau x)\ +\ (f(x) - \alpha)_+ - \eta\ m. \end{aligned} \tag{11}$$

Using Corollary 1, we complete the proof. □

Let $\chi_A(x)$ denote the characteristic function of a set A.

LEMMA 4. *For all $n \geq 1$ we have*

$$w_{\eta,\alpha,n}(x) \leq w_{\eta,\alpha,n-1}(\tau x) + \chi_{\{f(x)-\alpha\geq\eta\}}(x)\ \chi_{\{f(\tau x)-\alpha\leq\eta\}}(x). \tag{12}$$

Proof. We start with the following

OBSERVATION. *Let $0 \leq u, v \leq n$ be given integers. If $b_{v,\eta}(x) \geq a_u(x)$, then $b_{v-1,\eta}(\tau x) \geq a_{u-1}(\tau x)$.*

Let

$$P(x) = \{-1 \leq s_1(x) < t_1(x) \leq s_2(x) < \cdots < t_m(x) \leq n\} \in \mathcal{P}_0^n(x,\eta)$$

with $m = w_{\eta,\alpha,n}(x)$. (We will suppress the parameter x in $t_i(x)$ and $s_i(x)$ when convenient.) The above observation implies

$$w_{\eta,\alpha,n}(x) \leq w_{\eta,\alpha,n-1}(\tau x) + \chi_{\{s_1(x)=-1\}}(x). \tag{13}$$

It follows from (13) that it is enough to consider the case when $s_1(x) = -1$ in the rest of the proof. To simplify the notation let $A = \{f(x) - \alpha \geq \eta\}$ and $B = \{f(\tau x) - \alpha < \eta\}$. We consider the following two cases:

Case (I)*:* $\chi_B(x) = 0$. In this case we have $s_2(x) \geq 2$, for otherwise we obtain a contradiction with the upcrossing condition

$$-\sum_{j=0}^{s_2}(f(\tau^j x) - \alpha) + \sum_{j=0}^{t_1}(f(\tau^j x) - \alpha - \eta) \geq 0.$$

The condition $\chi_B(x) = 0$ and $s_2(x) \geq 2$ imply $t_1(x) \geq 1$. Define

$$P' = \{-1 \leq s'_1 < t'_1 < s'_2 < \cdots < t'_m \leq n-1\},$$

where $t'_i = t_i - 1$, $s'_i = s_i - 1$ for $i \geq 2$, $s'_1 = -1$, and $t'_1 = t_1 - 1$ if $\sum_{j=0}^{t_1-1}(f(\tau^{j+1}x) - \alpha - \eta) \geq 0$, and $t'_1 = 0$ otherwise. We claim that

$$P' \in \mathcal{P}_0^{n-1}(\tau x, \eta). \tag{14}$$

This will prove (12).

If $t'_1 = t_1 - 1$, then, by the above observation and since $P \in \mathcal{P}_0^n(x,\eta)$, in order to prove (14) we only need to show that $b_{t'_1,\eta}(\tau x) \geq 0$. But this holds by our choice of t'_1. If $t'_1 = 0$, then $b_{t'_1,\eta}(\tau x) \geq 0$ follows from the assumption $\chi_B(x) = 0$. Moreover, by the above observation and since $P \in \mathcal{P}_0^n(x,\eta)$, it remains to show that

$$-\sum_{j=0}^{s'_2}(f(\tau^j x) - \alpha) + \sum_{j=0}^{t'_1}(f(\tau^j x) - \alpha - \eta) \geq 0.$$

This again follows from the observation, our choice of t'_1, and the assumption $\chi_B(x) = 0$.

Case (II)*:* $\chi_B(x) = 1$. By (13) it is enough to consider the case when $\chi_A(x) = 0$. Under this condition we have $t_1(x) \geq 1$. Hence we can define

$$P' = \{-1 \leq s'_1 < t'_1 < s'_2 < \cdots < t'_m \leq n-1\}$$

by $t'_i = t_i - 1$, $s'_i = s_i - 1$ for $i \geq 2$, $s'_1 = -1$ and $t'_1 = t_1 - 1$. We claim that $P' \in \mathcal{P}_0^{n-1}(\tau x, \eta)$, which will complete the proof of (12). By the above observation and the fact that $s_1 = -1$, we only need to check that $b_{t'_1,\eta}(\tau x) \geq 0$. But since $P \in \mathcal{P}_0^n(x, \eta)$ we have

$$0 \leq b_{t_1,\eta}(x) = b_{t'_1,\eta}(\tau x) + f(x) - \alpha - \eta. \tag{15}$$

Hence $b_{t'_1,\eta}(\tau x) \geq 0$ follows from the assumption $\chi_A(x) = 0$. □

The following lemma complements Lemma 4.

Lemma 5. *Let α and η be real numbers ($\eta > 0$). Then for all $\eta' \leq \eta/2$ and $n \geq 1$ we have*

$$w_{\eta',\alpha,n}(x) \geq w_{\eta,\alpha,n-1}(\tau x) \; + \chi_{\{f(\tau x)-\alpha \leq -\eta'\}}(x) \; \chi_{\{f(x)-\alpha \geq \eta'\}}(x). \tag{16}$$

Proof. We start with the following

Observation. *Let $-1 \leq u_1 < v \leq u_2$ be given integers. If $b_{v,\eta}(\tau x) \geq a_{u_1}(\tau x)$ and $b_{v,\eta}(\tau x) \geq a_{u_2}(\tau x)$, then, since $\eta' \leq \eta/2$, it follows that $b_{v+1,\eta'}(x) \geq a_{u_1+1}(x)$ and $b_{v+1,\eta'}(x) \geq a_{u_2+1}(x)$.*

We may assume throughout the proof that $w_{\eta,\alpha,n-1}(\tau x) \geq 1$. Then, by Corollary 1, there exists a set

$$P = \{-1 \leq s_1 < t_1 \leq \cdots < t_m \leq n-1\} \in \mathcal{P}_2^{n-1}(\tau x, \eta),$$

with $m = |P| = w_{\eta,\alpha,n-1}(\tau x)$. Also, by Lemma 1(i) we have $P \in \mathcal{P}_0^{n-1}(\tau x, \eta)$. Define

$$P' = \{-1 < s'_1 < t'_1 \leq \cdots < t'_m \leq n\}$$

by $s'_i = s_i + 1, t'_i = t_i + 1$, $i = 1, \ldots, m$. The above observation implies $P' \in \mathcal{P}_0^n(x, \eta')$. Since $|P'| = m = w_{\eta,\alpha,n-1}(\tau x)$, it follows that $w_{\eta',\alpha,n}(x) \geq w_{\eta,\alpha,n-1}(\tau x)$. Therefore, to establish (16) we may assume for the rest of the proof that x satisfies $(f(\tau x) - \alpha) \leq -\eta'$ and $(f(x) - \alpha) \geq \eta'$. The inequality $(f(\tau x) - \alpha) \leq -\eta'$ implies

$$s_1 \geq 0 \text{ and } -\left(\sum_{j=0}^{s_1} (f(\tau^{j+1}x) - \alpha)\right) \geq -(f(\tau x) - \alpha) \geq \eta'. \tag{17}$$

To prove (17), notice that the conditions $s_1 = -1$ and $f(\tau x) - \alpha - \eta \leq -\eta' - \eta < 0$ contradict the fact that $P \in \mathcal{P}_1^{n-1}(\tau x, \eta)$. Similarly, the inequality

$$-\left(\sum_{j=0}^{s_1} (f(\tau^{j+1}x) - \alpha)\right) < -(f(\tau x) - \alpha)$$

is impossible when $s_1 = 0$, and for the other possible values of s_1 it contradicts the fact that $P \in \mathcal{P}_1^{n-1}(\tau x, \eta)$.

Define now

$$P'' = \{s_0 = -1 < t_0 = 0 \leq s_1' < t_1' \leq \cdots < t_m' \leq n\}.$$

Notice that $|P''| = |P'| + 1 = w_{\eta,\alpha,n-1}(\tau x) + 1$. Therefore, to finish the proof we need to show that $P'' \in \mathcal{P}_0^n(x, \eta')$. Since, as indicated earlier, $P' \in \mathcal{P}_0^n(x, \eta')$, and $f(x) - \alpha - \eta' \geq 0$, we only need to prove that

$$-\left(\sum_{j=0}^{s_1'} (f(\tau^j x) - \alpha)\right) + f(x) - \alpha - \eta' \geq 0.$$

But this follows from (17), since

$$\eta' \leq -\left(\sum_{j=0}^{s_1} (f(\tau^{j+1} x) - \alpha)\right) = -\left(\sum_{j=0}^{s_1'} (f(\tau^j x) - \alpha)\right) + f(x) - \alpha. \qquad \square$$

3. Integral inequalities for generalized upcrossings

The following upper bound can be found in [3].

THEOREM 1. *Assume that α and η ($\eta > 0$) are given real numbers, τ is a measure preserving transformation, and A an invariant subset. Then, if $\chi_A\ (f - \alpha)_+ \in L^1$, we have*

$$\int_A \eta\ w_{\eta,\alpha}(x)\ d\mu(x) \leq \int_A\ (f(x) - \alpha)_+\ d\mu(x). \tag{18}$$

Proof. The result follows by integrating (10) (after multiplication with χ_A) and noticing that under our hypothesis $\chi_A(x)\lambda_{\eta,\alpha,n}(x) \in L^1$ for all n. $\square$

LEMMA 6. *Assume that τ is a measure preserving transformation, $f \in L^1$, and that real numbers α and η ($\eta > 0$) are given. Then for each x for which $(\alpha + \eta) > \lim_{n\to\infty} A_n(f)(x)$ the following limit exists as a real number:*

$$\lim_{n\to\infty} \lambda_{\eta,\alpha,n}(x). \tag{19}$$

Proof. Consider x such that $(\alpha + \eta) > \lim_{n\to\infty} A_n(f)(x)$. We show first that there are positive integers $t_1 > t_0$ such that

$$\sum_{j=0}^{t} (f(\tau^j x) - \alpha - \eta) < 0 \text{ for all } t \geq t_0, \tag{20}$$

$$\sum_{j=0}^{t} (f(\tau^j x) - \alpha - \eta) - \sum_{j=0}^{s} (f(\tau^j x) - \alpha) < 0 \text{ for all } t > s \geq t_0, \tag{21}$$

and

$$\sum_{j=0}^{t}(f(\tau^j x)-\alpha-\eta)\leq\sum_{j=0}^{t_0}(f(\tau^j x)-\alpha-\eta) \text{ for all } t>t_1. \tag{22}$$

Let $g(x):=\lim_{n\to\infty}A_n(f)(x)$. Take $\epsilon=\min(\eta/2,(\alpha+\eta-g(x))/2)$ and t_0 such that $|A_t f(x)-g(x)|<\epsilon$ for all $t\geq t_0$. Then

$$A_t f(x)-\alpha-\eta<\epsilon+g(x)-\alpha-\eta\leq\frac{g(x)-\alpha-\eta}{2}<0. \tag{23}$$

This proves (20).

Let now $t>s\geq t_0$. For convenience set $\eta'=(g(x)-\alpha)$ and notice that $\epsilon\leq(\eta-\eta')/2$. We consider two cases: If $\eta'\geq 0$, then

$$\begin{aligned}\sum_{j=0}^{t}(f(\tau^j x)-\alpha-\eta)&-\sum_{j=0}^{s}(f(\tau^j x)-\alpha)\\&<(t-s)\,(g(x)-\alpha)-(t+1)\;\eta+(t+s+2)\;\epsilon\\&<\frac{(\eta-\eta')}{2}(2+2t)+(t+1)(\eta'-\eta)=0.\end{aligned} \tag{24}$$

If $\eta'<0$, then

$$\begin{aligned}\sum_{j=0}^{t}(f(\tau^j x)-\alpha-\eta)&-\sum_{j=0}^{s}(f(\tau^j x)-\alpha)\\&<(t-s)\,(g(x)-\alpha)-(t+1)\;\eta+(t+s+2)\;\epsilon\\&<2(t+1)\frac{\eta}{2}-(t+1)\eta=0.\end{aligned} \tag{25}$$

Hence (21) is proven.

We now prove (22). Define $t_1=3\;t_0$ and take $t>t_1+1$. To simplify the notation let $y=A_{t_0}f(x)-\alpha-\eta$ and $z=A_t f(x)-\alpha-\eta$. Since $t>t_0$ we have $|y-z|\leq 2\;\epsilon$, so

$$\left|\frac{y}{z}\right|\leq\frac{2\;\epsilon}{|z|}+1. \tag{26}$$

Moreover, from $|A_t f(x)-g(x)|<\epsilon$ we obtain $|z|>-\;\epsilon+|g(x)-\alpha-\eta|\geq\epsilon$, where the last inequality follows from our choice of ϵ. Hence (26) gives

$$\left|\frac{y}{z}\right|\leq\frac{2\;\epsilon}{|g(x)-\alpha-\eta|-\;\epsilon}+1\leq 3.$$

By (20) we have $y<0$ and $z<0$. Hence $(t_0+1)\;y\geq(t+1)\;z$, which is (22). Equations (21) and (22) prove that $\lambda_{\eta,\alpha,n-1}(x)=\lambda_{\eta,\alpha,n}(x)$ for all $n>t_1+1$. Hence (19) is proven. □

The following lower bound is our main result.

THEOREM 2. *Assume that $f \in L^1$ and that α and η ($\eta > 0$) are given real numbers. Let τ be a measure preserving transformation and A an invariant subset with $\mu(A) < \infty$. Then if $(\alpha + \eta) > \lim_{n\to\infty} A_n f(x)$ on A,*

$$\int_A (f(x) - \alpha - \eta)_+ \, d\mu(x) \le \int_A \eta \, w_{\eta,\alpha}(x) \, d\mu(x). \tag{27}$$

Proof. We use the notation $g_n(x) = (\lambda_{\eta,\alpha,n}(x) - \lambda_{\eta,\alpha,n-1}(x)) \, \chi_A(x)$. One can check that $0 \le g_n(x) \le (f(\tau^n x) - \alpha - \eta)_+ \, \chi_A(x) \le T^n(f(x) - \alpha - \eta)_+ \, \chi_A(x)$. We show next that $h_n(x) = T^n(f(x) - \alpha - \eta)_+ \, \chi_A(x)$ (with $h(x) = h_0(x) = (f(x) - \alpha - \eta)_+ \, \chi_A(x)$) is a uniformly integrable sequence, and therefore $g_n(x)$ is also uniformly integrable. Since there exists a constant $a > 0$, independent of n, such that $||h_n||_1 \le a \, ||f||_1$, to prove the uniform integrability it is enough to verify that for all $\epsilon > 0$ there exists a constant K_ϵ which satisfies $\int_X (h_n - K_\epsilon)_+ \, d\mu(x) < \epsilon$ for all n. To this end, take $\epsilon > 0$ and choose K_ϵ such that $\int_X (h - h \wedge K_\epsilon) \, d\mu(x) < \epsilon$. Then

$$\begin{aligned} \int_X (h_n - K_\epsilon)_+ \, d\mu(x) &= \int_X (h_n - h_n \wedge K_\epsilon) \, d\mu(x) \\ &= \int_X (T^n h - T^n h \wedge K_\epsilon) \, d\mu(x) \\ &\le \int_X T^n(h - h \wedge K_\epsilon) \, d\mu(x) \\ &= \int_X (h - h \wedge K_\epsilon) \, d\mu(x) < \epsilon. \end{aligned} \tag{28}$$

Multiply (7) by $\chi_A(x)$ and integrate to obtain

$$\begin{aligned} \int_A (f(x) - \alpha - \eta)_+ \, d\mu(x) \le \int_A & w_{\eta,\alpha,n-1}(x) \, \eta \, d\mu(x) \\ &+ \int_A (\lambda_{\eta,\alpha,n}(x) - \lambda_{\eta,\alpha,n-1}(x)) \, d\mu(x). \end{aligned} \tag{29}$$

We now apply Theorem 5 of the Appendix to the uniformly integrable sequence $g_n(x)$. Notice that, by Lemma 6, $\lim_{n\to\infty} g_n(x) = 0$ a.e. on A, and by Lebesgue's monotone convergence theorem we have $\lim_{n\to\infty} \int_A w_{\eta,\alpha,n-1}(x) = \int_A w_{\eta,\alpha}(x)$. Hence, taking the limit as $n \to \infty$ in (29) and using (50) gives (27). □

REMARK 1. The condition $\alpha + \eta > \lim_{n\to\infty} A_n f(x)$ seems to be needed because we are dealing with upcrossings. It can be removed once we introduce downcrossings as we will do in Section 5. In the case when τ is ergodic and $\mu(X) < \infty$ the condition becomes $(\alpha + \eta)\mu(X) > \int_X f$.

4. Generalized upcrossings and spatial oscillations

In this section we discuss the geometric meaning of generalized upcrossings and establish a connection with the usual (geometric) upcrossings and with oscillations (or jumps).

Definition 4 (Upcrossings). Given a function $f(x)$, an integer $n \geq 0$, real numbers α and η ($\eta > 0$) and $x \in X$, define

$$U_{\eta,\alpha,n}(x) = \max\{k : \zeta = (u_r, v_r)_{r=1,\ldots,k.}\}, \tag{30}$$

where the sequence ζ satisfies

$$-1 \leq u_1 < v_1 < u_2 < \cdots < v_k \leq n, \tag{31}$$

and

$$A_{u_r} f(x) \leq \alpha \quad \text{and} \quad A_{v_r} f(x) \geq (\alpha + \eta) \tag{32}$$

for $r = 1, \ldots, k$. The sequence ζ is called an *n-upcrossing sequence at x* and the space of these sequences is denoted by $\mathcal{U}_0^n(x, \eta, \alpha)$. The function $U_{\eta,\alpha}(x) = \lim_{n\to\infty} U_{\eta,\alpha,n}(x)$ will be referred to as the number of upcrossings through the interval $[\alpha, \alpha + \eta]$ (see [3]).

The following simple proposition serves as a key motivation for the study of upper bounds for $w_{\eta,\alpha}$.

Proposition 1. *We have $\mathcal{U}_0^n(x, \eta, \alpha) \subseteq \mathcal{P}_0^n(x, \eta, \alpha)$, and hence*

$$U_{\eta,\alpha,n}(x) \leq w_{\eta,\alpha,n}(x). \tag{33}$$

Notice that in general $\lim_{\eta\to 0} U_{\eta,\alpha}(x) < \infty$, unless, for example, in the ergodic case (and finite measure), $\alpha = \int f/\mu(X)$. The following proposition gives information on what happens to $w_{\eta,\alpha}(x)$ as $\eta \to 0$.

Proposition 2. *Assume that τ is an ergodic transformation. For any $p \geq 0$ define the measurable sets $A_{p,\infty} = \{x \mid \liminf_{\eta\to 0} \eta^p\ w_{\eta,\alpha}(x) = \infty\}$. Then*

$$\mu(A_{p,\infty}) = \mu(X) \ \text{or}\ \mu(A_{p,\infty}) = 0. \tag{34}$$

If, in addition, $(f - \alpha)_+ \in L^1$, then for any $p \geq 1$

$$\mu(A_{p,\infty}) = 0. \tag{35}$$

Proof. Consider first the case $p = 0$. In this case, $\liminf_{\eta\to 0} w_{\eta,\alpha}(x) = \lim_{\eta\to 0} w_{\eta,\alpha}(x)$. From Lemmas 4 and 5 it follows that $\tau^{-1} A_{0,\infty} = A_{0,\infty}$. Therefore (34) follows from the ergodicity of τ.

Now consider the case $p > 0$. For each integer M define the sets $A_{p,M} = \{x \mid \liminf_{\eta\to 0} \eta^p\ w_{\eta,\alpha}(x) > M\}$. From Lemmas 4 and 5 it follows that

$$A_{p,M} \subseteq \tau^{-1}(A_{p,M}) \subseteq A_{p,M/2^p}. \tag{36}$$

Notice that for any k, $A_{p,\infty} = \bigcap_{M=k}^{\infty} A_{p,M}$. Hence (36) gives $A_{p,\infty} \subseteq \tau^{-1}A_{p,\infty} \subseteq A_{p,\infty}$. This proves that $A_{p,\infty}$ is an invariant subset. Therefore (34) follows from the ergodicity of τ. To prove (35), we just need to consider $p = 1$. Applying Fatou's theorem to (18) and to the invariant set $A = A_{1,\infty}$ in that equation, we obtain $\liminf_{\eta\to\infty} \eta w_{\eta,\alpha}(x) < \infty$ a.e. Hence $\mu(A_{1,\infty}) = 0$. $\square$

Remark 2. In view of the above proposition and Theorem 2 it is natural to expect that under rather general conditions we have, in fact, $\mu(A_{p,\infty}) = \mu(X)$ for $0 \leq p < 1$.

Definition 5 (Jumps). Given a function $f(x)$, a fixed integer $n \geq 0$, a real number $\eta > 0$ and $x \in X$, define

$$J_{\eta,n}(x) = \max\{k : \xi = (t_r)_{r=0,\ldots,k}\},$$

where ξ satisfies

$$-1 \leq t_0 < t_1 < t_2 < \cdots < t_k \leq n$$

and

$$|A_{t_{r+1}} f(x) - A_{t_r} f(x)| \geq \eta, \text{ for all } r = 0, \ldots, k-1. \tag{37}$$

Also define

$$J_\eta(x) = \sup\{J_{\eta,n}(x) : n \geq 0\}.$$

The function J_η will be referred to as the number of η-jumps.

Taken together, the proof of item (3) in the next lemma and Proposition 1 give a rather complete picture of the geometric meaning of generalized upcrossings.

Lemma 7. *For given real numbers α and η ($\eta > 0$) and any integer $n \geq 0$ we have:*

(1) *If $P = (s_1, t_1, \ldots, s_m, t_m) \in \mathcal{P}_0^n(x)$, then $t_i < s_{i+1}$ for $i = 1, \ldots, m-1$.*
(2) *$w_{\eta,\alpha,n+1}(x) = w_{\eta,\alpha,n}(x)$ or $w_{\eta,\alpha,n+1}(x) = w_{\eta,\alpha,n}(x) + 1$.*
(3) *If $P = (s_1, t_1, \ldots, s_m, t_m) \in \mathcal{P}_0^n(x)$ with $m = w_{\eta,\alpha,n}(x)$, then there exists a sequence $-1 \leq \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_m \leq t_m$ such that*

$$|A_{\theta_{i+1}} f(x) - A_{\theta_i} f(x)| \geq \frac{\eta}{2} \text{ for all } i = 0, \ldots, m-1. \tag{38}$$

Proof. (1) If there exists i such that $t_i = s_{i+1}$ we have $b_{t_i} - a_{s_{i+1}} = -(t_i + 1)\eta < 0$.

(2) Suppose that $w_{\eta,\alpha,n+1}(x) > w_{\eta,\alpha,n}(x)$. Let $P = (s_1, t_1, \ldots, s_r, t_r) \in \mathcal{P}_2^{n+1}(x)$. By Lemma 1 we have $(s_1, t_1, \ldots, s_{r-1}, t_{r-1}) \in \mathcal{P}_0^n(x)$ and $w_{\eta,\alpha,n}(x) \geq r - 1$.

(3) We will prove the assertion by induction on n. By (2) we may assume $w_{\eta,\alpha,n}(x) = w_{\eta,\alpha,n-1}(x) + 1 = m$. Let $P = (s_1, t_1, \ldots, s_m, t_m) \in \mathcal{P}_0^n(x)$. This implies

$$b_{t_i}(x) - a_{s_i}(x) \geq 0 \text{ for } i = 1, \ldots, m \tag{39}$$

and

$$b_{t_i}(x) - a_{s_{i+1}}(x) \geq 0 \text{ for } i = 1, \ldots, m-1. \tag{40}$$

We first prove that $A_{t_{m-1}}f(x) - A_{s_m}f(x) \geq \eta$ or $A_{t_m}f(x) - A_{s_m}f(x) \geq \eta$. To this end we first assume that $A_{s_m}f(x) > \alpha$ and $A_{t_{m-1}}f(x) - A_{s_m}f(x) < \eta$. In this case we have

$$\begin{aligned} 0 &> t_{m-1}(A_{t_{m-1}}f(x) - \alpha - \eta) - t_{m-1}(A_{s_m}f(x) - \alpha) \\ &\geq t_{m-1}(A_{t_{m-1}}f(x) - \alpha - \eta) - s_m(A_{s_m}f(x) - \alpha) \\ &= b_{t_{m-1}}(x) - a_{s_m}(x), \end{aligned} \tag{41}$$

which contradicts (40).

Now, assume that $A_{s_m}f(x) \leq \alpha$ and $A_{t_m}f(x) - A_{s_m}f(x) < \eta$. Then

$$\begin{aligned} 0 &> t_m(A_{t_m}f(x) - \alpha - \eta) - t_m(A_{s_m}f(x) - \alpha) \\ &\geq t_m(A_{t_m}f(x) - \alpha - \eta) - s_m(A_{s_m}f(x) - \alpha) \\ &= b_{t_m}(x) - a_{s_m}(x), \end{aligned} \tag{42}$$

which contradicts (39).

Since $(s_1, t_1, \ldots, s_{m-1}, t_{m-1}) \in \mathcal{P}_0^{n-1}(x)$, by the inductive hypothesis there exists a sequence $-1 \leq \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{m-1} \leq t_{m-1}$ such that (38) holds. We now need to define θ_m. It suffices to consider the following two cases:

(i) $A_{t_{m-1}}f(x) - A_{s_m}f(x) \geq \eta$: If $\theta_{m-1} = t_{m-1}$, we take $\theta_m = s_m < t_m$. Suppose therefore that $\theta_{m-1} < t_{m-1}$. If $|A_{t_{m-1}}f(x) - A_{\theta_{m-1}}f(x)| < \eta/2$, it follows that $|A_{\theta_{m-1}}f(x) - A_{s_m}f(x)| \geq \eta/2$, so we can again take $\theta_m = s_m$. Otherwise, if $|A_{t_{m-1}}f(x) - A_{\theta_{m-1}}f(x)| \geq \eta/2$, we take $\theta_m = t_{m-1}$.

(ii) $A_{t_m}f(x) - A_{s_m}f(x) \geq \eta$: If $|A_{t_m}f(x) - A_{\theta_{m-1}}f(x)| < \eta/2$, then $|A_{s_m}f(x) - A_{\theta_{m-1}}f(x)| \geq \eta/2$, and since clearly $\theta_{m-1} < s_m$, we can take $\theta_m = s_m < t_m$. If $|A_{t_m}f(x) - A_{\theta_{m-1}}f(x)| \geq \eta/2$, we take $\theta_m = t_m$. □

The following corollary, which follows from item (3) in Lemma 7, permits us to transfer all of our lower bound inequalities for $w_{\eta,\alpha,n}(x)$ to lower bound inequalities for $J_{\eta,n}(x)$. From the results in [4] one can expect that the inequalities obtained in this way will in general not be tight.

Corollary 2.

$$\sup_\alpha (w_{\eta,\alpha,0}(x)) \leq J_{\eta,0}(x),$$

$$\sup_\alpha (w_{\eta,\alpha,n}(x)) \leq J_{\eta',n}(x) \textit{ for all } \eta' \leq \eta/2 \textit{ and } n \geq 1.$$

5. Integral inequalities for generalized downcrossings

Given our techniques, it is of interest to consider generalized downcrossings. These could easily be related to quantities introduced previously and to geometric downcrossings, but we will not do so here.

For real numbers α and η ($\eta > 0$) and given x we specialize the sequences $\{a_i\}$ and $\{b_i\}$ given in Definition 2 as follows:

$$b_t = b_t^d(x) = b_{t,\alpha}^d(x) = - \sum_{j=0}^{t} (f_j(x) - \alpha),$$

$$a_s = a_s^d(x) = a_{s,\eta,\alpha}^d(x) = - \sum_{j=0}^{s} (f_j(x) - \alpha - \eta).$$

The set $\mathcal{P}_0^n$ in Definition 2 specializes to the set of n-generalized downcrossing sequences, denoted by $\mathcal{P}_{d,0}^n(x, \eta, \alpha)$.

For a nonempty admissible sequence $P = (s_1, t_1, \ldots, s_m, t_m)$, we define

$$S_d(P)(x) = \sum_{i=1}^{|P|} \left(b_{t_i}^d(x) - a_{s_i}^d(x) \right). \tag{43}$$

We let $\mathcal{P}_{d,1}^n(x)$ be the set of n-admissible sequences P with $S_d(P)(x)$ maximal in $\mathcal{P}^n$, and $\mathcal{P}_{d,2}^n(x)$ the set of sequences P in $\mathcal{P}_{d,1}^n(x)$ with $|P|$ maximal. As we did for upcrossings, we introduce the following notation for generalized downcrossings.

Definition 6. For a given integer $n \geq 0$ we define

$$\lambda_{\eta,\alpha,n}^d(x) = \max_{P \in \mathcal{P}^n} S_d(P)(x) = S_d(P_1)(x), \tag{44}$$

where P_1 is any element in $\mathcal{P}_{d,1}^n(x)$. The (maximal) *number of n-generalized downcrossings* at x is given by

$$w_{\eta,\alpha,n}^d(x) = \max\{|P| : P \text{ is an } n\text{-generalized downcrossing sequence at } x\}. \tag{45}$$

Also, we define the number of generalized downcrossings at x by

$$w_{\eta,\alpha}^d(x) = \lim_{n \to \infty} w_{\eta,\alpha,n}^d(x).$$

With the above definitions, Lemma 1 is immediately applicable. Results analogous to Corollary 1 and to Lemmas 2 and 3 can be obtained for the quantities defined above. Finally, we have the following dual theorems for generalized downcrossings.

Theorem 3. *Assume $\chi_A \ (\alpha + \eta - f)_+ \in L^1$, where A is an invariant subset with respect to τ, a measure preserving transformation. If α and η*

($\eta > 0$) are given real numbers, then

$$\int_A \eta\ w^d_{\eta,\alpha}(x)\ d\mu(x) \le \int_A\ (\alpha + \eta - f(x))_+\ d\mu(x). \tag{46}$$

THEOREM 4. *Assume that $f \in L^1$ and that α and η ($\eta > 0$) are given real numbers. Let τ be a measure preserving transformation and A an invariant subset with $\mu(A) < \infty$. Then if $(\alpha + \eta) \le \lim_{n\to\infty} A_n f(x)$ on A,*

$$\int_A\ (\alpha - f(x))_+\ d\mu(x) \le \int_A \eta\ w^d_{\eta,\alpha}(x)\ d\mu(x). \tag{47}$$

By combining the results for generalized downcrossings and upcrossings we can remove the hypothesis on α. To this end define $m_{\eta,\alpha}(x) = \max(w^d_{\eta,\alpha}(x), w_{\eta,\alpha}(x))$. To simplify the statement, we assume in the following corollary that X has finite measure.

COROLLARY 3. *Let $f \in L^1$, $\mu(X) < \infty$, and let α and η ($\eta > 0$) be real numbers. If τ is a measure preserving transformation and A an invariant set, then*

$$\min\left(\int_A (f - \alpha - \eta)_+ d\mu(x), \int_A (\alpha - f)_+ d\mu(x)\right) \le \int_A\ \eta\ m_{\eta,\alpha} d\mu(x). \tag{48}$$

Appendix: Background material

Here we mention a known result used in the paper. It is an extension of Lebesgue's dominated convergence theorem to the setting of uniformly integrable functions.

DEFINITION 7. A sequence of measurable functions g_n in a finite measure space (A, μ) is said to be uniformly integrable if

$$\int_{\{|g_n| \ge c\}} |g_n|\ d\mu \to 0 \text{ as } c\ \to\ \infty, \text{ uniformly in } n. \tag{49}$$

THEOREM 5 ([1, p. 295]). *In the above setting we have: If $g_n(x) \to g(x)$ a.e., then g is integrable and*

$$\int_A \lim_{n\to\infty}\ g_n\ d\mu = \lim_{n\to\infty}\ \int_A g_n\ d\mu. \tag{50}$$

Acknowledgement. We thank Dr. John Baxter for the present version of Lemma 1 and for his continuous encouragement.

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