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LINEAR SYSTEMS OF PLANE CURVES WITH IMPOSED MULTIPLE POINTS

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Abstract. A conjecture of Harbourne and Hirschowitz implies that $r \geq 9$ general points of multiplicity m impose independent conditions to the linear system of curves of degree d when $d(d+3) \geq rm(m+1)-2$. In this paper we prove that the conditions are independent provided $d+2 \geq (m+1)(\sqrt{r+1.9}+\pi/8)$.

Introduction

Let $p_1, p_2, \ldots, p_r$ be points of $\mathbb{P}^2$. Consider the linear system $\mathcal{L}$ consisting of plane curves of degree d with multiplicity at least m at every p_i. For each point p_i the requirement to have an m–fold point at p_i imposes $m(m+1)/2$ conditions on curves of degree d; we say that the r points impose independent conditions when the dimension of $\mathcal{L}$ is $d(d+3)/2 - rm(m+1)/2$.

The dimension of $\mathcal{L}$ depends on the position of the points, and achieves its minimal value for a general set of points. We will be interested in this minimum and will suppose henceforth that the points are general. A conjecture of Harbourne [9] and Hirschowitz [14] predicts for which degrees and multiplicities the conditions are independent; the conjecture implies independence whenever $r \geq 9$ and $d(d+3)/2 - rm(m+1)/2 \geq -1$ (note that the latter is a necessary condition). The paper [18] is a very nice survey on this subject, giving an overview of known results and describing the present state of conjectures. The conjecture of Harbourne and Hirschowitz is known to hold for small multiplicities (namely $m \leq 12$), and also in some other particular cases, e.g., when the number of points is $r = 4^h$ (cf. [4], respectively [7]). In the general case, various sufficient conditions on d for the independence of the linear conditions can be found in the literature. According to [17], the best known bounds are Ballico's bound [1]

$$\frac{d(d+3)}{2} - r\frac{m(m+1)}{2} \geq d(m-1)-1$$

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for large values of r, and Xu's bound [21]

$$d+3>(m+1)\sqrt{\frac{10}{9}r},$$

valid when $r<360$ and m is relatively large. In this paper we use a specialization similar to that of [20], and the semicontinuity theorem to prove an H^1-vanishing result which implies that

$$d+2\geq(m+1)(\sqrt{r+1.9}+\pi/8)$$

is enough to have independent linear conditions. This is better than both of the above-mentioned bounds whenever $r\geq 108$, $m\geq 28$ and $m^2\geq r$.

Recently Harbourne, Holay and Fitchett [12] proved that

$$d\geq m\lceil\sqrt{r}\rceil+\frac{\lceil\sqrt{r}\rceil-3}{2},$$

where $\lceil x\rceil$ stands for the least integer greater or equal to x, is a sufficient condition for independence. The closer r is to a square, the better this bound is; the bound that we prove here is better for large m whenever $\sqrt{r+1.9}+\pi/8<\lceil\sqrt{r}\rceil$, that is, for approximately 60 % of the values of r.

1. Preliminaries

Let k be an algebraically closed field, $\mathbb{P}^2$ the projective plane over k. Consider a sequence

$$S_r\xrightarrow{\sigma_r}S_{r-1}\longrightarrow\cdots\xrightarrow{\sigma_2}S_1\xrightarrow{\sigma_1}S_0=\mathbb{P}^2,$$

where σ_i is the blowing-up of a point $p_i\in S_{i-1}$. The set $\{p_1,p_2,\ldots,p_r\}$ is a *cluster* and the sequence $K=(p_1,p_2,\ldots,p_r)$ is an *ordered cluster*. We write $S_K=S_r$ and $\pi_K:S_K\to\mathbb{P}^2$ for the composition of the blowing-ups. If $E_i\subset S_i$ is the exceptional divisor of σ_i, we also denote by E_i its total transform (pullback) in S_K when no confusion arises. When considering more than one cluster at a time we will write $p_i(K)$ and $S_i(K)$ for the i–th point of the cluster K and the surface obtained by blowing up the first i points of K. We now review some well–known facts about clusters and Enriques diagrams (a convenient graphical device to represent these) and refer the reader to [2] and [3] for proofs and details.

A point p_j is said to be *proximate* to p_i, $j>i$, if and only if $j=i+1$ and p_j lies on the exceptional divisor of blowing up p_i, or $j>i+1$ and p_j lies on the *strict* transform of this exceptional divisor. A point which is proximate to no other point can be naturally identified with a point of $\mathbb{P}^2$; these points are the *roots* of K. A point can be proximate to at most two points, one of which must be proximate to the other. If p_i is proximate to two points it is called *satellite*; otherwise it is called *free*. Thus roots are free.

The combinatorial structure of clusters (that is, the proximity relations between their points) can be conveniently represented by means of *Enriques* *diagrams*. As is customary, a tree is a directed connected graph without loops and a forest is a disjoint union of trees. The Enriques diagram $\mathbf{D}$ of a cluster K is a forest with a vertex $\dot{p}_i$ for each point $p_i \in K$. One usually carries the attributes of the points of the cluster over to the vertices of the Enriques diagram; thus, we say that $\dot{p}_j$ is proximate to $\dot{p}_i$ whenever p_j is proximate to p_i, and we speak of roots, satellites or free vertices in $\mathbf{D}$. When necessary, we also consider the points of $\mathbf{D}$ to be ordered in the same way as the corresponding points on K. The edges of the diagram reflect the proximity relations of K. For every vertex $\dot{p}_j$ which is not a root, let $\dot{p}_i$ be the last vertex to which $\dot{p}_j$ is proximate; then there is an edge from $\dot{p}_i$ to $\dot{p}_j$. The sequence of edges connecting a maximal succession of free vertices are drawn as a smooth curve, whereas the sequence of edges connecting a maximal succession of vertices that are proximate to the same vertex $\dot{p}_i$ are drawn as a line segment, orthogonal to the edge joining $\dot{p}_i$ with the first vertex of the sequence.

A detailed account of Enriques diagrams can be found in [3, 3.9]; Figures 1 and 2 show the few diagrams that will be used in the sequel. One of these is the diagram consisting of r roots and no other point; we denote this diagram by $\mathbf{D}_0(r)$, or simply $\mathbf{D}_0$, if r is clear from the context. The other diagrams form a series, $\mathbf{D}_k(r)$, or simply $\mathbf{D}_k$, $k = 1, 2, \ldots, r-1$, so that in $\mathbf{D}_k$ the k points $\dot{p}_2, \dot{p}_3, \ldots, \dot{p}_{k+1}$ are proximate to $\dot{p}_1$ and, moreover, $\dot{p}_i$ is proximate to $\dot{p}_{i-1}$ for all $i > 1$ (in particular, $\dot{p}_3, \dot{p}_4, \ldots, \dot{p}_{k+1}$ are satellites).

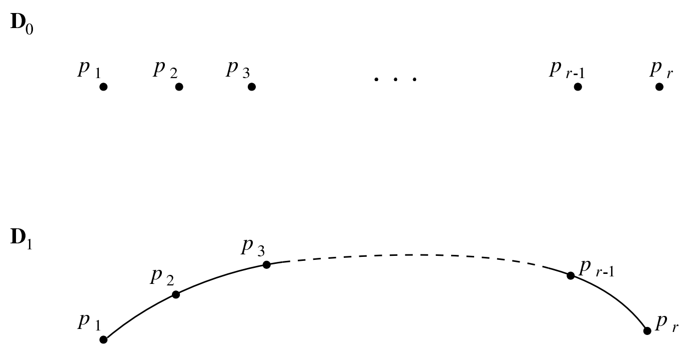

FIGURE 1. Enriques diagrams $\mathbf{D}_0$ and $\mathbf{D}_1$.

The clusters of one point of $\mathbb{P}^2$ are naturally identified with the points of $\mathbb{P}^2$. The clusters of two points whose second point is infinitely near the first one are naturally identified with the points of the tangent bundle of $\mathbb{P}^2$. For clusters with more than two points, there exist varieties of higher dimension whose points are naturally identified with the desired clusters. These varieties are known as *iterated blowing-ups* and were introduced by Steven L. Kleiman in [15] and [16]. Let X_{r-1} be the variety of all ordered clusters of r points of $\mathbb{P}^2$

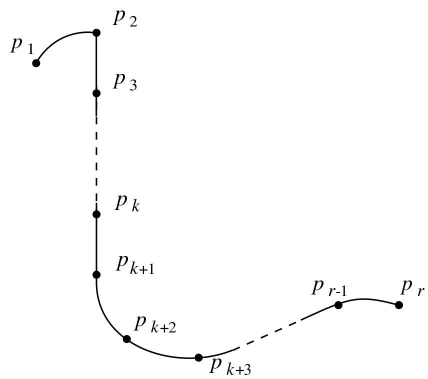

FIGURE 2. Enriques diagrams $\mathbf{D}_k$, $k > 1$.

(cf. [10], [20]). There is a family of surfaces parametrized by it, $X_r \to X_{r-1}$, and relative divisors $F_0, F_1, \dots, F_r$ in X_r such that the fiber over a given cluster K is the surface S_K, and the pullback (or restriction) of F_i to S_K is the total transform E_i of the exceptional divisor of blowing up $p_i(K)$.

For every Enriques diagram $\mathbf{D}$ the subset of X_{r-1} containing exactly the clusters K whose Enriques diagram is $\mathbf{D}$ is an irreducible smooth locally closed subvariety $U(\mathbf{D})$. This was proved for the Enriques diagrams $\mathbf{D}_i$ in [20]; the general case will be treated elsewhere. Also, by [20] we have the sequence of inclusions

$$X_r = \overline{U(\mathbf{D}_0)} \supset \overline{U(\mathbf{D}_1)} \supset \cdots \supset \overline{U(\mathbf{D}_{r-1})}.$$

A *system of multiplicities* for an ordered cluster K or Enriques diagram $\mathbf{D}$ with r points is a sequence of integers $\mathbf{m} = (m_1, m_2, \dots, m_r)$. A pair $(K, \mathbf{m})$ (resp. $(\mathbf{D}, \mathbf{m})$) where K is an ordered cluster (resp. $\mathbf{D}$ is an ordered Enriques diagram) and $\mathbf{m}$ a system of multiplicities is called a *weighted cluster* (resp. *weighted Enriques diagram*). A system of multiplicities of the form

$$(\overbrace{m, m, \dots, m}^{r})$$

will be denoted by (m^r). To each weighted cluster $(K, \mathbf{m})$, we associate a divisor $D_{K,\mathbf{m}} = -m_1E_1 - m_2E_2 - \cdots - m_rE_r$. Denote by E_0 the pullback in S_K of a line in $\mathbb{P}^2$. The linear system $\mathcal{L}$ of the plane curves of degree d with (virtual, see [3]) multiplicity at least m_i at $p_i \in K$ transforms to $\mathcal{L}' = |dE_0 + D_{K,\mathbf{m}}| = \mathbb{P}(H^0(\mathcal{O}_{S_K}(dE_0 + D_{K,\mathbf{m}})))$ on S_K. This allows us to compute $\dim \mathcal{L} = \dim \mathcal{L}'$ by the Riemann–Roch theorem. It is well–known that $K = -3E_0 + E_1 + E_2 + \cdots + E_r$ is a canonical divisor on S_K. The intersection in $\operatorname{Pic}(S_K)$ is given by $E_i \cdot E_j = 0$ for $i \neq j$, $E_0^2 = 1$, $E_i^2 = -1$ for $i > 0$, $p_a(S_K) = 0$, and $H^2(\mathcal{O}_{S_K}(dE_0 + D_{K,\mathbf{m}})) = 0$ if $d \geq -2$. Hence the

Riemann–Roch formula gives

$$
\begin{aligned}
&h^0(\mathcal{O}_{S_K}(dE_0 + D_{K,\mathbf{m}})) - h^1(\mathcal{O}_{S_K}(dE_0 + D_{K,\mathbf{m}})) \\
&\qquad\qquad = \frac{d(d+3)}{2} - \sum \frac{m_i\,(m_i+1)}{2} + 1 \ ,
\end{aligned}
$$

and the independence of the linear conditions imposed by the points of the weighted cluster is equivalent to the vanishing of $H^1(\mathcal{O}_{S_r}(dE_0 + D_{K,\mathbf{m}}))$. Therefore our goal, as stated in the introduction, is to prove an H^1-vanishing theorem for the invertible sheaf $\mathcal{O}_{S_K}(dE_0+D_{K,\mathbf{m}})$ when K is general in $U(\mathbf{D}_0)$ (that is, when K consists of r roots in general position) and when $\mathbf{m} = (m^r)$.

It turns out that for some systems of multiplicities $\mathbf{m} \neq \mathbf{m}'$ the linear systems $|dE_0 + D_{K,\mathbf{m}}|$ and $|dE_0 + D_{K,\mathbf{m}'}|$ agree for every d, up to fixed components which are components of the exceptional divisors. Therefore the corresponding linear systems of plane curves are equal; in particular, $h^0(\mathcal{O}_{S_K}(dE_0 + D_{K,\mathbf{m}})) = h^0(\mathcal{O}_{S_K}(dE_0 + D_{K,\mathbf{m}'}))$. In this case we will say that $(K, \mathbf{m})$ and $(K, \mathbf{m}')$ are equivalent weighted clusters. Given an arbitrary weighted cluster $(K, \mathbf{m})$, it is possible to remove some components from $D_{K,\mathbf{m}}$ to obtain an equivalent system of multiplicities $\mathbf{m}'$ such that $|dE_0 + D_{K,\mathbf{m}}|$ has no fixed part for $d \gg 0$. This procedure is called *unloading* (see [3, 4], [5, IV.II], or [2]), and we next describe how to obtain $\mathbf{m}'$ from $\mathbf{m}$.

The *proximity inequality* at p_i is

$$
m_i \geq \sum_{p_j \text{ prox. to } p_i} m_j .
$$

If $\tilde{E}_i \subset S_K$ is the strict transform of the exceptional divisor of blowing-up p_i, then the proximity inequality is equivalent to $\tilde{E}_i \cdot D_{K,\mathbf{m}} \geq 0$. A weighted cluster or Enriques diagram is called *consistent* if and only if it satisfies the proximity inequalities at all of its points; in this case $\tilde{E}_i \cdot D_{K,\mathbf{m}} \geq 0$ for all $i > 0$ and $|dE_0 + D_{K,\mathbf{m}}|$ has no fixed part for $d \gg 0$. In each step of the unloading procedure, one *unloads* part of the multiplicity of a point p_i whose proximity inequality is not satisfied from the points proximate to it (which implies that $\tilde{E}_i$ is a fixed part of every $|dE_0 + D_{K,\mathbf{m}}|$). One takes the minimal integer n with $\tilde{E}_i \cdot (D - n\tilde{E}_i) \geq 0$ and replaces D by $D - n\tilde{E}_i$. This increases the multiplicity of p_i by n and decreases the multiplicity of every point proximate to p_i by n. The resulting weighted cluster is equivalent to $(K, \mathbf{m})$ and satisfies the proximity inequality at p_i. A finite number of unloading steps lead to the desired equivalent consistent cluster $(K, \mathbf{m}')$. An unloading step applied to a point p_i for which $m_i = \sum m_j - 1$ (where the sum is taken over all points p_j proximate to p_i) is called *tame* (see [3, 4.6]). Tame unloadings are of particular interest to us, since they preserve independence of the conditions, that is, we have $h^1(\mathcal{O}_{S_K}(dE_0 + D_{K,\mathbf{m}})) = h^1(\mathcal{O}_{S_K}(dE_0 + D_{K,\mathbf{m}'}))$ after a tame unloading.

Recall that, for any cluster $K \in X_{r-1}$ and $i \geq 0$, the pullback to the surface S_K of F_i is the same as the class E_i of the exceptional divisor of blowing up p_i. Given an integer d we define, as in [20],

$$\mathcal{J}_{d,m} = \mathcal{O}_{X_r}(dF_0 - m_1F_1 - m_2F_2 - \cdots - m_rF_r) .$$

Then $\mathcal{J}_{d,\mathbf{m}}$ is an invertible sheaf on X_r and therefore flat over X_{r-1}, so the function

$$K \longmapsto h^1\left(\mathcal{J}_{d,\mathbf{m}} \otimes_{X_{r-1}} k(K)\right) = h^1\left(\mathcal{O}_{S_K}(dE_0 + D_{K,\mathbf{m}})\right)$$

is upper–semicontinuous on X_{r-1}. Therefore, if Y is a closed subvariety of X_{r-1}, $K \in Y$ is a cluster such that $(K, \mathbf{m})$ imposes independent conditions on curves of degree d, then there is an open subset $V \subset Y$ such that every cluster $K' \in V$ imposes independent conditions on curves of degree d.

2. Specialization and H^1-vanishing

Given a system of multiplicities $\mathbf{m} = (m_1, m_2, \ldots, m_r)$ we define $s_{j,k}(\mathbf{m}) = \sum_{i=j}^{k} m_i$. Consider systems of the form $\mathbf{m} = (m_1, m_r+1, m_r+1, \ldots, m_r+1, m_r, \ldots, m_r)$; such a system is determined by m_1 and $s = s_{2,r}(\mathbf{m})$, namely

$$\mathbf{m}(m_1, s) = (m_1, \overbrace{m_r+1, m_r+1, \ldots, m_r+1}^{t}, m_r, \ldots, m_r),$$

where t and m_r are chosen so that $s_{2,r}(\mathbf{m}(m_1, s)) = s$; i.e., by Euclidean division, we have $s = m_r(r-1) + t$ with $0 \leq t < r-1$. In particular, we have $(m^r) = \mathbf{m}(m, (r-1)m)$.

Lemma 2.1. *Let $\mathbf{m} = \mathbf{m}(x, y)$, $\mathbf{m}' = \mathbf{m}(x+1, y-k)$ and assume that (x, y) satisfy $x \geq s_{2,k+1}(\mathbf{m}) - 1$. Let K be a cluster with Enriques diagram $\mathbf{D}_k$ such that $(K, \mathbf{m}')$ imposes independent conditions on curves of degree d. Then $(K, \mathbf{m})$ also imposes independent conditions on curves of degree d.*

Proof. Let $z = x - s_{2,k+1}(\mathbf{m}) + 1$. By the assumption on (x, y), we have $z \geq 0$. Choose z points $q_1, q_2, \ldots, q_z$ on the strict transform $\tilde{E}_1 \subset S_K$ of the exceptional divisor of blowing up $p_1(K)$. Define a system of multiplicities $\bar{\mathbf{m}} = (m_1, m_2, \ldots, m_r, 1, \ldots, 1)$ for the new cluster $\bar{K} = (p_1, p_2, \ldots, p_r, q_1, \ldots, q_z)$; that is, add to the weighted cluster $(K, \mathbf{m})$ z points of multiplicity one in the first neighborhood of the root $p_1(K)$. Each new point adds a linear condition to the conditions of $(K, \mathbf{m})$, which may or may not be independent; however, clearly, if all conditions imposed by $(\bar{K}, \bar{\mathbf{m}})$ on curves of degree d are independent, then those imposed by $(K, \mathbf{m})$ must be independent too. Now $(\bar{K}, \bar{\mathbf{m}})$ is not consistent; if we perform the unloading procedure to $(\bar{K}, \bar{\mathbf{m}})$, we see that the equivalent consistent cluster is $(K, \mathbf{m}')$ (plus the points q_i with multiplicity zero, which does not affect the argument), and all unloading steps are tame. Therefore $(\bar{K}, \bar{\mathbf{m}})$ imposes independent conditions, because $(K, \mathbf{m}')$ does. $\square$

We now define recursively systems of multiplicities $\mathbf{m}_1, \mathbf{m}_2, \dots \mathbf{m}_{r-1}$, as follows. We set $\mathbf{m}_1 = (m^r) = \mathbf{m}(m, (r-1)m)$. Suppose we have defined $\mathbf{m}_k = \mathbf{m}(x_k, y_k)$. For each integer $n \geq 0$, consider the system $\mathbf{m}_k(n) = \mathbf{m}(x_k + n, y_k - nk)$. Clearly $f_k(n) = x_k + n - s_{2,k+2}(\mathbf{m}_k(n))$ is an increasing function. We now choose n_k as the minimum integer with $n_k \geq 0$ and $f_k(n_k) \geq -1$, and define $\mathbf{m}_{k+1} = \mathbf{m}_k(n_k)$.

THEOREM 2.2. *Let $\mathbf{m}_k = \mathbf{m}(x_k, y_k)$ be the systems of multiplicities defined above, and let d be an integer, $d \geq s_{1,2}(\mathbf{m}_{r-1}) - 1$. Then for K general in $U(\mathbf{D}_k)$, the weighted cluster $(K, \mathbf{m}_k)$ imposes independent conditions on curves of degree d.*

This has the following immediate corollary:

COROLLARY 2.3. *$d_1(m, r) = s_{1,2}(\mathbf{m}_{r-1}) - 1$ is a sufficient degree for independence of the conditions imposed by r general m–fold points. In other words, let $p_1, p_2, \dots, p_r$ be points in general position, and assume $d \geq s_{1,2}(\mathbf{m}_{r-1}) - 1$. Then the linear system $\mathcal{L}$ of curves of degree d which have multiplicity at least m at each point has the expected dimension, namely*

$$\dim \mathcal{L} = \frac{d(d+3)}{2} - r\frac{m\,(m+1)}{2} .$$

Proof. We have already mentioned that it is enough to find a cluster $K \in X_r$ such that $(K, \mathbf{m})$ imposes independent conditions on curves of degree d, with $\mathbf{m} = (m, m, \dots, m)$. Theorem 2.2, with $k = 1$, says that a general cluster in $U(\mathbf{D}_1)$ has this property. □

Proof of Theorem 2.2. We use descending induction on k. Let $k = r - 1$ and $K \in U(\mathbf{D}_{r-1})$. Observe that we can assume that $(K, \mathbf{m}_{r-1})$ is consistent. Indeed, by construction $x_k \geq s_{2,r}(\mathbf{m}_k) - 1$, so either $(K, \mathbf{m}_{r-1})$ is consistent, or an equivalent consistent system of multiplicities is reached by a single tame unloading step at $p_1(K)$, which does not change the value $s_{1,2}(\mathbf{m}_{r-1})$. Now for consistent weighted clusters $(K, \mathbf{m})$ with $K \in U(\mathbf{D}_{r-1})$ the dimension of the linear system $|dE_0 + D_{K,\mathbf{m}'}|$ is known for all d (cf. [6]), and in particular it is known that the conditions are independent if and only if $d \geq s_{1,2}(\mathbf{m}) - 1$.

Let now $k < r - 1$ and suppose we have proved that for K general in $U(\mathbf{D}_{k+1})$, the weighted cluster $(K, \mathbf{m}_{k+1})$ also imposes independent conditions on curves of degree d. As $\overline{U(\mathbf{D}_{k+1})} \subset \overline{U(\mathbf{D}_k)}$, by semicontinuity we obtain that $(K, \mathbf{m}_{k+1})$ also imposes independent conditions on curves of degree d when K is general in $U(\mathbf{D}_k)$. We will be done if we prove that, for any given $K \in U(\mathbf{D}_k)$, if $(K, \mathbf{m}_{k+1})$ imposes independent conditions, then $(K, \mathbf{m}_k)$ imposes independent conditions too. But by definition $(x_{k+1}, y_{k+1}) = (x_k + n_0, y_k - n_0 k)$, so applying Lemma 2.1 n_0 times gives the desired result. □

Let $d_1(m,r) = s_{1,2}(\mathbf{m}_{r-1}) - 1$ be the minimal degree for which Corollary 2.3 guarantees independence of the conditions imposed by r general m–fold points. We have given an algorithm to compute this degree, but it is still not clear whether the resulting values are better than the known bounds on the degree. It is therefore of interest to give upper bounds for $d_1(m,r)$, and in particular to determine whether the value $d_1(m,r)$ is "asymptotically correct", that is, of the same order as the known necessary conditions for independence, in the sense that

$$\lim_{m\to\infty} \lim_{r\to\infty} \frac{d_1(m,r)}{m\sqrt{r}} = 1\,.$$

Let $A_k = kx_k + y_k = kx_{k+1} + y_{k+1}$. By definition,

$$d_1(m,r) + 1 = \left\lceil \frac{A_{r-1}}{r-1} \right\rceil \le \frac{A_{r-1} + r - 2}{r-1}\,,$$

and we want to bound A_{r-1} above. To do this, we will give recursive upper bounds on n_k, x_k and A_k. It is not difficult to see that, given arbitrary integers x, y,

$$s_{2,k+2}(\mathbf{m}(x,y)) \le \frac{k+1}{r-1}(y-k-1) + k + 1$$

with equality holding if and only if $y \equiv k+1 \pmod{r-1}$. Therefore

$$f_k(n) \ge x_k - \frac{k+1}{r-1} y_k + n\frac{r-1+k(k+1)}{r-1} - (k+1)\frac{r-1-(k+1)}{r-1}\,,$$

and, by the definition of n_k,

$$\begin{aligned} n_k &< \frac{-x_k + \frac{k+1}{r-1}y_k + (k+1)\frac{r-1-(k+1)}{r-1} - 1}{\frac{r-1+k(k+1)}{r-1}} + 1 \\ &= (A_k + r - 2)(\alpha_{k+1} - 1) - x_k\,, \end{aligned}$$

where

$$\alpha_i = \frac{r-1+i^2}{r-1+i^2-i}\,.$$

The definition of x_{k+1} then gives $x_{k+1} < (A_k + r - 2)(\alpha_{k+1} - 1)$ and $A_{k+1} = A_k + x_{k+1} < (A_k + r - 2)\alpha_{k+1} - (r-2)$. Thus we obtain

$$\begin{aligned} d_1(m,r) + 1 = \left\lceil \frac{A_{r-1}}{r-1} \right\rceil &\le \frac{A_{r-1} + r - 2}{r-1} < \frac{A_{r-2} + r - 2}{r-1}\alpha_{r-1} \\ &< \frac{A_{r-3} + r - 2}{r-1}\alpha_{r-2}\alpha_{r-1} < \cdots < \frac{A_1 + r - 2}{r-1}\prod_{i=2}^{r-1}\alpha_i \\ &= \frac{rm + r - 2}{r-1}\prod_{i=2}^{r-1}\alpha_i, \end{aligned}$$

and we have proved the following result:

Corollary 2.4. *The value*

$$d_2(m,r) = \left\lceil \frac{r(m+1)-2}{r-1} \prod_{i=2}^{r-1} \frac{r-1+i^2}{r-1+i^2-i} \right\rceil - 2$$

is a sufficient degree for independence of the conditions imposed by r general m–fold points.

The product $P(r) = (r-1)\prod_{i=2}^{r-1} \alpha_i^{-1}$ has already been studied in the literature (see [20] and [11]). In particular, it is known that $P(r) = \sqrt{r} - \pi/8 + O(\sqrt{r}^{-1})$, so defining $\alpha(r) = r/P(r)$ we have $d_1(m,r)+1 < (m+1)\alpha(r)$ and $\alpha(r) = \sqrt{r} + \pi/8 + O(\sqrt{r}^{-1})$; therefore $d_1(m,r)$ is of the desired order (and asymptotically correct). It would be interesting to have a more precise estimate for $P(r)$. Using the upper bound of [20], we prove:

Corollary 2.5. *The value*

$$d_3(m,r) = \left\lceil (m+1)\left(\sqrt{r+1.9} + \frac{\pi}{8}\right) \right\rceil - 2$$

is a sufficient degree for independence of the conditions imposed by r general m–fold points.

Proof. By [20, 5.1] we have $P(r) > \sqrt{r-1} - \pi/8$ for $r \geq 10$, so

$$d_1(m,r) + 1 < (m+1)\frac{r}{P(r)} < \frac{r(m+1)}{\sqrt{r-1}-\pi/8},$$

and it is easy to see that

$$\frac{r}{\sqrt{r-1}-\pi/8} = \sqrt{r-1} + \frac{\pi}{8} + \frac{\pi^2+64}{64\sqrt{r-1}-8\pi}\ .$$

To complete the proof, note that the constant 1.9 has been chosen so that, for $r \geq 10$,

$$\sqrt{r+1.9} - \sqrt{r-1} > \frac{\pi^2+64}{64\sqrt{r-1}-8\pi}\ .$$

□

To compare $d_3(m,r)$ with previously known sufficient degrees, note first that if $m \geq 13$ (which we may assume, because for smaller multiplicities the dimensions of the linear systems are known) and $r \geq 108$, then

$$(m+1)\left(\sqrt{r+1.9}+\frac{\pi}{8}\right) < (m+1)\left(\sqrt{\frac{10}{9}r} - \frac{1}{13}\right) \leq (m+1)\sqrt{\frac{10}{9}r} - 1,$$

so for $r \geq 108$ it is better to use $d_3(m,r)$ than the sufficient degree given by Xu. The sufficient degree obtained by Ballico equals

$$\frac{-5+2\,m+\sqrt{17-20\,m+4\,m^2+4\,m\,r+4\,m^2\,r}}{2},$$

that is, $(m+1)(\sqrt{r+1}+1)+O(\sqrt{m})$. For fixed $r \geq 12$ and large m, this expression is greater than $d_3(m,r)$ since $\sqrt{r+1}+1 > \sqrt{r+1.9}+\pi/8$. The reader may check that this holds whenever $m \geq 28$, $m^2 \geq r$. Altogether, $d_3(m,r)$ is better than the previous known bounds provided $r \geq 108$, $m \geq 28$ and $m^2 \geq r$.

3. Comments and remarks

We conclude this paper with some remarks on the method and the results obtained.

First, the specialization does not depend at all on the homogeneity of the multiplicities involved, which has been used only to simplify the calculations leading to the explicit sufficient conditions. It is not difficult to see how to adapt Lemma 2.1 to the case when $\mathbf{m} = (m_1, m_2, \ldots, m_r)$ is arbitrary; then one would define analogous systems of multiplicities $\mathbf{m}_k$ and finally obtain a value $d_1(\mathbf{m})$ such that, for $d \geq d_1(\mathbf{m})$, r points in general position with multiplicities $m_1, m_2, \ldots, m_r$ impose independent conditions on curves of degree d. The algorithm can be implemented and the results obtained are significant; however, it is not clear how to obtain explicit sufficient degrees analogous to those in Corollaries 2.4 and 2.5.

Secondly, although $d_1(m,r)$ behaves asymptotically like $(m+1)(\sqrt{r+1.9}+\pi/8)$, if m is small compared to r (i.e., for m around $\sqrt{r}$ or smaller) the difference is important, so it is worthwhile to effectively compute $d_1(m,r)$. This is of interest because $m \leq \sqrt{r}$ is the range in which Ballico's sufficient degree is lower than $(m+1)\left(\sqrt{r+1.9}+\pi/8\right)$, so $d \geq d_1(m,r)$ might be a better sufficient condition for independence also in these cases. As an example, consider the case $m = 20$, $r = 401$. The conjectures predict that the linear system of curves of degree d through 401 general 20–fold points is empty for $d \leq 408$ and the conditions are independent for $d \geq 409$. The minimal degree for which we have just proved independence is $d_1(20, 401) = 417$, and $d_3(20, 401) = 428$, whereas the sufficient degrees of [12], [1] and [21] are 429, 429 and 441, respectively. The importance of these differences becomes even more evident when we recall that for $d \leq 401$ the linear system is known to be empty [11], so the degrees for which we do not (yet) know its dimension reduce almost to the half by taking $d_1(m,r)$ instead of the other known sufficient degrees. We would like to thank B. Harbourne for some inspired remarks that lead us to this example.

Finally, we note that any H^1-vanishing result like that proved in this paper can be applied to prove the existence of curves of low degree with prescribed singularities, using methods of [8], [1] or [19]. In particular, from Corollary 2.5 one deduces the existence of irreducible curves of every degree d satisfying

$$d+1 \geq (m+1)\left(\sqrt{r+1.9}+\frac{\pi}{8}\right),$$

with r ordinary singularities of multiplicity m and no other singularity. This improves previously known results. In the example mentioned above we can use $d_1(20, 401)$ to show the existence of irreducible plane curves of degree 418 with 401 ordinary 20–fold points and no other singularity.

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