

## QUANTUM BOHR COMPACTIFICATION

PIOTR M. SOLTAN

ABSTRACT. We introduce a non commutative analog of the Bohr compactification. Starting from a general quantum group  $G$  we define a compact quantum group  $\mathfrak{b}G$ , which has a universal property such as the universal property of the classical Bohr compactification for topological groups. We study the object  $\mathfrak{b}G$  in the special cases when  $G$  is a classical locally compact group, the dual of a classical group, a discrete or compact quantum group, and a quantum group arising from a manageable multiplicative unitary. We also use our construction to give new examples of compact quantum groups.

### 1. Introduction

Let  $A$  be a  $C^*$ -algebra and let  $\Delta \in \text{Mor}(A, A \otimes A)$  be a coassociative morphism. Then  $G = (A, \Delta)$  is a non commutative analog of a semigroup. If  $A$  is unital and the sets

$$(1.1) \quad \begin{aligned} &\{\Delta(\mathbf{a})(I \otimes \mathbf{b}) : \mathbf{a}, \mathbf{b} \in A\}, \\ &\{(\mathbf{a} \otimes I)\Delta(\mathbf{b}) : \mathbf{a}, \mathbf{b} \in A\} \end{aligned}$$

are linearly dense in  $A \otimes A$ , then  $G$  is a *compact quantum group* ([33]). In the papers [29], [30], [33] S.L. Woronowicz brought the understanding of compact quantum groups to a very satisfactory level. His theory is a cornerstone of the theory of locally compact quantum groups, the latter being still at the development stage. There are different approaches to defining general quantum groups. The common agreement is that a quantum group is described by a pair  $(A, \Delta)$  of a  $C^*$ -algebra and a comultiplication on  $A$  and possessing some additional properties. The linear density (and containment) of the sets (1.1) in  $A \otimes A$  is generally accepted. A very successful definition of a reduced  $C^*$ -algebraic quantum group is due to Kustermans and Vaes ([9]). A related notion of an algebraic quantum group was introduced by van Daele (see, e.g.,

---

Received May 12, 2005; received in final form July 3, 2005.

2000 *Mathematics Subject Classification*. Primary 46L89. Secondary 58B32, 22D25.

Research partially supported by KBN grants nos. 2P03A04022 and 115/E-343/SPB/6.PRUE/DIE50/2005-2008.

[22], [10]). Some experts have suggested that quantum groups should be defined as the objects related to manageable multiplicative unitaries ([34], [18]). In some sense this last approach contains the other ones. However, to include into the picture, e.g., the universal quantum groups ([8]) one must go beyond the scheme of manageable multiplicative unitaries.

In this paper we develop a procedure of obtaining a universal compactification of a (quite general) quantum group. This universal object is a non commutative (or *quantum*) analog of the Bohr compactification for topological groups ([27], [11], [6]). It is a compact quantum group with a universal property which mirrors that of the classical Bohr compactification. This construction provides a functorial passage from quantum (semi)groups to compact quantum groups. With the thorough understanding of the latter class we can, in some cases, limit the study of an unfamiliar object to a much better behaved one. We can also use our procedure to construct new examples of compact quantum groups (see Section 4). In addition, our approach to quantum Bohr compactification allows easy generalizations of the standard notions of harmonic analysis, such as almost periodic functions, the mean of an almost periodic function, etc. However, we do not deal with these aspects in this paper.

The notion of a universal compactification has been studied by the author in the algebraic context of discrete quantum groups (in the sense of Van Daele [21]) in [15]. However, the algebraic approach is not satisfactory from the point of view of harmonic analysis. For example, it does not allow the existence of a mean for almost periodic elements. Moreover, the procedure of constructing the quantum Bohr compactification proposed in [15] does not carry over to the more general quantum groups on the  $C^*$ -algebra level (although it can be extended to  $C^*$ -algebraic discrete quantum groups, cf. [16] and Section 4.3).

In non commutative geometry objects of geometric nature are replaced by non commutative algebras which play the role of algebras of functions on non commutative spaces or *quantum spaces*. In this sense locally compact quantum spaces are simply the objects of the category dual to the category of  $C^*$ -algebras (cf. [31]). Let us stress that we always treat non isomorphic  $C^*$ -algebras as different quantum spaces. In particular, if  $\Gamma$  is a non amenable discrete group, then the universal and reduced  $C^*$ -algebras  $C^*(\Gamma)$  and  $C_r^*(\Gamma)$  are different and we treat them as algebras of functions on different compact quantum groups.

As mentioned above, the purpose of this paper is to define and study a non commutative analog of the Bohr compactification for quantum groups. In order to encompass as many definitions of quantum groups as possible, we shall deal with objects of the form  $G = (A, \Delta)$ , where  $A$  is a  $C^*$ -algebra and  $\Delta \in \text{Mor}(A, A \otimes A)$  is coassociative. We shall call these objects *quantum (semi)groups* to indicate that our main interest lies in the group structure of  $G$ . For any such object  $G$  we shall define a compact quantum group  $\mathfrak{b}G$

possessing a certain universal property (cf. Section 3). We shall also deal with many special cases and examples of quantum Bohr compactifications.

The generality of dealing with “quantum semigroups” means, in particular, that the object  $\mathfrak{b}G$  will be trivial in many cases. On the other hand, this object can be trivial even for classical groups (so-called *minimally almost periodic groups*, see, e.g., [24]). In Section 4 we shall specialize our construction to some well studied classes of quantum (semi)groups.

Let us now describe the contents of the paper. Section 2 is devoted to the definition of the quantum Bohr compactification. First the notion of a bounded representation of a quantum (semi)group will be discussed. Then, in Subsection 2.2, we shall define admissible representations which will be crucial for the definition of quantum Bohr compactification. In Subsections 2.3 and 2.4 we shall conduct an analysis of the sets of matrix elements of bounded and admissible representations. The definition of quantum Bohr compactification is given in Subsection 2.4. In Section 3 we discuss the universal property of quantum Bohr compactification. We use it to turn the compactification into a functor from the category of quantum (semi)groups to the full subcategory of compact quantum groups. The last section contains a discussion of the quantum Bohr compactification in several special cases. We study the quantum Bohr compactification of a classical locally compact group (Subsection 4.1), the dual of a classical group (Subsection 4.2), and compact and discrete quantum groups (Subsection 4.3). Then we use our construction to produce new examples of compact quantum groups which resemble the profinite groups of classical harmonic analysis (Subsection 4.4). In Subsection 4.5 we discuss the notion of maximal almost periodicity for quantum groups and state a theorem which links this notion with other properties of quantum groups for the case of discrete quantum groups. In Subsection 4.6 we determine the additional elements of the structure of the canonical Hopf  $*$ -algebra of the quantum Bohr compactification for quantum groups arising from manageable multiplicative unitaries.

We shall freely use the established language of the theory of quantum groups on the  $C^*$ -algebraic level. We refer to [31], [32] for notions such as morphisms of  $C^*$ -algebras, multipliers, affiliated elements, etc. All vector spaces will be over the field of complex numbers. The algebra of  $\mathfrak{n} \times \mathfrak{n}$  matrices over  $\mathbb{C}$  will be denoted by  $M_{\mathfrak{n}}$ . Also, for any vector space  $\mathfrak{X}$ , the space of  $\mathfrak{n} \times \mathfrak{n}$  matrices with entries from  $\mathfrak{X}$  will be denoted by  $M_{\mathfrak{n}}(\mathfrak{X})$ . The algebra of all linear maps  $\mathfrak{X} \rightarrow \mathfrak{X}$  will be denoted by  $\text{End}(\mathfrak{X})$ .

The author would like to express his gratitude to the referee for helpful remarks and suggestions.

## 2. Definition of quantum Bohr compactification

**2.1. Bounded representations.** Let  $G = (A, \Delta)$  be a quantum (semi)-group. We shall deal with elements of tensor products of finite dimensional vector spaces with  $M(A)$ . If  $\mathfrak{Z}$  is a finite dimensional vector space and  $X \in \mathfrak{Z} \otimes M(A)$  is any element, then we define elements  $X_{12}$  and  $X_{13}$  of  $\mathfrak{Z} \otimes M(A) \otimes M(A)$  by

$$\begin{aligned} X_{12} &= (\text{id} \otimes \phi_2)X, \\ X_{13} &= (\text{id} \otimes \phi_3)X, \end{aligned}$$

where the maps  $\phi_k: M(A) \rightarrow M(A) \otimes M(A)$  ( $k = 2, 3$ ) are given by

$$\begin{aligned} \phi_2(\mathfrak{m}) &= (\mathfrak{m} \otimes I_A), \\ \phi_3(\mathfrak{m}) &= (I_A \otimes \mathfrak{m}) \end{aligned}$$

for all  $\mathfrak{m} \in M(A)$ .

In what follows the space  $\mathfrak{Z}$  will be equal to  $\text{End}(\mathfrak{X})$  for some finite dimensional vector space  $\mathfrak{X}$ ; in particular,  $\mathfrak{Z}$  will be a unital algebra. Let  $\mathfrak{Z}_1$  and  $\mathfrak{Z}_2$  be two unital algebras and  $X \in \mathfrak{Z}_1 \otimes M(A)$  and  $Y \in \mathfrak{Z}_2 \otimes M(A)$ . Then we can define elements  $X_{13}$  and  $Y_{23}$  of  $\mathfrak{Z}_1 \otimes \mathfrak{Z}_2 \otimes M(A)$  by

$$\begin{aligned} X_{13} &= (\psi_1 \otimes \text{id})X, \\ Y_{23} &= (\psi_2 \otimes \text{id})Y, \end{aligned}$$

where the maps  $\psi_k: \mathfrak{Z}_k \rightarrow \mathfrak{Z}_1 \otimes \mathfrak{Z}_2$  ( $k = 1, 2$ ) are given by

$$\begin{aligned} \psi_1(x) &= (x \otimes I_{\mathfrak{Z}_2}), \\ \psi_2(y) &= (I_{\mathfrak{Z}_1} \otimes y) \end{aligned}$$

for all  $x \in \mathfrak{Z}_1$  and all  $y \in \mathfrak{Z}_2$ .

**DEFINITION 2.1.** Let  $G = (A, \Delta)$  be a quantum (semi)group and let  $\mathfrak{X}$  be a finite dimensional vector space. A bounded representation of  $G$  on  $\mathfrak{X}$  is an element  $T \in \text{End}(\mathfrak{X}) \otimes M(A)$  satisfying

- (1)  $(\text{id} \otimes \Delta)T = T_{12}T_{13}$ ,
- (2)  $T$  is an invertible element of  $\text{End}(\mathfrak{X}) \otimes M(A)$ .

Let  $\mathfrak{X}$  be a finite dimensional vector space and let  $G = (A, \Delta)$  be a quantum (semi)group. The element  $I_{\text{End}(\mathfrak{X})} \otimes I_A \in \text{End}(\mathfrak{X}) \otimes M(A)$  is a bounded representation of  $G$  on  $\mathfrak{X}$  called the *trivial* representation. There may be few other representations if no further hypothesis is put on  $G$ . It is known that compact quantum groups have many representations ([33]).

There are several natural operations which can be performed on representations of quantum (semi)groups. We shall use two of them.

**2.1.1. Direct sum.** Let  $G = (A, \Delta)$  be a quantum (semi)group and let  $\mathfrak{X}$  and  $\mathfrak{Y}$  be finite dimensional vector spaces. Let  $T \in \text{End}(\mathfrak{X}) \otimes M(A)$  and  $S \in \text{End}(\mathfrak{Y}) \otimes M(A)$  be bounded representations of  $G$  on  $\mathfrak{X}$  and  $\mathfrak{Y}$ , respectively. The direct sum  $T \oplus S$  of  $T$  and  $S$  is an element of  $\text{End}(\mathfrak{X} \oplus \mathfrak{Y}) \otimes M(A)$  defined by

$$(2.1) \quad T \oplus S = (\iota_{\mathfrak{X}} \otimes I_A) T (\pi_{\mathfrak{X}} \otimes I_A) + (\iota_{\mathfrak{Y}} \otimes I_A) S (\pi_{\mathfrak{Y}} \otimes I_A),$$

where  $\iota_{\mathfrak{X}}$  and  $\iota_{\mathfrak{Y}}$  are canonical inclusions of  $\mathfrak{X}$  and  $\mathfrak{Y}$  into  $\mathfrak{X} \oplus \mathfrak{Y}$  and  $\pi_{\mathfrak{X}}$  and  $\pi_{\mathfrak{Y}}$  are the canonical projections from  $\mathfrak{X} \oplus \mathfrak{Y}$  onto the summands. It is very easy to see that  $T \oplus S$  is a bounded representation of  $G$  on  $\mathfrak{X} \oplus \mathfrak{Y}$ .

**2.1.2. Tensor product.** As before let  $G = (A, \Delta)$  be a quantum (semi)group and let  $T \in \text{End}(\mathfrak{X}) \otimes M(A)$  and  $S \in \text{End}(\mathfrak{Y}) \otimes M(A)$  be bounded representations of  $G$  on finite dimensional vector spaces  $\mathfrak{X}$  and  $\mathfrak{Y}$ , respectively. The tensor product  $S \otimes T$  of  $S$  and  $T$  is the element of  $\text{End}(\mathfrak{X} \otimes \mathfrak{Y}) \otimes M(A) = \text{End}(\mathfrak{X}) \otimes \text{End}(\mathfrak{Y}) \otimes M(A)$  defined by

$$T \otimes S = T_{13} S_{23}.$$

Just as easily as for direct sums, one can prove that  $T \otimes S$  is a bounded representation of  $G$  on  $\mathfrak{X} \otimes \mathfrak{Y}$ .

**2.2. Admissible representations.** Let  $\mathfrak{X}$  be a finite dimensional vector space and let  $G = (A, \Delta)$  be a quantum (semi)group. Let  $T \in \text{End}(\mathfrak{X}) \otimes M(A)$  be a bounded representation of  $G$  on  $\mathfrak{X}$ . Denote by  $\mathfrak{X}'$  the dual space of  $\mathfrak{X}$  and let  $\theta: \text{End}(\mathfrak{X}) \rightarrow \text{End}(\mathfrak{X}')$  be the mapping of an operator to its adjoint. Define  $T^\top \in \text{End}(\mathfrak{X}') \otimes M(A)$  by

$$T^\top = (\theta \otimes \text{id})T.$$

We call  $T^\top$  the *transpose* of  $T$ . The transpose of a representation is not a representation. First of all formula (1) of Definition 2.1 needs to be modified, but more importantly,  $T^\top$  might not be invertible. In what follows we shall restrict to situations when this does not happen.

**DEFINITION 2.2.** Let  $G = (A, \Delta)$  be a quantum (semi)group and let  $\mathfrak{X}$  be a finite dimensional vector space. Let  $T$  be a bounded representation of  $G$  on  $\mathfrak{X}$ . Then  $T$  is *admissible* if  $T^\top$  is an invertible element of  $\text{End}(\mathfrak{X}') \otimes M(A)$ .

**REMARK 2.3.**

- (1) The trivial representation of a quantum (semi)group  $G$  on a finite dimensional vector space  $\mathfrak{X}$  is admissible. It may happen that  $G$  has no other admissible representations (or even bounded finite dimensional ones, even if  $G$  is a classical group, cf. [24]).
- (2) Let us note that any finite dimensional representation of a compact quantum group is admissible (one need not add the adjective

“bounded” in the compact case). Indeed, let  $G = (A, \Delta)$  be a compact quantum group. Then there exists a family  $(U_\alpha)$  of finite dimensional representations such that any representation is equivalent to a direct sum of some of the  $U_\alpha$ 's. It is therefore enough to show that each  $U_\alpha$  is admissible. Each  $U_\alpha$  can be treated as a unitary element of the  $C^*$ -algebra  $M_{N^\alpha}(A) = \text{End}(\mathbb{C}^n) \otimes A$  (cf. Subsection 2.3). Let  $\kappa$  be the coinverse of  $G$ . Then by [33, Equation (6.12)] we have  $U_\alpha^\top = [(T \otimes \kappa)U_\alpha]^*$ . Since  $\kappa$  is antimultiplicative,  $T \otimes \kappa$  is an algebra antihomomorphism and it follows that  $U_\alpha^\top$  is invertible (cf. also [25, Section 2]).

**PROPOSITION 2.4.** *Let  $T$  and  $S$  be admissible representations of a quantum (semi)group  $G = (A, \Delta)$  on finite dimensional vector spaces  $\mathfrak{X}$  and  $\mathfrak{Y}$ , respectively. Then the representation  $T \oplus S$  of  $G$  on  $\mathfrak{X} \otimes \mathfrak{Y}$  is admissible.*

*Proof.* Clearly

$$(T \oplus S)^\top = (\iota_{\mathfrak{X}'} \otimes I_A) T^\top (\pi_{\mathfrak{X}'} \otimes I_A) + (\iota_{\mathfrak{Y}'} \otimes I_A) S^\top (\pi_{\mathfrak{Y}'} \otimes I_A).$$

In particular, the inverse of  $(T \oplus S)^\top$  in  $\text{End}(\mathfrak{X}' \oplus \mathfrak{Y}') \otimes M(A)$  is

$$(\iota_{\mathfrak{X}'} \otimes I_A) (T^\top)^{-1} (\pi_{\mathfrak{X}'} \otimes I_A) + (\iota_{\mathfrak{Y}'} \otimes I_A) (S^\top)^{-1} (\pi_{\mathfrak{Y}'} \otimes I_A).$$

It follows that  $T \oplus S$  is admissible. □

Later we will show that the tensor product of admissible representations is admissible.

**2.3. Matrix elements.** Let  $G = (A, \Delta)$  be a quantum (semi)group and let  $\mathfrak{X}$  be a finite dimensional vector space. Let  $T \in \text{End}(\mathfrak{X}) \otimes M(A)$  be a bounded finite dimensional representation of  $G$  on  $\mathfrak{X}$ . For any linear functional  $\varphi$  on  $\text{End}(\mathfrak{X})$  the element

$$(\varphi \otimes \text{id})T \in M(A)$$

is called a *matrix element* of  $T$ .

**PROPOSITION 2.5.** *Let  $G = (A, \Delta)$  be a quantum (semi)group. The set of matrix elements of bounded finite dimensional representations of  $G$  is a unital subalgebra of  $M(A)$ .*

*Proof.* Let  $T \in \text{End}(\mathfrak{X}) \otimes M(A)$  and  $S \in \text{End}(\mathfrak{Y}) \otimes M(A)$  be bounded representations of  $G$  on finite dimensional vector spaces. Let  $\varphi$  be a functional on  $\text{End}(\mathfrak{X})$  and  $\psi$  a functional on  $\text{End}(\mathfrak{Y})$ . The functional

$$\text{End}(\mathfrak{X}) \oplus \text{End}(\mathfrak{Y}) \ni \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix} \longmapsto \varphi(m) + \psi(n) \in \mathbb{C}$$

has an extension  $f$  to  $\text{End}(\mathfrak{X} \oplus \mathfrak{Y})$  and clearly

$$(f \otimes \text{id})(T \oplus S) = (\varphi \otimes \text{id})T + (\psi \otimes \text{id})S.$$

Therefore the set of matrix elements of bounded finite dimensional representations of  $G$  is a vector subspace of  $M(A)$ .

The unit of  $M(A)$  is a matrix element of any trivial representation of  $G$ . Finally, if  $\varphi$  and  $\psi$  are as above, then

$$(\varphi \otimes \text{id})T(\psi \otimes \text{id})S = ((\varphi \otimes \psi) \otimes \text{id})(T \oplus S),$$

which shows that the set of matrix elements of bounded finite dimensional representations of  $G$  is a subalgebra of  $M(A)$ .  $\square$

Let  $\mathfrak{X}$  be a finite dimensional vector space. The space  $\text{End}(\mathfrak{X})$  is naturally isomorphic to  $\mathfrak{X} \otimes \mathfrak{X}'$  (where  $\mathfrak{X}'$  is the dual space of  $\mathfrak{X}$ ) via the map

$$\mathfrak{X} \otimes \mathfrak{X}' \ni x \otimes x' \longmapsto t_{x,x'} \in \text{End}(\mathfrak{X}),$$

where  $t_{x,x'}(y) = \langle x', y \rangle x$ . If  $e_1, \dots, e_n$  is a basis of  $\mathfrak{X}$  and  $e'_1, \dots, e'_n$  the dual basis of  $\mathfrak{X}'$ , then we can write any element  $X \in \text{End}(\mathfrak{X}) \otimes M(A)$  in the form

$$X = \sum_{k,l=1}^n (e_k \otimes e'_l) \otimes x^{kl}.$$

Thus  $X$  can be identified with an  $n \times n$  matrix of elements of  $M(A)$ . The multiplication of matrices corresponds to the product in  $\text{End}(\mathfrak{X}) \otimes M(A)$ . One quickly finds that an invertible element  $T \in \text{End}(\mathfrak{X}) \otimes M(A)$  is a bounded representation of  $G$  if and only if the matrix representation  $T = (t^{kl})_{k,l=1,\dots,n}$  satisfies the familiar formula

$$(2.2) \quad \Delta(t^{kl}) = \sum_{p=1}^n t^{kp} \otimes t^{pl}.$$

Moreover, having chosen a basis in  $\mathfrak{X}$  we naturally identify  $\text{End}(\mathfrak{X})$  and  $\text{End}(\mathfrak{X}')$  with  $M_n$  in such a way that  $T^\top$  becomes the transpose matrix of  $T$ . Thus  $T$  is admissible if and only if the matrix

$$(t^{lk})_{k,l=1,\dots,n}$$

is invertible.

Let us also note the following lemma.

LEMMA 2.6. *The linear set of all matrix elements of  $T$  coincides with the span of  $\{t^{kl} : k, l = 1, \dots, n\}$ .*

Then we have:

PROPOSITION 2.7. *Let  $G = (A, \Delta)$  be a quantum (semi)group and let  $T$  be an admissible representation of  $G$  on a finite dimensional vector space  $\mathfrak{X}$ . Let  $B_T$  be the unital  $C^*$ -subalgebra of  $M(A)$  generated by all matrix elements*

of  $\Gamma$ . Let  $\Delta_\Gamma$  be the restriction of  $\Delta$  to  $B_\Gamma$ . Then  $G_\Gamma = (B_\Gamma, \Delta_\Gamma)$  is a compact quantum group.

*Proof.* Let us fix a basis of  $\mathfrak{X}$  and treat  $\Gamma$  as a matrix of elements of  $M(A)$ . Then clearly  $\Gamma$  belongs to  $M_n(B_\Gamma)$ . Moreover, formula (2.2) shows that  $\Delta_\Gamma(B_\Gamma) \subset B_\Gamma \otimes B_\Gamma$ .

We know that  $\Gamma^\top$  is invertible in  $M_n(M(A))$ , but since it belongs to  $M_n(B_\Gamma)$  it is also invertible in this smaller  $C^*$ -algebra ([1, Chapter 1]). By the results of [30]  $G_\Gamma = (B_\Gamma, \Delta_\Gamma)$  is a compact quantum group.  $\square$

REMARK 2.8. Let  $G = (A, \Delta)$  be a quantum (semi)group and let  $\Gamma$  be an admissible representation of  $G$ . Let  $B_\Gamma$  be the  $C^*$ -algebra defined in Proposition 2.7. The inclusion map  $\chi_\Gamma$  of  $B_\Gamma$  into  $M(A)$  is a morphism of quantum (semi)groups (see Section 3). Therefore  $G_\Gamma$  is a quotient quantum group of  $G$ . This is fully analogous to the construction performed for bounded representations of locally compact groups in [27, §19 and Chapitre VII].

Let  $G = (A, \Delta)$  be a quantum (semi)group and let  $\mathfrak{H}$  be a finite dimensional Hilbert space. A representation  $\Gamma$  of  $G$  on  $\mathfrak{H}$  is called *unitary* if  $\Gamma$  is a unitary element of the  $C^*$ -algebra  $\text{End}(\mathfrak{H}) \otimes M(A)$ . If  $\Gamma$  is a unitary representation, then choosing an orthonormal basis in  $\mathfrak{H}$  and representing  $\Gamma$  as matrix as before we see that

$$(\mathfrak{t}^{kl})_{k,l=1,\dots,n}$$

is a unitary element of  $M_n(M(A))$ . Conversely, if a bounded representation  $\Gamma$  of  $G$  on some finite dimensional vector space  $\mathfrak{X}$  can, upon some choice of a basis in  $\mathfrak{X}$ , be represented by a unitary matrix, then declaring this basis to be orthonormal provides  $\mathfrak{X}$  with a Hilbert space structure such that  $\Gamma$  becomes unitary.

We shall say that a given representation  $\Gamma$  of  $G$  on a finite dimensional vector space  $\mathfrak{X}$  is *similar to a unitary representation* if there exists a Hilbert space structure on  $\mathfrak{X}$  such that  $\Gamma$  is a unitary element of  $\text{End}(\mathfrak{X}) \otimes M(A)$ .

COROLLARY 2.9. Let  $G$  be a quantum (semi)group and let  $\Gamma$  be an admissible representation of  $G$ . Then  $\Gamma$  is similar to a unitary representation.

*Proof.* First let us observe that from the proof of Proposition 2.7 we see that  $G_\Gamma = (B_\Gamma, \Delta_\Gamma)$  is in fact a compact *matrix* quantum group ([29], [30]). Therefore the results of [29] and [30] are directly applicable to our situation.

We know that  $\Gamma \in \text{End}(\mathfrak{X}) \otimes M(A)$  is a representation of  $G$ , but we can also view  $\Gamma$  as an element of  $\text{End}(\mathfrak{X}) \otimes B_\Gamma$ . In this way  $\Gamma$  becomes a bounded representation of  $G_\Gamma$ . Now by [29, Theorem 5.2]  $\Gamma$  is similar to a unitary representation.  $\square$

REMARK 2.10.



- (1) Let  $T$  be an admissible representation of a quantum (semi)group  $G = (A, \Delta)$  on a finite dimensional vector space  $\mathfrak{X}$ . Having chosen a basis in  $\mathfrak{X}$  we can represent  $T$  as an  $n \times n$  matrix of elements of  $M(A)$ . Corollary 2.9 says that there exists an invertible  $m \in M_n(\mathbb{C})$  such that  $(m \otimes I_A)^{-1}T(m \otimes I_A)$  is unitary in  $M_n(M(A))$ .
- (2) It is important to point out that for a representation of a quantum (semi)group being similar to a unitary representation is not equivalent to being admissible. An appropriate counterexample due to S.L. Woronowicz is given in [25, Section 4].

COROLLARY 2.11. *Let  $G$  be a quantum (semi)group and let  $T$  and  $S$  be two admissible representations of  $G$ . Then  $T \oplus S$  is admissible.*

*Proof.* We know that  $T \oplus S$  is admissible, so we can carry out the construction described in Proposition 2.7 for  $T \oplus S$ . The algebra  $B_{T \oplus S}$  coincides with the unital  $C^*$ -algebra generated by all matrix elements of  $T$  and  $S$  inside  $M(A)$ . If the carrier spaces of  $T$  and  $S$  are  $\mathfrak{X}$  and  $\mathfrak{Y}$ , then it follows that  $T$  and  $S$  are elements of  $\text{End}(\mathfrak{X}) \otimes B_{T \oplus S}$  and  $\text{End}(\mathfrak{Y}) \otimes B_{T \oplus S}$ , respectively. Therefore they are both bounded representations of the compact quantum group  $G_{T \oplus S}$ . Thus their tensor product is a representation of  $G_{T \oplus S}$ , which is admissible as pointed out in Remark 2.3 (2). In other words,

$$(T \oplus S)^\top$$

is an invertible element of  $\text{End}((\mathfrak{X} \otimes \mathfrak{Y})') \otimes B_{T \oplus S}$ . Therefore it is also invertible in  $\text{End}((\mathfrak{X} \otimes \mathfrak{Y})') \otimes M(A)$ , which means that  $T \oplus S$  is admissible.  $\square$

PROPOSITION 2.12. *The set of all matrix elements of all admissible representations of a quantum (semi)group  $G = (A, \Delta)$  is a unital  $*$ -subalgebra of  $M(A)$ .*

*Proof.* Since the operations of direct sum and tensor product do not lead out of the class of admissible representations (Proposition 2.4 and Corollary 2.11) and any trivial representation is admissible, we see that the set of matrix elements of all admissible representations of  $G$  is a unital subalgebra of  $M(A)$ . What we need to show now is that this set is  $*$ -invariant.

Let  $T$  be an admissible representation of  $G$ . By Corollary 2.9  $T$  can be treated as a unitary element of  $M_n(M(A))$ ,

$$T = (t^{kl})_{k,l=1,\dots,n} .$$

Let  $\bar{T}$  be the  $n \times n$  matrix whose  $(k, l)$ -entry is  $t^{kl*}$ . Then clearly

$$\Delta(t^{kl*}) = \sum_{p=1}^n t^{kp*} \otimes t^{pl*} .$$

Since  $T$  is admissible, the transpose matrix  $T^\top$  of  $T$  has an inverse  $X$  in the  $C^*$ -algebra  $M_n(M(A)) = M_n \otimes M(A)$ . Applying  $*$  to the equalities

$$T^\top X = XT^\top = I_n \otimes I_A$$

we see that  $X^*$  is the inverse of  $\bar{T}$ . Therefore  $\bar{T}$  is a bounded representation of  $G$ . It is also admissible because the inverse of  $\bar{T}^\top$  is  $T$  (by unitarity).

Thus we have shown that for any admissible representation  $T$  of  $G$  there exists another admissible representation  $\bar{T}$  whose matrix elements are adjoints of the matrix elements of  $T$ . Consequently the set of matrix elements of all admissible representations of  $G$  is  $*$ -invariant.  $\square$

**2.4. Almost periodic elements and quantum Bohr compactification.** In this subsection we shall give the definition of the quantum Bohr compactification of a quantum (semi)group. The first step will consist in defining the set of almost periodic elements for a quantum (semi)group.

**PROPOSITION 2.13.** *Let  $G = (A, \Delta)$  be a quantum (semi)group and let  $\mathbb{A}\mathbb{P}(G)$  be the closure in  $M(A)$  of the set of matrix elements of all admissible representations of  $G$ . Let  $\Delta_{\mathbb{A}\mathbb{P}(G)}$  be the restriction of  $\Delta$  to  $\mathbb{A}\mathbb{P}(G)$ . Then  $\Delta_{\mathbb{A}\mathbb{P}(G)}$  is an element of  $\text{Mor}(\mathbb{A}\mathbb{P}(G), \mathbb{A}\mathbb{P}(G) \otimes \mathbb{A}\mathbb{P}(G))$  and  $(\mathbb{A}\mathbb{P}(G), \Delta_{\mathbb{A}\mathbb{P}(G)})$  is a compact quantum group.*

*Proof.* From Proposition 2.12 we know that  $\mathbb{A}\mathbb{P}(G)$  is the closure of a unital  $*$ -subalgebra of  $M(A)$ , so  $\mathbb{A}\mathbb{P}(G)$  is a unital  $C^*$ -subalgebra of  $M(A)$ . Since  $\mathbb{A}\mathbb{P}(G)$  is generated by matrix elements of representations, formula (2.2) shows that  $\Delta$  maps  $\mathbb{A}\mathbb{P}(G)$  into  $\mathbb{A}\mathbb{P}(G) \otimes \mathbb{A}\mathbb{P}(G)$ . As  $\Delta$  extended to  $M(A)$  is a unital map, we see that  $\Delta_{\mathbb{A}\mathbb{P}(G)}$  is a morphism. It is clearly coassociative.

To see that  $(\mathbb{A}\mathbb{P}(G), \Delta_{\mathbb{A}\mathbb{P}(G)})$  is a compact quantum group notice that  $\mathbb{A}\mathbb{P}(G)$  is the closure of

$$(2.3) \quad \bigcup B_T,$$

where the sum is taken over all admissible representations of  $G$  (cf. Proposition 2.7). This follows from the fact that the operations of direct sum and tensor product do not lead out of the class of admissible representations. Take  $b, c$  from (2.3). Then there exists an admissible representation  $T$  such that  $b, c \in B_T$ . (The easiest choice of  $T$  is the direct sum of representations  $T_1$  and  $T_2$  such that  $b$  is a matrix element of  $T_1$  and  $c$  is a matrix element of  $T_2$ .) By Proposition 2.7 we know that  $(b \otimes c)$  is in the closure of the span of elements of the form  $\Delta_T(a_1)(I \otimes a_2)$  with  $a_1, a_2 \in B_T \subset \mathbb{A}\mathbb{P}(G)$ . It follows that

$$\{\Delta_{\mathbb{A}\mathbb{P}(G)}(a)(I \otimes b) : a, b \in \mathbb{A}\mathbb{P}(G)\}$$

is dense in  $\mathbb{A}\mathbb{P}(G) \otimes \mathbb{A}\mathbb{P}(G)$ . The other density condition is verified in the same way.  $\square$

The elements of  $\mathbb{A}\mathbb{P}(\mathbb{G})$  will be called *almost periodic* for  $\mathbb{G}$ . There is a clear analogy between the classical notion of an almost periodic function on a topological group ([11], [27]) and the notion of an almost periodic element for a quantum (semi)group. This is explained in Subsection 4.1.

We are now in position to give the definition of the main object of this paper.

DEFINITION 2.14. Let  $\mathbb{G} = (\mathbb{A}, \Delta)$  be a quantum (semi)group and let  $\mathbb{A}\mathbb{P}(\mathbb{G})$  be the algebra of almost periodic elements for  $\mathbb{G}$ . The compact quantum group  $(\mathbb{A}\mathbb{P}(\mathbb{G}), \Delta_{\mathbb{A}\mathbb{P}(\mathbb{G})})$  described in Proposition 2.13 will be called the *quantum Bohr compactification* of  $\mathbb{G}$ . We will denote it by the symbol  $\mathfrak{b}\mathbb{G}$ .

Let  $\chi_{\mathbb{G}}$  denote the inclusion of  $\mathbb{A}\mathbb{P}(\mathbb{G})$  into  $M(\mathbb{A})$ . Then  $\chi_{\mathbb{G}}$  is clearly an element of  $\text{Mor}(\mathbb{A}\mathbb{P}(\mathbb{G}), M(\mathbb{A}))$ . Moreover, we have

$$(\chi_{\mathbb{G}} \otimes \chi_{\mathbb{G}}) \circ \Delta_{\mathbb{A}\mathbb{P}(\mathbb{G})} = \Delta \circ \chi_{\mathbb{G}}.$$

As  $\mathfrak{b}\mathbb{G} = (\mathbb{A}\mathbb{P}(\mathbb{G}), \Delta_{\mathbb{A}\mathbb{P}(\mathbb{G})})$  is a compact quantum group, we have the canonical dense Hopf  $*$ -algebra inside  $\mathbb{A}\mathbb{P}(\mathbb{G})$ . Let us denote this Hopf  $*$ -algebra by  $\mathcal{A}\mathcal{P}(\mathbb{G})$ . It is easy to see that  $\mathcal{A}\mathcal{P}(\mathbb{G})$  is simply the set of matrix elements of admissible representations of  $\mathbb{G}$ . The elements of  $\mathcal{A}\mathcal{P}(\mathbb{G})$  are analogs of almost invariant functions on a topological group (see [11]).

COROLLARY 2.15. Let  $\mathbb{G} = (\mathbb{A}, \Delta)$  be a quantum (semi)group and let  $\mathcal{A}\mathcal{P}(\mathbb{G})$  be the set of matrix elements of admissible representations of  $\mathbb{G}$ . Denote by  $\Delta_{\mathcal{A}\mathcal{P}(\mathbb{G})}$  the restriction of  $\Delta$  to  $\mathcal{A}\mathcal{P}(\mathbb{G})$ . Then

- (1)  $\Delta_{\mathcal{A}\mathcal{P}(\mathbb{G})}(\mathcal{A}\mathcal{P}(\mathbb{G})) \subset \mathcal{A}\mathcal{P}(\mathbb{G}) \otimes_{\text{alg}} \mathcal{A}\mathcal{P}(\mathbb{G})$ ,
- (2)  $(\mathcal{A}\mathcal{P}(\mathbb{G}), \Delta_{\mathcal{A}\mathcal{P}(\mathbb{G})})$  is a Hopf  $*$ -algebra.

REMARK 2.16. It is possible to give a direct proof of the result contained in Corollary 2.15 without using Proposition 2.13. Then Proposition 2.13 is a consequence of this result.

### 3. Universality and functoriality

For  $k = 1, 2$  let  $\mathbb{G}_k = (\mathbb{A}_k, \Delta_k)$  be a quantum (semi)group. A morphism of quantum (semi)groups from  $\mathbb{G}_2$  to  $\mathbb{G}_1$  is an element  $\Phi \in \text{Mor}(\mathbb{A}_1, \mathbb{A}_2)$  such that

$$(\Phi \otimes \Phi) \circ \Delta_1 = \Delta_2 \circ \Phi.$$

In particular, let  $\mathbb{G} = (\mathbb{A}, \Delta)$  be a quantum (semi)group and let  $\mathfrak{b}\mathbb{G} = (\mathbb{A}\mathbb{P}(\mathbb{G}), \Delta_{\mathbb{A}\mathbb{P}(\mathbb{G})})$  be its quantum Bohr compactification. Then the morphism  $\chi_{\mathbb{G}}$  described after Definition 2.14 is a morphism of quantum (semi)groups from  $\mathbb{G}$  to  $\mathfrak{b}\mathbb{G}$ . The class of all quantum (semi)groups with morphisms of (semi)groups forms a category  $\mathfrak{Q}\mathfrak{G}$ . Let us also denote the full subcategory of compact quantum groups by  $\mathfrak{C}\mathfrak{Q}\mathfrak{G}$ .

**THEOREM 3.1.** *Let  $G = (A, \Delta)$  be a quantum (semi)group and let  $K = (B, \Delta_K)$  be a compact quantum group. If  $\Phi \in \text{Mor}(B, A)$  is a morphism of quantum (semi)groups from  $G$  to  $K$ , then there exists a unique morphism of quantum (semi)groups  $\mathfrak{b}\Phi \in \text{Mor}(B, \mathbb{A}\mathbb{P}(G))$  such that*

$$(3.1) \quad \Phi = \chi_G \circ \mathfrak{b}\Phi.$$

*Proof.* Let  $\mathfrak{X}$  be a finite dimensional vector space and let  $T \in \text{End}(\mathfrak{X}) \otimes B$  be a finite dimensional representation of  $K$ . Then  $T$  is admissible (cf. Remark 2.3, (2)) and  $(\text{id} \otimes \Phi)T \in \text{End}(\mathfrak{X}) \otimes M(A)$  is an admissible representation of  $G$ . Consequently its matrix elements belong to  $\mathbb{A}\mathbb{P}(G)$ . Since the set of matrix elements of finite dimensional representations of  $K$  is dense in  $B$ , we have  $\Phi(B) \subset \mathbb{A}\mathbb{P}(G)$ .

Let  $\mathfrak{b}\Phi$  be the map  $\Phi$  regarded as a  $*$ -homomorphism from  $B$  to  $\mathbb{A}\mathbb{P}(G)$ . Clearly  $\mathfrak{b}\Phi$  is a morphism of quantum (semi)groups and (3.1) is satisfied. The uniqueness of  $\mathfrak{b}\Phi$  follows from the injectivity of  $\chi_G$ .  $\square$

The universal property described in Theorem 3.1 implies the uniqueness of the quantum Bohr compactification: given a quantum (semi)group  $G$ , any compact quantum group with a morphism to  $A$  possessing the universal property of  $(\mathfrak{b}G, \chi_G)$  is isomorphic to  $\mathfrak{b}G$ .

Let  $G_1 = (A_1, \Delta_1)$  and  $G_2 = (A_2, \Delta_2)$  be quantum (semi)groups with quantum Bohr compactifications  $\mathfrak{b}G_1 = (\mathbb{A}\mathbb{P}(G_1), \Delta_{\mathbb{A}\mathbb{P}(G_1)})$  and  $\mathfrak{b}G_2 = (\mathbb{A}\mathbb{P}(G_2), \Delta_{\mathbb{A}\mathbb{P}(G_2)})$ . We also have the quantum (semi)group morphisms  $\chi_{G_1} \in \text{Mor}(\mathbb{A}\mathbb{P}(G_1), A_1)$  and  $\chi_{G_2} \in \text{Mor}(\mathbb{A}\mathbb{P}(G_2), A_2)$ .

Now let  $\Psi \in \text{Mor}(A_1, A_2)$  be a morphism of quantum (semi)groups. We will now define  $\mathfrak{b}\Psi \in \text{Mor}(\mathbb{A}\mathbb{P}(G_1), \mathbb{A}\mathbb{P}(G_2))$ , which will be a morphism of quantum (semi)groups from  $\mathfrak{b}G_2$  to  $\mathfrak{b}G_1$ . The map  $\Psi \circ \chi_1 \in \text{Mor}(\mathbb{A}\mathbb{P}(G_1), A_2)$  is a morphism of quantum (semi)groups, so by Theorem 3.1 there exists a unique  $\mathfrak{b}\Psi \in \text{Mor}(\mathbb{A}\mathbb{P}(G_1), \mathbb{A}\mathbb{P}(G_2))$  such that

$$\Psi \circ \chi_1 = \chi_2 \circ \mathfrak{b}\Psi.$$

In case  $G_1$  is a compact quantum group, the definition of  $\mathfrak{b}\Psi$  coincides with the one given in Theorem 3.1 (cf. the beginning of Subsection 4.3). We shall refer to the resulting morphism  $\mathfrak{b}\Psi$  of compact quantum groups as the *compactification* of the morphism  $\Psi$ . Thus we obtain the following theorem:

**THEOREM 3.2.** *The passage from a quantum (semi)group  $G$  to its quantum Bohr compactification and from a morphism  $\Psi$  to its compactification  $\mathfrak{b}\Psi$  is a covariant functor from the category  $\mathfrak{Q}\mathfrak{G}$  to its full subcategory  $\mathfrak{C}\mathfrak{Q}\mathfrak{G}$ .*

## 4. Examples and special cases

**4.1. Classical groups.** In this subsection we shall describe the quantum Bohr compactification of quantum (semi)groups of the form  $G = (C_0(G), \Delta_G)$ ,

where  $G$  is a locally compact group and  $\Delta_G$  is the morphism dualizing the group operation in  $G$ . Our general setup allows  $G$  to be merely a semigroup, but as mentioned in Section 1 our construction is aimed at quantum groups rather than semigroups. Therefore we shall not devote any attention to the subject of classical semigroups.

**PROPOSITION 4.1.** *Let  $G$  be a locally compact group. Identify  $G$  with the quantum group  $(C_0(G), \Delta_G)$ . Then  $\mathfrak{b}G$  is a classical group<sup>1</sup> isomorphic to the classical Bohr compactification of  $G$ .*

*Proof.* The algebra  $\mathbb{A}\mathbb{P}(G)$  is contained in the Abelian  $C^*$ -algebra  $C_b(G)$ . It follows that it is commutative. On the other hand, the universal property of  $\mathfrak{b}G$  implies the universal property of the classical Bohr compactification. Therefore  $\mathfrak{b}G$  is a compact group which, in particular, has the universal property of the Bohr compactification.  $\square$

**REMARK 4.2.** It is easily seen that for any quantum group  $G = (A, \Delta)$  the algebra  $\mathbb{A}\mathbb{P}(G)$  is contained in the  $C^*$ -subalgebra of  $M(A)$  consisting of those elements  $a \in M(A)$  for which  $\Delta(a) \in M(A) \otimes M(A)$  (the minimal  $C^*$ -tensor product). However, if  $A$  is commutative, i.e., when  $G$  is a classical locally compact group, we have

$$\mathbb{A}\mathbb{P}(G) = \{a \in M(A) : \Delta(a) \in M(A) \otimes M(A)\}.$$

It is not known if this equality holds for general quantum groups.

**4.2. Duals of classical groups.** An important class of examples of quantum groups are the duals of locally compact groups. They are quantum groups of the form  $G = (C^*(H), \Delta)$ , where  $H$  is a locally compact group and  $\Delta$  is defined uniquely by  $\Delta(U_x) = U_x \otimes U_x$ , where

$$H \ni x \mapsto U_x \in M(C^*(H))$$

is the universal representation of  $H$ . In particular,  $G$  is a *cocommutative* quantum group. The quantum Bohr compactification of the dual of a locally compact group can be easily described.

**PROPOSITION 4.3.** *Let  $H$  be a locally compact group and let  $G = (C^*(H), \Delta)$  be the universal dual of  $H$ . Then  $\mathbb{A}\mathbb{P}(G)$  is the  $C^*$ -algebra generated inside  $M(C^*(H))$  by the image of the universal representation. For any  $x \in H$  we have  $\Delta_{\mathbb{A}\mathbb{P}(G)}(U_x) = U_x \otimes U_x$ .*

*Proof.* First let us notice that any finite dimensional unitary representation of  $G$  is admissible. Indeed, any such representation corresponds to a finite dimensional representation of the algebra of functions on  $H$ . Every such

---

<sup>1</sup>i.e., a quantum group described by an Abelian  $C^*$ -algebra.

representation is, in turn, similar to a direct sum of one dimensional representations. It is clear that the transpose of a diagonal invertible matrix is invertible. In view of Corollary 2.9 we can conclude that  $\mathbb{A}\mathbb{P}(\mathbb{G})$  is generated by matrix elements of one dimensional unitary representations of  $\mathbb{G}$  or, in other words, elements  $U_x$  with  $x \in H$  of the image of the universal representation of  $H$ . The comultiplication on  $\mathbb{A}\mathbb{P}(\mathbb{G})$  is the restriction of  $\Delta$  on  $C^*(H)$  and hence we have the formula  $\Delta_{\mathbb{A}\mathbb{P}(\mathbb{G})}(U_x) = U_x \otimes U_x$ .  $\square$

If  $H$  is any group, then we can make it into a topological group by putting on it the discrete topology. We shall denote the group  $H$  with discrete topology by  $H_d$ . We shall use this notation in the statements of the remaining results of this subsection.

Recall that if  $\mathbb{G}$  is an Abelian topological group, then the Bohr compactification  $\mathfrak{b}\mathbb{G}$  of  $\mathbb{G}$  is naturally isomorphic to the dual of  $\widehat{\mathbb{G}}_d$  ([6]). The following corollary of Proposition 4.3 provides a generalization of this result in context of locally compact groups:

**COROLLARY 4.4.** *Let  $H$  be a locally compact group such that  $H_d$  is amenable and let  $\mathbb{G} = (C^*(H), \Delta)$  be the universal dual of  $H$ . Then  $\mathfrak{b}\mathbb{G}$  is the universal dual of  $H_d$ .*

*Proof.* By Proposition 4.3 the  $C^*$ -algebra  $\mathbb{A}\mathbb{P}(\mathbb{G})$  contains the Hopf  $*$ -algebra generated by elements  $\{U_x : x \in H\}$ . This Hopf  $*$ -algebra is precisely the canonical Hopf  $*$ -algebra dense in  $C^*(H_d)$ . It is known that if  $H_d$  is amenable, then the Hopf  $*$ -algebra generated by  $\{U_x : x \in H\}$  admits a unique quantum group completion (see, e.g., [3]). Therefore  $\mathbb{A}\mathbb{P}(\mathbb{G})$  must be the universal group  $C^*$ -algebra of  $H_d$  and consequently  $\mathfrak{b}\mathbb{G}$  is the universal dual of  $H_d$ .  $\square$

**4.3. Compact and discrete quantum groups.** If  $\mathbb{G} = (A, \Delta)$  is a compact quantum group, then  $\mathbb{A}\mathbb{P}(\mathbb{G}) = A$ . Therefore in this case  $\mathfrak{b}\mathbb{G} = \mathbb{G}$ . In particular, this shows that any compact quantum group can be obtained as the quantum Bohr compactification of a quantum (semi)group. Therefore quantum Bohr compactification can have all “quantum” features like a non tracial Haar measure or a non trivial scaling group.

The case of discrete quantum groups is far more interesting. There are several ways to define discrete quantum groups (see, e.g., [21]). We shall use the definition adopted in [13], according to which a discrete quantum group is a dual of a compact quantum group (see also [33, Theorem 2.5]). The quantum Bohr compactification for discrete quantum groups has been introduced in [16]. In this section we shall summarize the results of that paper.

In order to formulate some statements about the quantum Bohr compactification of discrete quantum groups we need to recall some terminology. A compact quantum group  $\mathbb{G}$  is of *Kac type* if its Haar measure is a trace. This

is equivalent to the fact that its dual  $\widehat{G}$  is a *unimodular* discrete quantum group, i.e., its left and right Haar measures coincide. Another equivalent condition is that the coinverse of  $G$  (or  $\widehat{G}$ ) is bounded and this is further equivalent to involutivity of the coinverse and, further still, to the coinverse being a  $*$ -antiautomorphism. The next theorem gives some information on the quantum Bohr compactification of a discrete quantum group.

**THEOREM 4.5** ([16]). *Let  $G = (A, \Delta)$  be a discrete quantum group. The algebra of almost periodic elements for  $G$  coincides with the closed linear span of matrix elements of finite dimensional unitary representations of  $G$ . The quantum Bohr compactification  $bG$  of  $G$  is a compact quantum group of Kac type.*

Before proving the above theorem let us comment on the special situation of discrete quantum groups in the context of the quantum Bohr compactification. Let  $G = (A, \Delta)$  be a discrete quantum group with universal dual  $\widehat{G} = (B, \Delta_B)$ . Then any unitary representation<sup>2</sup>  $T \in M(\mathcal{K}(\mathfrak{H}) \otimes A)$  of  $G$  on a Hilbert space  $\mathfrak{H}$  is obtained as

$$T = \sigma(\text{id} \otimes \pi_T)u^*,$$

where  $u$  is the universal bicharacter  $u \in M(A \otimes B)$  describing the duality between  $G$  and  $\widehat{G}$ ,  $\pi_T$  is a (uniquely determined) representation of  $B$  on  $\mathfrak{H}$  and  $\sigma$  is the flip  $A \otimes \mathcal{K}(\mathfrak{H}) \rightarrow \mathcal{K}(\mathfrak{H}) \otimes A$  (cf. [13, Section 3]). However, it is possible to find a smaller compact quantum group which carries all the information about *finite dimensional* unitary representations of  $G$ . This is the canonical Kac quotient of  $\widehat{G}$ . We have described it in detail in the Appendix.

Let  $G = (A, \Delta)$  be a discrete quantum group and let  $K = (B, \Delta_B)$  be its universal dual. The canonical Kac quotient  $K_{\text{KAC}}$  of  $K$  is a compact quantum group  $K_{\text{KAC}} = (B_{\text{KAC}}, \Delta_{\text{KAC}})$  of Kac type, together with a quantum (semi)group morphism  $\pi \in \text{Mor}(B, B_{\text{KAC}})$  which is a surjection. Moreover, any finite dimensional representation of  $B$  factors through  $\pi$ .

Now if  $T$  is a finite dimensional unitary representation of  $G$ , then it arises from a finite dimensional representation of  $B$  which factors through  $\pi: B \rightarrow B_{\text{KAC}}$ . Let  $G^{\text{KAC}} = (A^{\text{KAC}}, \Delta^{\text{KAC}})$  be the dual of  $K_{\text{KAC}}$ . It follows that  $A^{\text{KAC}}$  injects into  $A$  with a non degenerate homomorphism and any finite dimensional unitary representation of  $G$  is in fact a representation of  $G^{\text{KAC}}$ . In particular, all matrix elements of finite dimensional unitary representations of  $G$  are contained in  $M(A^{\text{KAC}}) \subset M(A)$  and are matrix elements of finite dimensional representations of  $G^{\text{KAC}}$ . The crucial fact here is that  $G^{\text{KAC}}$  is *unimodular* as it is the dual of a compact quantum group of Kac type. In particular, its coinverse is bounded.

---

<sup>2</sup>A general (not finite dimensional) unitary representation  $T$  of  $G$  is an element of  $M(\mathcal{K}(\mathfrak{H}) \otimes A)$  as opposed to  $\text{End}(\mathfrak{H}) \otimes M(A)$  for finite dimensional representations (cf. [33, Section 4]).

We have thus shown that any finite dimensional unitary representation of a discrete quantum group is, in fact, a representation of a “smaller” and (more importantly) unimodular discrete quantum group. A direct proof of the fact that any finite dimensional unitary representation of a unimodular discrete quantum group is admissible is lacking and we cannot yet conclude that any finite dimensional unitary representation of a discrete quantum group is admissible. We can, however, obtain the result that *the set* of matrix elements of finite dimensional unitary representations is equal to *the set* of matrix elements of admissible representations by proving first that the former set is  $*$ -invariant and that the  $C^*$ -algebra obtained as its closure, together with the restricted comultiplication, is a compact quantum group.

**PROPOSITION 4.6.** *Let  $G = (A, \Delta)$  be a discrete quantum group. The closure of the set of matrix elements of all finite dimensional unitary representations of  $G$  is a unital  $C^*$ -subalgebra of  $M(A)$  and with comultiplication inherited from  $M(A)$  it is a compact quantum group.*

*Proof.* Let  $G^{\text{KAC}} = (A^{\text{KAC}}, \Delta^{\text{KAC}})$  be the dual of the canonical Kac quotient of the universal dual of  $G$  and let  $\mathfrak{S}$  be the set of those  $x$  affiliated with  $A^{\text{KAC}}$  for which  $\Delta(x)$  is a finite sum of tensor products of elements affiliated with  $A^{\text{KAC}}$ . Let  $\mathfrak{S}_0$  be the set of elements of  $\mathfrak{S}$  which are in  $M(A^{\text{KAC}})$ . Then the set of matrix elements of finite dimensional unitary representations of  $G$  is contained in  $\mathfrak{S}_0$ . It was shown in [15] that  $\mathfrak{S}$  is in fact a Hopf  $*$ -algebra.<sup>3</sup> Now the canonical maps

$$\begin{aligned} \mathfrak{S} \otimes_{\text{alg}} \mathfrak{S} \ni (a \otimes b) &\longmapsto \Delta(a)(I \otimes b) \in \mathfrak{S} \otimes_{\text{alg}} \mathfrak{S}, \\ \mathfrak{S} \otimes_{\text{alg}} \mathfrak{S} \ni (a \otimes b) &\longmapsto (a \otimes I)\Delta(b) \in \mathfrak{S} \otimes_{\text{alg}} \mathfrak{S} \end{aligned}$$

preserve  $\mathfrak{S}_0 \otimes_{\text{alg}} \mathfrak{S}_0$  because the coinverse (which gives the inverses of the canonical maps) is *bounded*. Therefore  $\mathfrak{S}_0$  is a Hopf  $*$ -algebra and its closure, with comultiplication inherited from  $M(A^{\text{KAC}}) \subset M(A)$ , is a compact quantum group.

Any finite dimensional representation of this compact quantum group is a finite dimensional representation of  $G$  and vice versa. It follows from the theory of compact quantum groups that the set of matrix elements of finite dimensional unitary representations of  $G$  is dense in the closure of  $\mathfrak{S}_0$ . In particular, its closure is a  $C^*$ -algebra.  $\square$

Now let  $G = (A, \Delta)$  be a discrete quantum group. It is easy to see that the compact quantum group described in Proposition 4.6 has the universal property of the quantum Bohr compactification. It also follows from the proof of Proposition 4.6 that its coinverse is bounded and therefore it is of Kac type. This concludes the proof of Theorem 4.5.

<sup>3</sup>If  $G = (A, \Delta)$  is a discrete quantum group, then  $A$  is a direct sum matrix algebras. Therefore the set of elements affiliated with  $A$  forms a  $*$ -algebra.



The construction of the canonical Kac quotient is also helpful in determining the quantum Bohr compactification of some discrete quantum groups. For example, we have (cf. [28] and [16, Section 7]):

PROPOSITION 4.7. *Choose a real parameter  $q$  with  $0 < |q| < 1$  and let  $G$  be the dual of the compact quantum group  $S_qU(2)$ . Then  $\widehat{\mathfrak{b}G}$  is isomorphic to the commutative compact group obtained as the classical Bohr compactification of the group of integers.*

It would be interesting to have an answer to the following question:

QUESTION 1. *Which compact quantum groups of Kac type can be quantum Bohr compactifications of discrete quantum groups?*

We conjecture that a quantum Bohr compactification of a discrete quantum group must be a universal compact quantum group in the sense of [8].

**4.4. Profinite quantum groups.** In this subsection we want to present a new family of compact quantum groups which has a nice universal property. The name “profinite quantum groups” has been suggested to the author by Shuzhou Wang.

In [23] A. Van Daele and S. Wang introduced a family of compact matrix quantum groups called universal compact quantum groups. The universal property of this family is the following: any compact matrix quantum group is a subgroup of one of the universal ones.<sup>4</sup>

The universal compact quantum groups are “parameterized” by non singular complex matrices, but this correspondence is many-to-one (cf. [2], [26]). Let  $Q$  be such a matrix. The corresponding universal compact quantum group will be denoted by  $G_Q$ . The  $C^*$ -algebra describing the quantum space of  $G_Q$  is generated by elements  $\{v^{kl} : k, l = 1, \dots, m\}$  (where  $m$  is the size of the matrix  $Q$ ) such that the matrix  $v$  with entries  $v^{kl}$  satisfies

$$\begin{aligned}
 vv^* &= I_m = v^*v, \\
 v^T Q \bar{v} Q^{-1} &= I_m = Q \bar{v} Q^{-1} v^T.
 \end{aligned}$$

In the paper [23] this quantum group was denoted by  $A_u(Q)$ . The subscript “ $u$ ” relates to the unitarity of the matrix  $v$  (i.e., the first condition above).

For any  $Q \in GL_m(\mathbb{C})$  the dual  $\widehat{G_Q}$  is a discrete quantum group. The family of profinite quantum groups we wish to describe is the family of quantum Bohr compactifications

$$(4.1) \quad \{\widehat{\mathfrak{b}G_Q}\}_{Q \in GL_m(\mathbb{C}), m \in \mathbb{N}}$$

---

<sup>4</sup>In [25] S. Wang described a family of compact matrix quantum groups of Kac type with the corresponding universal property for compact matrix quantum groups of Kac type.

It is clear that (4.1) is a family of compact quantum groups. The universal property of this family is the following:

**THEOREM 4.8.** *Let  $F = (A, \Delta)$  be a finite quantum group. Then there exists an element  $G_F = (B, \Delta_B)$  of the family (4.1) such that  $F$  is a quotient of  $G_F$ , i.e., there is an injective morphism from  $A$  to  $B$  preserving the comultiplications.*

*Proof.* If  $F$  is a finite quantum group, then so is its dual  $\widehat{F}$ . In particular,  $\widehat{F}$  is a compact matrix quantum group. Therefore it is a subgroup of some  $G_Q$  for some matrix  $Q \in GL_m(\mathbb{C})$ . By duality  $F$  is a quotient of  $\widehat{G_Q}$ . If we write  $\widehat{G_Q} = (C, \Delta_C)$ , then this situation is described by an injective morphism  $\lambda_F \in \text{Mor}(A, C)$  which preserves the comultiplication. But  $F$  is also a compact quantum group, so by the universal property of the Bohr compactification we obtain an injective morphism  $\mathfrak{b}\lambda_F$  from  $A$  to  $\mathbb{A}\mathbb{P}(\widehat{G_Q})$  which preserves comultiplication. This means that  $F$  is a quotient of  $G_F = \mathfrak{b}\widehat{G_Q}$ .  $\square$

Let us remark that since all finite quantum groups are of Kac type (see, e.g., [29, Appendix 2]), in the above proof we could have taken  $Q$  to be the identity matrix. For other non singular matrices  $Q$  the profinite quantum groups  $\mathfrak{b}\widehat{G_Q}$  can be more complicated than for the identity matrix, but the isomorphism class of  $\mathfrak{b}\widehat{G_Q}$  depends only on a part of the information carried by the matrix  $Q$ . This is described in the next theorem, which relies on the fact that given  $Q \in GL_m(\mathbb{C})$  the canonical Kac quotient of  $G_Q$  is the free product compact quantum group  $G_{I_{n_1}} * \cdots * G_{I_{n_k}}$  (cf. [25]), where  $n_1, \dots, n_k$  are the multiplicities of the singular values of  $Q$ . The precise formulation is the following:

**THEOREM 4.9** ([16, Section 7]). *Let  $Q \in GL_m(\mathbb{C})$  and let  $n_1, \dots, n_k$  be the multiplicities of different singular values of  $Q$  (eigenvalues of  $|Q|$ ). Then  $\mathfrak{b}\widehat{G_Q}$  is the Bohr compactification of the dual of the free product compact quantum group*

$$G_{I_{n_1}} * \cdots * G_{I_{n_k}}.$$

*In particular, if all singular values of  $Q$  are different, then  $\mathfrak{b}\widehat{G_Q}$  is isomorphic to the Bohr compactification of the free group on  $\mathfrak{m}$  generators, where  $\mathfrak{m}$  is the size of  $Q$ . Moreover, the  $C^*$ -algebras describing the profinite quantum groups are non separable.*

The last statement of the above theorem shows that the profinite quantum groups are definitely new examples of compact quantum groups. Let us remark that in the original definitions ([29], [33]) a compact quantum group was assumed to be described by a separable  $C^*$ -algebra. It was A. Van Daele who

in [20] (see also [12]) satisfactorily extended the definition to include cases with non separable  $C^*$ -algebras.

A more precise determination of the  $C^*$ -algebras  $\mathbb{A}\mathbb{P}(\widehat{G_Q})$  could be carried out if we knew that the  $C^*$ -algebras  $A_u(\mathfrak{m})$  defined in [25, Section 4] were residually finite dimensional (cf. Subsection 4.5). As likely as this seems, we have not found a proof of this hypothesis.

**4.5. MAP quantum groups.** A topological group  $G$  is said to be a *maximally almost periodic group*, or an *MAP group*, if the set of almost periodic functions on  $G$  separates points of  $G$ . This is equivalent to saying that the canonical homomorphism from  $G$  to its Bohr compactification is injective. It is easy to give examples of groups which are not maximally almost periodic (see, e.g., [24]), as well as examples of MAP groups. Clearly a topological group  $G$  is maximally almost periodic if and only if  $G$  is injectable into a compact group ([27, §32]).

In what follows we shall call a quantum (semi)group  $G = (A, \Delta)$  *maximally almost periodic* if the range of the canonical map  $\chi_G$  from  $\mathbb{A}\mathbb{P}(G)$  to  $M(A)$  is strictly dense in  $M(A)$ . This condition clearly corresponds to the injectivity of the homomorphism from a topological group into its Bohr compactification.

We shall now give a proposition which ties the concept of maximal almost periodicity to some other properties for the case of a discrete quantum group. It is known that a discrete group  $\Gamma$  is maximally almost periodic if the  $C^*$ -algebra  $C^*(\Gamma)$  is residually finite dimensional ([5, Remark 4.2(iii)]). It turns out that we can generalize this result to discrete quantum groups.

PROPOSITION 4.10.

- (1) *Let  $G = (A, \Delta)$  be a discrete quantum group such that its universal dual  $\widehat{G} = (B, \Delta_B)$  has the property that  $B$  is residually finite dimensional. Then  $G$  is unimodular and maximally almost periodic.*
- (2) *Any maximally almost periodic discrete quantum group is unimodular.*

*Proof.* (1) By Corollary A.3  $G$  is unimodular. The algebra  $\mathbb{A}\mathbb{P}(G)$  is generated by matrix elements of finite dimensional unitary representations of  $G$ . Every such representation is of the form  $\sigma(\text{id} \otimes \pi)u^*$ , where  $u$  is the universal bicharacter for the duality of  $G$  and  $\widehat{G}$  and  $\pi$  is a finite dimensional representation of  $B$  and  $\sigma$  is the tensor product flip (cf. [13, Section 3] and Subsection 4.3). Let  $\varphi$  be a non zero continuous functional on  $A$ . Then there exists an  $x \in \mathbb{A}\mathbb{P}(G)$  such that  $\varphi(x)$  is non zero (we use here the canonical extension of continuous functionals on  $A$  to  $M(A)$ ). Indeed, if  $\varphi(x) = 0$  for all  $x$  of the form  $x = (\text{id} \otimes (\psi \circ \pi))u^*$  with  $\pi$  a finite dimensional representation of  $B$  and  $\psi$  a functional on the image of  $\pi$ , then the element  $(\varphi \otimes \text{id})u^*$  is in the intersection of kernels of all finite dimensional representations of  $B$ . It follows that  $(\varphi \otimes \text{id})u^* = 0$  and consequently  $\varphi = 0$ . Since continuous functionals on

$\mathcal{A}$  are precisely the strictly continuous functionals on  $M(\mathcal{A})$  and  $M(\mathcal{A})$  is the strict completion of  $\mathcal{A}$ , it follows that  $\mathbb{A}\mathbb{P}(\mathbb{G})$  is strictly dense in  $M(\mathcal{A})$ .

(2) Let  $\mathbb{G} = (\mathcal{A}, \Delta)$  be an MAP discrete quantum group. Then the range of  $\chi_{\mathbb{G}}$  is strictly dense in  $M(\mathcal{A})$ . The coinverse of  $M(\mathcal{A})$  extends that of  $\mathfrak{b}\mathbb{G}$  (cf. Subsection 4.6). Since  $\mathfrak{b}\mathbb{G}$  is of Kac type, its coinverse is bounded. Now the coinverse is strictly closable and its canonical extension to  $M(\mathcal{A})$  coincides with its strict closure. The closure of a map which is bounded on a dense set is bounded. In particular, the coinverse of  $\mathbb{G}$  is bounded and therefore  $\mathbb{G}$  must be unimodular.  $\square$

The second part of Proposition 4.10 sheds some light on the extent to which the quantum Bohr compactification remembers the original quantum group. The canonical map from a discrete quantum group to its Bohr compactification cannot be injective for non unimodular discrete quantum groups.

**4.6. Quantum groups arising from manageable multiplicative unitaries.** Let us now describe some of the features of the quantum Bohr compactifications of quantum groups arising from manageable multiplicative unitaries ([34], see also [18]). All reduced  $C^*$ -algebraic quantum groups fall within that class ([9]). Let us recall that if  $\mathcal{H}$  is a Hilbert space and  $W$  is a manageable multiplicative unitary, then the quantum group arising from  $W$  is  $\mathbb{G} = (\mathcal{A}, \Delta)$ , where  $\mathcal{A}$  is a non degenerate  $C^*$ -subalgebra of  $B(\mathcal{H})$  defined as the closure of

$$(4.2) \quad \{(\omega \otimes \text{id})W : \omega \in B(\mathcal{H})_*\}$$

and  $\Delta$  is introduced by

$$\Delta(a) = W(a \otimes I)W^*.$$

There is a lot of extra structure that can be obtained from  $W$ . A thorough account of this extra structure can be found in the fundamental paper [34]. We shall only recall some elements of the beautiful theory of manageable multiplicative unitaries. The coinverse  $\kappa$  is a closed linear operator  $\mathcal{A} \rightarrow \mathcal{A}$  such that (4.2) is a core for  $\kappa$  and

$$\kappa((\omega \otimes \text{id})W) = (\omega \otimes \text{id})W^*.$$

The coinverse has a polar decomposition

$$(4.3) \quad \kappa = R \circ \tau_{i/2},$$

where  $R$  is a  $*$ -antiautomorphism of  $\mathcal{A}$  and  $\tau_{i/2}$  is the analytic generator of a one parameter group of automorphisms of  $\mathcal{A}$ . The  $*$ -antiautomorphism  $R$  is called the *unitary coinverse*. The mappings  $\kappa$ ,  $R$  and the automorphisms  $\tau_t$  have canonical extensions to  $M(\mathcal{A})$ . We shall denote them by  $\tilde{\kappa}$ ,  $\tilde{R}$  and  $(\tilde{\tau}_t)_{t \in \mathbb{R}}$ .

In this subsection we shall relate this extra structure of  $\mathbb{G}$  to the corresponding structure of the quantum Bohr compactification  $\mathfrak{b}\mathbb{G}$ . Let us briefly

comment on this. Let  $K = (B, \Delta_B)$  be a compact quantum group and let  $\mathcal{B}$  be the canonical Hopf  $*$ -algebra sitting inside  $B$ . Using the results of [33] or [7] one can show that there is a scaling group  $(\tau_t)_{t \in \mathbb{R}}$  and a unitary coinverse  $R$  in  $\mathcal{B}$  such that (4.3) holds on this subalgebra of  $B$ . We have:

PROPOSITION 4.11. *Let  $G = (A, \Delta)$  be a quantum group arising from a manageable multiplicative unitary and let  $\kappa, R$  and  $(\tau_t)_{t \in \mathbb{R}}$  be the coinverse, unitary coinverse and the scaling group of  $G$ . Let  $\mathfrak{b}G = (\mathbb{A}\mathcal{P}(G), \Delta_{\mathbb{A}\mathcal{P}(G)})$  be its quantum Bohr compactification. Denote by  $\mathfrak{b}\kappa, \mathfrak{b}R$  and  $(\mathfrak{b}\tau_t)_{t \in \mathbb{R}}$  the corresponding objects for  $\mathfrak{b}G$ . Then  $\mathfrak{b}\kappa, \mathfrak{b}R$  and  $\mathfrak{b}\tau_t$  are the restrictions of  $\tilde{\kappa}, \tilde{R}$  and  $\tilde{\tau}_t$  to  $\mathcal{AP}(G) \subset \mathbb{A}\mathcal{P}(G) \subset M(A)$ .*

*Proof.* For any admissible representation  $T$  of  $G$  and any  $t \in \mathbb{R}$  the element  $(\text{id} \otimes \tilde{\tau}_t)T$  is an admissible representation of  $G$ . It follows that the group  $(\tilde{\tau}_t)_{t \in \mathbb{R}}$  preserves  $\mathcal{AP}(G)$ .

Moreover, if  $x \in \mathcal{AP}(G)$ , then  $x$  is a matrix element of a unitary representation  $U$ :

$$x = (\varphi \otimes \text{id})U$$

(by Corollary 2.9). By [34, Theorem 1.6 4.]  $x$  is in the domain of  $\tilde{\kappa}$  and applying  $*$  to  $\tilde{\kappa}(x)$  yields

$$(4.4) \quad \tilde{\kappa}(x)^* = (\bar{\varphi} \otimes \text{id})U \in \mathcal{AP}(G).$$

Since  $\mathcal{AP}(G)$  is  $*$ -invariant, we have that  $\tilde{\kappa}(x) \in \mathcal{AP}(G)$ . In particular, the mapping  $\tilde{\kappa}$  preserves the subset  $\mathcal{AP}(G)$  of  $M(A)$ . By the decomposition  $\tilde{\kappa} = \tilde{R} \circ \tilde{\tau}_{i/2}$  we see that  $R(\mathcal{AP}(G)) = \mathcal{AP}(G)$ .

It remains to show that the restrictions of  $\tilde{\kappa}, \tilde{R}$  and  $\tilde{\tau}_t$  to  $\mathcal{AP}(G)$  coincide with the maps  $\mathfrak{b}\kappa, \mathfrak{b}R$  and  $\mathfrak{b}\tau_t$ . Let us begin with the observation that the counit  $e$  of  $(\mathcal{AP}(G), \Delta_{\mathcal{AP}(G)})$  must satisfy

$$e(t^{kl}) = \delta_{kl}$$

for any matrix entry  $t^{kl}$  of an admissible representation  $T$  viewed as a matrix  $T = (t^{kl})_{k,l=1,\dots,n}$ . Now  $T$  can be chosen unitary and then  $\tilde{\kappa}(t^{kl}) = t^{lk*}$  (cf. (4.4)). It follows that

$$m(\tilde{\kappa} \otimes \text{id})\Delta_{\mathcal{AP}(G)}(t^{kl}) = m(\text{id} \otimes \tilde{\kappa})\Delta_{\mathcal{AP}(G)}(t^{kl}) = e(t^{kl}) I_{\mathcal{AP}(G)}.$$

Therefore  $\tilde{\kappa}$  must coincide with  $\mathfrak{b}\kappa$  on  $\mathcal{AP}(G)$ . Now the uniqueness of the polar decomposition of  $\mathfrak{b}\kappa$  shows that the maps  $\tilde{R}$  and  $\tilde{\tau}_t$  coincide with  $\mathfrak{b}R$  and  $\mathfrak{b}\tau_t$  when restricted to  $\mathcal{AP}(G)$ .  $\square$

Let  $G = (A, \Delta)$  be a quantum group arising from a manageable multiplicative unitary  $W$ . It is a tempting prospect to define  $\mathfrak{b}G$  directly in terms of  $W$  without passing through the construction of  $(A, \Delta)$  first. This raises the following question:

QUESTION 2. *Let  $G$  be a quantum group arising from a manageable multiplicative unitary. Is  $\mathfrak{b}G$  a reduced compact quantum group?*

If the answer to Question 2 were positive, then one could hope to establish a procedure of constructing the multiplicative unitary for  $\mathfrak{b}G$  directly from the one for  $G$ . The examples of discrete and compact quantum groups show that this procedure would be very interesting. In some cases it would remove a lot of information, while in others it would fully preserve the multiplicative unitary.

Finally, let us give an example when we can determine the quantum Bohr compactification of a non trivial quantum group arising from a manageable multiplicative unitary.

PROPOSITION 4.12. *Let  $G$  be one of the quantum “ $\mathfrak{az} + \mathfrak{b}$ ” groups defined in [35] and [17] and let  $q$  be its deformation parameter. Then:*

- (1) *If  $q$  is real, then  $\mathfrak{b}G$  is isomorphic to  $\mathfrak{b}\mathbb{Z} \times \mathbb{T}$ .*
- (2) *If  $q$  is not real, then  $\mathfrak{b}G$  is isomorphic to  $\mathfrak{b}\mathbb{R} \times \mathbb{Z}_N$ .*

*Proof.* The quantum “ $\mathfrak{az} + \mathfrak{b}$ ” groups are defined by modular multiplicative unitaries ([18]), which means that they are reduced. Let us observe that they are at the same time universal. Let  $G = (A, \Delta)$  be one of them. The counit of  $G$  is continuous and so by [4, Theorem 3.1]  $G$  is universal. It is known that the reduced dual of  $G$  is anti isomorphic to  $G$ , which makes it universal as well. Therefore any unitary representation of  $G$  is obtained from a representation of  $A$  (for the case of real  $q$  this was proved in a more elementary way in [14]). In [35] and [17] the algebra  $A$  is identified as the crossed product  $C_0(\bar{\Gamma}) \rtimes \Gamma$ , where  $\Gamma$  is a multiplicative subgroup of  $\mathbb{C} \setminus \{0\}$  depending on the deformation parameter  $q$  and  $\bar{\Gamma}$  is the closure of  $\Gamma$  in  $\mathbb{C}$ . Thus the representations of  $A$  are fully classified.

Since admissible representations of  $G$  are similar to unitary ones, in order to find all admissible representations of  $G$ , we can concentrate on finite dimensional representations of  $A$ . They correspond to finite dimensional representations of  $\Gamma$  and are therefore admissible. In this way we found all admissible representations of  $G$ . It is easily seen that  $\mathfrak{b}G$  is isomorphic to  $\mathfrak{b}\Gamma$  and our conclusion follows from the construction of  $\Gamma$  from the deformation parameter  $q$  (cf. [35], [17]).  $\square$

## Appendix A. The canonical Kac quotient of a compact quantum group

In the appendix we shall describe an object which helps very much in dealing with representations of discrete quantum groups. This object is the

canonical Kac quotient of a compact quantum group. This notion is due to Stefan Vaes ([19]).

Let  $K = (B, \Delta_B)$  be a compact quantum group. Let  $J$  be the (closed two sided) ideal of  $B$  defined as the intersection of left kernels of all tracial states on  $B$ :

$$(A.1) \quad J = \{b \in B : \tau(b^*b) = 0 \text{ for any tracial state } \tau \text{ on } B\}$$

(in case there are no tracial states we set  $J = B$ ). Let  $B_{KAC}$  be the quotient  $B/J$  and let  $\pi$  be the corresponding quotient map.

PROPOSITION A.1. *Let  $K = (B, \Delta_B)$  be a compact quantum group and let  $B_{KAC}$  be the quotient of  $B$  by the ideal (A.1) with  $\pi$  the quotient map. Then:*

(1) *The equation*

$$(A.2) \quad \Delta_{KAC}(\pi(b)) = (\pi \otimes \pi)\Delta_B(b)$$

*defines a comultiplication on  $B_{KAC}$ ; with this comultiplication  $K_{KAC} = (B_{KAC}, \Delta_{KAC})$  becomes a compact quantum group.*

(2)  *$K_{KAC}$  is a compact quantum group of Kac type.*

*Proof.* First let us remark that  $B_{KAC}$  is evidently isomorphic to the image of  $B$  in the representation, which is the direct sum of GNS representations of  $B$  for all tracial states. In particular,  $B_{KAC}$  has a faithful family of tracial states (and if it is separable, then it possesses a faithful tracial state). Moreover,  $\ker(\pi \otimes \pi)$  consists of all  $x \in B \otimes B$  such that  $(\tau_1 \otimes \tau_2)(x^*x) = 0$  for any tracial states  $\tau_1, \tau_2$  of  $B$ .

(1) Let  $b \in B$  be such that  $\pi(b) = 0$ . Then for any two tracial states  $\tau_1, \tau_2$  on  $B$  we have

$$(\tau_1 \otimes \tau_2)(\Delta_B(b)^*\Delta_B(b)) = (\tau_1 * \tau_2)(b^*b),$$

which is equal to 0, since a convolution of two traces is a trace (and  $b \in J$ ). Therefore (A.2) defines a map  $B_{KAC} \rightarrow B_{KAC} \otimes B_{KAC}$ . It is now straightforward to check that  $\Delta_{KAC}$  is coassociative. Moreover, since for any  $a, b \in B$

$$\begin{aligned} (\pi \otimes \pi)(\Delta_B(a)(I \otimes b)) &= \Delta_{KAC}(\pi(a))(I \otimes \pi(b)), \\ (\pi \otimes \pi)((a \otimes I)\Delta_B(b)) &= (I \otimes \pi(a))\Delta_{KAC}(\pi(b)) \end{aligned}$$

and  $\pi \otimes \pi$  is surjective, we see that  $K_{KAC} = (B_{KAC}, \Delta_{KAC})$  is a compact quantum group.

(2) We shall show that the Haar measure of  $K_{KAC}$  is a trace and the conclusion will follow from the remarks preceding the statement of Theorem 4.5. To that end we shall repeat the procedure of constructing the Haar measure described in [12, Section 4] and use a slight modification of the argument in [33, Lemma 3.1] (for separable  $B_{KAC}$  it is enough to inspect the proof of [33, Theorem 2.3] or [29, Theorem 4.2]).

We shall use the following generalization of [33, Lemma 3.1]:

Let  $G = (D, \Delta_D)$  be a compact quantum group and let  $(\rho_\iota)_{\iota \in \mathbb{J}}$  be a faithful family of states of  $D$ . Let  $\mathfrak{h}$  be a state of  $D$  such that

$$\mathfrak{h} * \rho_\iota = \rho_\iota * \mathfrak{h} = \mathfrak{h}$$

for all  $\iota \in \mathbb{J}$ . Then  $\mathfrak{h}$  is the Haar measure of  $G$ .

The proof of the above fact is the same as [33, Lemma 3.1]. In the original formulation the family  $(\rho_\iota)_{\iota \in \mathbb{J}}$  consisted of a single element.

Now we can repeat the argument of A. Maes and A. Van Daele from [12]. For any tracial state  $\tau$  of  $B_{\text{KAC}}$  there exists a tracial state  $\mathfrak{h}$  such that  $\mathfrak{h} * \tau = \tau * \mathfrak{h} = \mathfrak{h}$ . This is [12, Lemma 4.2] combined with the fact that a convolution of traces is a trace.

Let  $\tau$  be any tracial state of  $B_{\text{KAC}}$  and let  $\mathfrak{h}$  be a tracial state on  $B_{\text{KAC}}$  such that  $\mathfrak{h} * \tau = \tau * \mathfrak{h} = \mathfrak{h}$ . Then if  $\omega$  is a positive functional on  $B_{\text{KAC}}$  such that  $\omega \leq \tau$ , then by [12, Lemma 4.3] we have that  $\mathfrak{h} * \omega = \omega * \mathfrak{h} = \omega(I)\mathfrak{h}$ .

For any tracial positive functional  $\rho$  we set

$$C_\rho = \{\mathfrak{h} : \mathfrak{h} \text{ is a tracial state of } B_{\text{KAC}} \text{ such that } \mathfrak{h} * \rho = \rho * \mathfrak{h} = \rho(I)\mathfrak{h}\}.$$

Then  $C_\rho$  is non empty and weakly compact. As in [12] we have  $C_{\rho_1 + \rho_2} \subset C_{\rho_1} \cap C_{\rho_2}$  and the family of all these sets has non empty intersection. Let  $\mathfrak{h}_{\text{KAC}}$  be an element of this intersection. Then  $\mathfrak{h}_{\text{KAC}}$  satisfies

$$\mathfrak{h}_{\text{KAC}} * \tau = \tau * \mathfrak{h}_{\text{KAC}} = \mathfrak{h}_{\text{KAC}}$$

for all tracial states  $\tau$ . Since  $B_{\text{KAC}}$  has a faithful family of tracial states, it follows that  $\mathfrak{h}_{\text{KAC}}$  is the Haar measure of  $K_{\text{KAC}} = (B_{\text{KAC}}, \Delta_{\text{KAC}})$ . By construction  $\mathfrak{h}_{\text{KAC}}$  is a trace.  $\square$

The compact quantum group  $K_{\text{KAC}}$  constructed in Proposition A.1 is called the *canonical Kac quotient* of the compact quantum group  $K$ . This terminology is not fully consistent with our approach to quantum groups because  $(B_{\text{KAC}}, \Delta_{\text{KAC}})$  corresponds to a *quantum subgroup* of  $K$ . Nevertheless we have decided to use it because the feature of  $K_{\text{KAC}}$  which is important for our purposes is related to properties of  $B_{\text{KAC}}$  as a  $C^*$ -algebra. This feature is the easy fact that any finite dimensional representation of the  $C^*$ -algebra  $B$  factors through the map  $\pi: B \rightarrow B_{\text{KAC}}$ .<sup>5</sup> Notice, however, that the map  $\pi \in \text{Mor}(B, B_{\text{KAC}})$  is a quantum (semi)group morphism.

REMARK A.2. In the proof of statement (2) of Proposition A.1 we have shown that if a compact quantum group  $K = (B, \Delta_B)$  has the property that  $B$  possesses a faithful family of tracial states, then  $K$  is its own canonical Kac quotient.

<sup>5</sup>Of course this is even true for any representation generating a finite von Neumann algebra.



A particular example of the situation described in Remark A.2 is the following: recall that a  $C^*$ -algebra  $B$  is called *residually finite dimensional* if it possesses a separating family of finite dimensional representations. Now let  $K = (B, \Delta_B)$  be a compact quantum group and let  $\pi$  be the quotient map from  $B$  onto  $B_{\text{KAC}}$ . If  $B$  is residually finite dimensional, then it possesses a faithful family of tracial states and consequently  $\pi$  is an isomorphism. As a corollary we get:

**COROLLARY A.3.** *Let  $K = (B, \Delta_B)$  be a compact quantum group with  $B$  a residually finite dimensional  $C^*$ -algebra. Then  $K$  is of Kac type.*

## REFERENCES

- [1] W. Arveson, *An invitation to  $C^*$ -algebras*, Springer-Verlag, New York, 1976. MR 0512360 (58 #23621)
- [2] T. Banica, *Le groupe quantique compact libre  $U(\mathfrak{n})$* , Comm. Math. Phys. **190** (1997), 143–172. MR 1484551 (99k:46095)
- [3] E. Bédos, G. J. Murphy, and L. Tuset, *Co-amenability of compact quantum groups*, J. Geom. Phys. **40** (2001), 130–153. MR 1862084 (2002m:46100)
- [4] E. Bédos and L. Tuset, *Amenability and co-amenability for locally compact quantum groups*, Internat. J. Math. **14** (2003), 865–884. MR 2013149 (2004k:46129)
- [5] M. B. Bekka and N. Louvet, *Some properties of  $C^*$ -algebras associated to discrete linear groups*,  $C^*$ -algebras (Münster, 1999), Springer, Berlin, 2000, pp. 1–22. MR 1796907 (2001m:46111)
- [6] P. Holm, *On the Bohr compactification*, Math. Ann. **156** (1964), 34–46. MR 0181700 (31 #5927)
- [7] J. Kustermans, *The analytic structure of algebraic quantum groups*, preprint, [arXiv:math.FA/9707010](https://arxiv.org/abs/math.FA/9707010).
- [8] ———, *Locally compact quantum groups in the universal setting*, Internat. J. Math. **12** (2001), 289–338. MR 1841517 (2002m:46108)
- [9] J. Kustermans and S. Vaes, *Locally compact quantum groups*, Ann. Sci. École Norm. Sup. (4) **33** (2000), 837–934. MR 1832993 (2002f:46108)
- [10] J. Kustermans and A. Van Daele,  *$C^*$ -algebraic quantum groups arising from algebraic quantum groups*, Internat. J. Math. **8** (1997), 1067–1139. MR 1484867 (99a:46130)
- [11] L.H. Loomis, *An introduction to abstract harmonic analysis*, Van Nostrand, Toronto-New York-London, 1953. MR 0054173 (14,883c)
- [12] A. Maes and A. Van Daele, *Notes on compact quantum groups*, Nieuw Arch. Wisk. (4) **16** (1998), 73–112. MR 1645264 (99g:46105)
- [13] P. Podleś and S. L. Woronowicz, *Quantum deformation of Lorentz group*, Comm. Math. Phys. **130** (1990), 381–431. MR 1059324 (91f:46100)
- [14] W. Pusz and P. M. Sołtan, *Functional form of unitary representations of the quantum ‘ $\mathfrak{az} + \mathfrak{b}$ ’ group*, Rep. Math. Phys. **52** (2003), 309–319. MR 2016222 (2005c:46108)
- [15] P.M. Sołtan, *Compactifications of discrete quantum groups*, to appear in Algebr. Represent. Theory.
- [16] ———, *Quantum Bohr compactification*, preprint des SFB 478 des Mathematisches Institut der Westfälischen Wilhelms-Universität Münster, 2003.
- [17] ———, *New quantum “ $\mathfrak{az} + \mathfrak{b}$ ” groups*, Rev. Math. Phys. **17** (2005), 313–364. MR 2144675
- [18] P. M. Sołtan and S. L. Woronowicz, *A remark on manageable multiplicative unitaries*, Lett. Math. Phys. **57** (2001), 239–252. MR 1862455 (2002i:46072)

- [19] S. Vaes, personal communication.
- [20] A. Van Daele, *The Haar measure on a compact quantum group*, Proc. Amer. Math. Soc. **123** (1995), 3125–3128. MR 1277138 (95m:46097)
- [21] ———, *Discrete quantum groups*, J. Algebra **180** (1996), 431–444. MR 1378538 (97a:16076)
- [22] ———, *An algebraic framework for group duality*, Adv. Math. **140** (1998), 323–366. MR 1658585 (2000g:16045)
- [23] A. Van Daele and S. Wang, *Universal quantum groups*, Internat. J. Math. **7** (1996), 255–263. MR 1382726 (97d:46090)
- [24] J. v. Neumann and E. P. Wigner, *Minimally almost periodic groups*, Ann. of Math. (2) **41** (1940), 746–750. MR 0002891 (2,127e)
- [25] S. Wang, *Free products of compact quantum groups*, Comm. Math. Phys. **167** (1995), 671–692. MR 1316765 (95k:46104)
- [26] ———, *Structure and isomorphism classification of compact quantum groups  $A_u(Q)$  and  $B_u(Q)$* , J. Operator Theory **48** (2002), 573–583. MR 1962472 (2004b:46083)
- [27] A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Herman, Paris, 1951.
- [28] S. L. Woronowicz, *Twisted SU(2) group. An example of a noncommutative differential calculus*, Publ. Res. Inst. Math. Sci. **23** (1987), 117–181. MR 890482 (88h:46130)
- [29] ———, *Compact matrix pseudogroups*, Comm. Math. Phys. **111** (1987), 613–665. MR 901157 (88m:46079)
- [30] ———, *A remark on compact matrix quantum groups*, Lett. Math. Phys. **21** (1991), 35–39. MR 1088408 (92k:46097)
- [31] ———, *Unbounded elements affiliated with  $C^*$ -algebras and noncompact quantum groups*, Comm. Math. Phys. **136** (1991), 399–432. MR 1096123 (92b:46117)
- [32] ———,  *$C^*$ -algebras generated by unbounded elements*, Rev. Math. Phys. **7** (1995), 481–521. MR 1326143 (96c:46056)
- [33] ———, *Compact quantum groups*, Symétries quantiques (Les Houches, 1995), North-Holland, Amsterdam, 1998, pp. 845–884. MR 1616348 (99m:46164)
- [34] ———, *From multiplicative unitaries to quantum groups*, Internat. J. Math. **7** (1996), 127–149. MR 1369908 (96k:46136)
- [35] ———, *Quantum “ $az + b$ ” group on complex plane*, Internat. J. Math. **12** (2001), 461–503. MR 1841400 (2002g:46120)

DEPARTMENT OF MATHEMATICAL METHODS IN PHYSICS, FACULTY OF PHYSICS, WARSAW UNIVERSITY, HOŻA 69, 00-681 WARSAW, POLAND

*E-mail address:* piotr.soltan@fuw.edu.pl