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PROBABILISTIC INVARIANT MEASURES FOR NON-ENTIRE FUNCTIONS WITH ASYMPTOTIC VALUES MAPPED ONTO ∞

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Abstract. We study the dynamics of transcendental meromorphic functions of the form $f(z) = R \circ \exp(z)$, where R is a non-constant rational map and both asymptotic values $R(0)$ and $R(\infty)$ are eventually mapped onto ∞. With each map f we associate its projection F on the cylinder $\mathcal{P}$. Let J_F^r consist of all points whose trajectory returns infinitely often to some compact set whose intersection with the postsingular set is empty, and let $h = \mathrm{HD}(J_F^r)$ be the Hausdorff dimension of J_F^r. We prove that the h-dimensional Hausdorff measure H^h of J_F^r is positive and finite, while the h-dimensional packing measure of J_F^r is locally infinite at every point of this set. We also prove that there exists a unique F-invariant Borel probability measure μ on J_F^r that is absolutely continuous with respect to the Hausdorff measure H^h, and that μ is ergodic and conservative.

1. Introduction

We consider the family $\mathcal{R}$ of transcendental meromorphic functions $f(z) : \mathbb{C} \to \overline{\mathbb{C}}$ of the form

$$f(z) = R \circ \exp(z), \tag{1.1}$$

where R is a non-constant rational map. The set of singularities $\mathrm{Sing}(f^{-1})$ consists of finitely many critical values and two asymptotic values

$$\xi_1 := R(0), \quad \xi_2 := R(\infty)\,.$$

Let $\mathcal{Q}^*$ be the class of non-entire functions from $\mathcal{R}$ such that both asymptotic values are mapped onto infinity, i.e., there exist numbers $q_i > 1$, $i = 1, 2$, such

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that $f^{q_i-1}(\xi_i)=\infty$, and

$$\operatorname{dist}_\chi(P^*(f),J_f)>0, \tag{1.2}$$

where J_f is the Julia set of f, χ is a chordal metric, and

$$P^*(f):=\overline{\Theta^+(\operatorname{Sing}(f^{-1})\setminus\Theta^+(\{\xi_1,\xi_2\})}\,.$$

Through the entire paper we assume that the considered functions belong to $\mathcal{Q}^*$. Then there are $N_i>0$, $i=1,2$, with the following properties: If $i=1$, then, for any $z\in\mathbb{C}$ with real part greater than N_1,

$$\begin{aligned} f^{q_1}(z)&=a_0e^{n_1z}+a_1e^{(n_1-1)z}+\cdots+a_{n_1}+a_{n_1+1}e^{-z}+\cdots\\ &=\sum_{j=0}^{\infty}a_je^{(n_1-j)z}, \end{aligned} \tag{1.3}$$

where $n_1>0$ and $a_0\neq 0$. If $i=2$, then, for any $z\in\mathbb{C}$ with real part smaller than $-N_2$,

$$\begin{aligned} f^{q_2}(z)&=b_0e^{-n_2z}+b_1e^{(-n_1+1)z}+\cdots+a_{n_2}+b_{n_2+1}e^{z}+\cdots\\ &=\sum_{j=0}^{\infty}b_je^{(-n_2+j)z}, \end{aligned} \tag{1.4}$$

where $n_2>1$ and $b_0\neq 0$. We can assume without lost of generality that

$$n_1\le n_2\,.$$

Following [7] we consider the map T_f defined by

$$T_f(z):=\begin{cases} f^{q_1}(z) & \text{if } \operatorname{Re}(z)>N_3,\\ f^{q_2}(z) & \text{if } \operatorname{Re}(z)<-N_3, \end{cases} \tag{1.5}$$

where $N_3:=\max\{N_1,N_2\}$. The following result was proved in [7] (see Lemma 2.2):

Proposition 1.1. *There exist $M_1,M_2>0$ and $M_3>N_3$ such that for every $z\in\mathbb{C}$ with $|\mathrm{Re}z|>M_3$ the following conditions hold:*

(i) $M_1e^{n(z)|\mathrm{Re}(z)|}\le|T_f(z)|\le M_2e^{n(z)|\mathrm{Re}(z)|}$,
(ii) $M_1e^{n(z)|\mathrm{Re}(z)|}\le|T_f'(z)|\le M_2e^{n(z)|\mathrm{Re}(z)|}$,

where

$$n(z):=\begin{cases} n_2 & \textit{if } \operatorname{Re}(z)<0,\\ n_1 & \textit{if } \operatorname{Re}(z)>0\,. \end{cases}$$

Since $f(z)$ is $2\pi i$-periodic, we consider it as a function on the cylinder rather than on $\mathbb{C}$. So let $\mathcal{P}$ be the quotient space (the cylinder)

$$\mathcal{P}=\mathbb{C}/\sim\,,$$

where $z_1 \sim z_2$ if and only if $z_1 - z_2 = 2k\pi i$ for some $k \in \mathbb{Z}$. Let $\pi : \mathbb{C} \to \mathcal{P}$ be the canonical projection. The function f projects down to a holomorphic map

$$F : \mathcal{P} \setminus \pi(f^{-1}(\infty)) \mapsto \mathcal{P}$$

so that $F \circ \pi = \pi \circ f$, i.e., the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C} \setminus B_0 & \xrightarrow{f} & \mathbb{C} \\ \pi \big\downarrow & & \big\downarrow \pi \\ \mathcal{P} \setminus B & \xrightarrow{F} & \mathcal{P} \end{array} \tag{1.6}$$

where $B_0 = f^{-1}(\infty)$ and $B = \pi(B_0)$. The Julia set J_F of F is defined to be

$$J_F := \pi(J_f \cap \mathbb{C}) .$$

Set $T_F = \pi(T_f)$. The next remark follows directly from Proposition 1.1.

REMARK 1.1. There exist $M_1, M_2 > 0$ and $M_3 > N_3$ such that for every $z \in \mathcal{P}$ with $|\mathrm{Re} z| > M_3$ the following conditions hold:

(i) $M_1 e^{n(z)|\mathrm{Re}(z)|} \le |T_F(z)| \le M_2 e^{n(z)|\mathrm{Re}(z)|}$,
(ii) $M_1 e^{n(z)|\mathrm{Re}(z)|} \le |T_F'(z)| \le M_2 e^{n(z)|\mathrm{Re}(z)|}$,

where $n(z)$ is defined as in Proposition 1.1.

Let

$$\zeta_i^j = \pi(f^{j-1}(\xi_i))$$

for $j = 1, \dots, q_i - 1$, $i = 1, 2$. Then for $n > 0$ we define the sets

$$W_n = \left\{ z \in \mathcal{P} : \ |\mathrm{Re}(z)| < n, \ |z - \zeta_i^j| > \frac{1}{n}, \ j = 1, \dots, q_i - 1, \ i = 1, 2 \right\} .$$

We also consider

$$K_n = \bigcap_{j \ge 0} F^{-j}(W_n) .$$

It was shown in [7] (see Lemma 3.1) that for $z \in K_n$

$$\lim_{k \to \infty} |(F^n)'(z)| = \infty. \tag{1.7}$$

Let m_n be the t_n-semiconformal measure supported on K_n, $t_n > 0$, i.e.,

$$m_n(F(A)) \ge \int_A |F(A)|^{t_n} dm_n$$

for every Borel set $A \subset \mathcal{P}$ such that $f_{|A}$ is 1-to-1. In [7] (see Lemma 5.4) the following result was shown:

Theorem 1.2. *For every $\epsilon > 0$ there exists N such that for all $n > n_0$ (with a suitable n_0)*

$$m_n(\{z \in J_F : |\mathrm{Re}(z)| > N\}) < \epsilon\,.$$

It follows that the sequence of measures $\{m_n\}$ is tight.

It was also shown in [7] that there exists $s > 1$ such that $t_n > s$ for all n large enough. In view of Theorem 1.2, there exists a subsequence $\{n_k\}$ such that the sequence $\{t_{n_k}\}$ converges. Let h denote the limit of this sequence. Then $h \in (1, 2]$. Passing to yet another subsequence we may assume that the sequence $\{m_{n_k}\}$ converges weakly to a measure m. This gives the next result, which was also proved in [7] (cf. Proposition 4.2 and Theorem 5.6). Let

$$J_F^r(n) = \{z \in J_F : \omega(z) \cap K_n \neq \emptyset\} \quad \text{and} \quad J_F^r = \bigcup_{n \geq N} J_F^r(n)\,.$$

Theorem 1.3. *There exists an h-conformal measure m on J_F such that m is atomless and $m(J_F^r) = 1$. If m' is a t-conformal measure for some $t > 1$, then $m' = m$ and $1 < h = \mathrm{HD}(J_F^r) < 2$. Moreover, there exists n_0 such that $m(J_F^r(n_0)) = 1$.*

Let H^h and P^h denote, respectively, the h-dimensional Hausdorff measure and the packing measure. In this paper we prove the following results.

Theorem A. *There exists a unique Borel probability F-invariant measure μ on J_F^r that is absolutely continuous with respect to a conformal measure m. This measure is ergodic and conservative.*

Theorem B. *We have:*

(i) $0 < \mathrm{H}^h(J_F^r) < \infty$.

(ii) $\mathrm{P}^h(J_F^r) = \infty$. *In fact,* $\mathrm{P}^h(J_F^r)$ *is locally infinite at every point of* J_F^r.

Corollary 1.4. $\mathrm{H}^h_{|J_F}$ *is equivalent to any measure with the properties in Theorem 1.3.*

Various versions of thermodynamic formalism and finer fractal geometry of transcendental entire and meromorphic functions have been explored since the mid 1990s, and more extensively since the year 2000. For an exposition of the main results obtained so far the reader is referred to the survey article [5].

2. An invariant measure equivalent to the conformal measure m

In this section we show the existence and uniqueness of an F-invariant Borel probability measure equivalent to m.

Analogously to Lemma 4.2 from [4] one can prove that for any open set $U \subset \mathcal{P}$ we have

$$\limsup_{n\to\infty} m(F^n(U)) = 1. \tag{2.1}$$

REMARK 2.1. The h-conformal measure m of Theorem 1.3 is ergodic and conservative.

The proof of Remark 2.1 has a long history going back to the papers [8], [9], [10]. The full proof carries over, with some obvious minor modifications, from the proof of Theorem 4.23 in [3].

LEMMA 2.1. *Up to a multiplicative constant there exists a unique F-invariant, σ-finite measure μ, which is conservative, ergodic and equivalent to the h-conformal measure m.*

The idea of the proof of Lemma 2.1 is to apply a general sufficient condition for the existence of a σ-finite absolutely continuous invariant measure, obtained in [6]. In order to formulate this condition, suppose that X is a σ-compact metric space, m is a Borel probability measure on X which is positive on open sets, and suppose that a measurable map $T : X \to X$ is given, with respect to which the measure m is quasi-invariant, i.e., $m \circ T^{-1} \ll m$. Moreover, assume the existence of a countable partition $\alpha = \{A_n : n \geq 0\}$ of subsets of X which are all σ-compact and of positive measure m, and such that $m(X \setminus \bigcup_{n\geq 0} A_n) = 0$. If, in addition, for all $m, n \geq 1$ there exists $k \geq 0$ such that

$$m(T^{-k}(A_m) \cap A_n) > 0, \tag{2.2}$$

then the partition α is called irreducible. The result of Martens, comprising Proposition 2.6 and Theorem 2.9 of [6], says the following:

THEOREM 2.2. *Suppose that $\alpha = \{A_n : n \geq 0\}$ is an irreducible partition for $T : X \to X$. Suppose that T is conservative and ergodic with respect to the measure m. If for every $n \geq 1$ there exists $K_n \geq 1$ such that for all $k \geq 0$ and all Borel subsets A of A_n*

$$K_n^{-1}\frac{m(A)}{m(A_n)} \leq \frac{m(T^{-k}(A))}{m(T^{-k}(A_n))} \leq K_n\frac{m(A)}{m(A_n)}, \tag{2.3}$$

then T has a σ-finite T-invariant measure μ that is absolutely continuous with respect to m. Additionally, μ is equivalent with m, conservative and ergodic, and unique up to a multiplicative constant.

Since in the sequel we will need a bit more than what is asserted in Lemma 2.1, namely a construction of the invariant measure claimed in Theorem 2.2, we briefly describe this construction. Following Martens, we consider the sequences of measures

$$S_k m = \sum_{i=0}^{k-1} m \circ T^{-i} \quad \text{and} \quad Q_k m = \frac{S_k m}{S_k m(A_0)}. \tag{2.4}$$

It was shown in [6] that each weak limit μ of the sequence $Q_k m$ has the properties required in Theorem 2.2, where a sequence $\{\nu_k : k \geq 1\}$ of measures on X is said to converge vaguely if for all $n \geq 1$ the measures ν_k converge weakly on all compact subsets of A_n. In fact, it turns out that the sequence $Q_k m$ converges and

$$\mu(F) = \lim_{n\to\infty} Q_k m(F)$$

for every Borel set $F \subset X$. Making use of (2.2) and (2.3) one can show (see Lemma 2.4 in [6]) the following:

LEMMA 2.3. *For every $n \geq 0$ we have $0 < \mu(A_n) < \infty$. Furthermore, the Radon-Nikodym derivative $d\mu/dm$ is bounded above and below on A_n.*

Now let us pass to the map $F : \mathcal{P} \setminus B \to \mathcal{P}$. The ergodicity and conservativity of the measure m follows from Remark 2.1. Thus, we only need to construct an irreducible partition α with the property (2.3). Indeed, set $Y = J(F) \setminus B$, and for every $y \in Y$ consider a ball $B(y, r(y)) \subset \mathcal{P}$ such that $r(y) > 0$, $m(\partial B(y, r(y))) = 0$, and $r(y) < (1/2)\min\{\pi/2, \operatorname{dist}(y, B)\}$. The balls $B(y, r(y))$, $y \in Y$, cover Y, and since Y is a metric separable metric, one can choose a countable cover, say $\{\tilde{A}_n : n \geq 0\}$, from these balls. We may additionally require that the family $\{\tilde{A}_n : n \geq 0\}$ is locally finite, i.e., that each point $x \in Y$ has an open neighborhood intersecting only finitely many balls $\tilde{A}_n$, $n \geq 0$. We now define the family $\alpha = \{A_n : n \geq 0\}$ inductively by setting

$$A_0 = \tilde{A}_0 \quad \text{and} \quad A_{n+1} = \tilde{A}_{n+1} \setminus \bigcup_{k=1}^{n} \overline{\tilde{A}_n}$$

(and throwing away empty sets). Obviously, α is a disjoint family and

$$\bigcup_{n\geq 1} A_n \supset J(F) \setminus B \setminus \bigcup_{n\geq 0} \partial\tilde{A}_n.$$

Hence $m\left(\bigcup_{n\geq 0} A_n\right) = 1$. The distortion condition (2.3) follows now from Koebe's distortion theorem with all constants K_n equal to some K, and the irreducibility of the partition α follows from the openness of the sets A_n and Theorem 1.3.

Let μ be an F-invariant measure that is absolutely continuous with respect to the measure m. Set

$$\mathcal{P}_M = \{z \in \mathcal{P} : \operatorname{Re}(z) > M\}$$

and

$$\mathcal{P}_{-M} = \{z \in \mathcal{P} : \mathrm{Re}(z) < -M\}$$

for $M \in \mathbb{R}$.

PROPOSITION 2.4. *There exists $M > 0$ such that*

$$\mu(P_{-M}) < \infty \quad \text{and} \quad \mu(P_M) < \infty. \tag{2.5}$$

To prove Proposition 2.4, take $k \in \mathbb{N}$ such that $k > M_3$, where M_3 is as defined in Remark 1.1, and consider the sets

$$X_k^- = \{z \in J_F : -(k+1) \le \mathrm{Re}(z) \le -k\}$$

and

$$X_k^+ = \{z \in J_F : k \le \mathrm{Re}(z) \le k+1\}.$$

LEMMA 2.5. *There exists a constant $C_1 > 0$ such that for n large enough and $k > M_3$ we have*

$$m(X_k^+) \le C_1 e^{n_1 k(1-h)} \quad \text{and} \quad m(X_k^-) \le C_1 e^{n_2 k(1-h)}.$$

Proof. It follows from Remark 1.1 that there exist universal constants D_+, D_- (independent of k) such that

$$|\mathrm{Im}(T_F(z)) - \mathrm{Im}(T_F(w))| \le D_\pm e^{n(z)|\mathrm{Re}(z)|} \tag{2.6}$$

for $z, w \in X_k^\pm$. This implies that if $k > M_3$ then T_F is $B_1 e^{n_1 k}$-to-1 on the set X_k^+, where B_1 depends on $D_\pm$, but is independent of k. Thus for every n large enough and all $k \ge M_3$ we have

$$\begin{aligned} 1 \ge m_n(T_F(X_K^+) &\ge (B_1 e^{n_1 k})^{-1} \int_{X_k^+} |T_F'|^{t_n} dm_n \\ &\ge (B_1 e^{n_1 k})^{-1} (M_1 e^{n_1 k})^{t_n} m_n(X_k^+) \\ &\ge (B_1)^{-1} M_1^{t_n} e^{n_1 k(t_n - 1)} m_n(X_k^+). \end{aligned}$$

Hence there exists a constant C_+ independent of k such that

$$m_n(X_k^+) \le C_+ e^{n_1 k(1-t_n)} \le C_+ e^{n_1 k(1-h)}.$$

Analogously, one can prove that for every n large enough

$$m_n(X_k^-) \le C_- e^{n_2 k(1-h)}$$

for some $C_- > 0$ and $k > M_3$. Setting $C_1 = \max\{C_+, C_-\}$, we obtain for the measure m

$$m(X_k^+) \le C_1 e^{n_1 k(1-h)} \quad \text{and} \quad m(X_k^-) \le C_1 e^{n_2 k(1-h)}$$

for $k > M_3$. □

Proof of Proposition 2.4. Set $A_0 = X^+_{M_3}$. Fix $k \ge M_3$, and let

$$S_k = [u+2, k+2] \times [-k/2, k/2] \subset \mathbb{C}\,,$$

where $u = \max\{|\xi_1|, |\xi_2|\}$. The set $\{z \in \mathbb{C} : \mathrm{Im} z = \pi\}$ is canonically embedded into $\mathbb{C}$. Thus each holomorphic inverse branch $F_*^{-j} : \mathcal{P} \setminus \pi(\{z \in \mathbb{C} : \mathrm{Im} z = \pi\}) \mapsto \mathcal{P}$ of F^j, $j \ge 1$, can be assumed to be defined on a subset of the complex plane $\mathbb{C}$. This map restricted to X_k^+ extends holomorphically to a univalent function on S_k. By Koebe's theorem there exists a constant C_2 such that, for every $j \ge 1$, every $x \in A_0$, and every $y \in X_k^+$, we have

$$\frac{|(F_*^{-j})'(y)|}{|(F_*^{-j})'(x)|} \le C_2 k^3\,.$$

Therefore

$$\frac{m(F_*^{-j}(X_k^+))}{m(F_*^{-j}(A_0))} \le C_2^h k^{3h} \frac{m(X_k^+)}{m(A_0)}\,.$$

Combining this with Lemma 2.5 we obtain

$$\frac{m(F_*^{-j}(X_k^+))}{m(F_*^{-j}(A_0))} \le C_1 C_2^h m(A_0)^{-1} k^{3h} e^{n_1 k(1-h)}\,.$$

Hence

$$\frac{m(F^{-j}(X_k^+))}{m(F^{-j}(A_0))} \le C_1 C_2^h m(A_0)^{-1} k^{3h} e^{n_1 k(1-h)}\,.$$

So, for every $n \ge 0$,

$$\frac{\sum_{j=0}^n m(F^{-j}(X_k^+))}{\sum_{j=0}^n m(F^{-j}(A_0))} \le C_1 C_2^h m(A_0)^{-1} k^{3h} e^{n_1 k(1-h)}\,.$$

Thus, applying Theorem 2.2, we get

$$\mu(X_k^+) = \lim_{n\to\infty} \frac{\sum_{j=0}^n m(F^{-j}(X_k^+))}{\sum_{j=0}^n m(F^{-j}(A_0))} \le C_1 C_2^h m(A_0)^{-1} k^{3h} e^{n_1 k(1-h)}\,.$$

Hence, if $M = M_3$, then

$$\mu(\mathcal{P}_M) \le \sum_{k=M_3}^{\infty} \mu(X_k^+) \le \sum_{k=M_3}^{\infty} C_1 C_2^h m(A_0)^{-1} k^{3h} e^{n_1 k(1-h)} < \infty\,.$$

Analogously, replacing n_1 by n_2 and taking $k > M_3$, we obtain that

$$\mu(\mathcal{P}_{-M}) \le \sum_{k=M_3}^{\infty} \mu(X_k^-) \le \sum_{k=M_3}^{\infty} C_1 C_2^h m(A_0)^{-1} k^{3h} e^{n_2 k(1-h)} < \infty\,.$$

This completes the proof. □

We recall that $\xi_1 = R(0)$, $\xi_2 = R(\infty)$. By our assumptions there are $q_i > 1$, $i = 1, 2$, such that $f^{q_i-1}(\xi_i) = \infty$. Let p_i denote the order of the pole $f^{q_i-2}(\xi_i)$. For every $k \geq 0$ let

$$R_k = \{z \in \mathcal{P} : \ \eta e^{-(k+1)} \leq |z - R(0)| \leq \eta e^{-k}\}$$

and

$$Q_k = \{z \in \mathcal{P} : \ \eta e^{-(k+1)} \leq |z - R(\infty)| \leq \eta e^{-k}\}$$

for some $\eta > 0$, where $R(0) \neq \infty$, $R(\infty) \neq \infty$. If $R(0) \neq R(\infty)$, then we choose η small enough so that $B(R(0), \eta) \cap B(R(\infty), \eta) = \emptyset$.

Proposition 2.6. *There exists $\epsilon > 0$ such that*

$$\mu(B(R(0), \epsilon)) < \infty \quad \textit{and} \quad \mu(B(R(\infty), \epsilon)) < \infty. \tag{2.7}$$

First we prove the following lemma.

Lemma 2.7. *There exist constants $C_3 > 0$, $r > 0$ and $p_1, p_2 \in \mathbb{N}$ such that for n large enough and all k satisfying $e^{-k} < r$ we have*

$$m(R_k) \leq C_3 e^{k[p_1 - (p_1+1)h]} \quad \textit{and} \quad m(Q_k) \leq C_3 e^{k[p_2 - (p_2+1)h]}.$$

Proof. Since $f^{q_1-2}(\xi_1)$ is a pole b_1 of multiplicity p_1, there exists $r_1 > 0$ such that

$$F^{q_1-2}(z) \asymp \frac{\kappa_1}{(z-\xi_1)^{p_1}} \tag{2.8}$$

for $z \in B(\xi_1, r_1) \subset \mathcal{P}$, $\kappa_1 \neq 0$. The comparability sign $\asymp$ means that

$$\begin{aligned} 0 &< \inf\left\{ \frac{|F^{q_1-2}(z)|}{|(z-\xi_1)^{p_1}|} : z \in B(\xi_1, r_1) \right\} \\ &\leq \sup\left\{ \frac{|F^{q_1-2}(z)|}{|(z-\xi_1)^{p_1}|} : z \in B(\xi_1, r_1) \right\} \\ &< \infty\,. \end{aligned}$$

This, in turn, implies the existence of a universal constant E_1 (independent of k) such that

$$|\mathrm{Im}(F^{q_1-2}(z)) - \mathrm{Im}(F^{q_1-2}(w))| \leq E_1 e^{kp_1}$$

for $z \in R_k$. Take K large enough so that $e^{-K} < r_1$. From (2.8) we obtain that for $k > K$

$$|(F^{q_1-1})'(z)| \geq F_1 e^{k(p_1+1)}$$

for some $F_1 > 0$. This implies that F^{q_1-1} is $G_1 e^{n_1 k}$-to-1 on the set R_k, where G_1 depends on F_1, but is independent of k. Thus, for every n large enough and all $k \geq K$,

$$\begin{aligned} 1 \geq m_n(F^{q_1-2}(R_k)) &\geq (G_1 e^{n_1 k})^{-1} \int_{R_k} |(F^{q_1-2})'|^{t_n} dm_n \\ &\geq (G_1 e^{p_1 k})^{-1} (F_1 e^{(p_1+1)k})^{t_n} m_n(R_k) \\ &\geq (G_1)^{-1} F_1^h e^{k[-p_1+(p_1+1)h]} m_n(R_k). \end{aligned}$$

Hence there exists a constant C' independent of k such that

$$m_n(R_k) \leq C' e^{k[p_1-(p_1+1)h]}$$

for $k > K$. Now let $f^{q_1-2}(\xi_2)$ be a pole b_2 of multiplicity p_2. Then there exists $r_2 > 0$ such that

$$F^{q_1-1}(z) \asymp \frac{\kappa_2}{(z-\xi_2)^{p_2}} \tag{2.9}$$

for $z \in B(\xi_2, r_2)$. Analogously one can prove that $m_n(Q_k) \leq C'' e^{k[p_2-(p_2+1)h]}$ for every n large enough and k such that $e^{-k} < r_2$. Consequently

$$m(R_k) \leq C_3 e^{k[p_1-(p_1+1)h]} \quad \text{and} \quad m(Q_k) \leq C_3 e^{k[p_2-(p_2+1)h]},$$

where $C_3 = \max\{C', C''\}$. □

Let f_0^{-1} denote a branch of the inverse map f^{-1} such that $0 \leq \operatorname{Im}(f_0^{-1}) < 2\pi$.

Lemma 2.8. *There is a universal constant $D > 0$ such that for all k large enough we have*

$$D^{-1} e^{-(k+1)} \leq |F'(z)| \leq D e^{-k}$$

if $z \in f_0^{-1}(R_k \cup Q_k)$.

Proof. First we estimate $f'(z)$ for $z \in f_0^{-1}(R_k)$. For simplicity we assume that $\pi(\xi_1) = \xi_1$. If $R'(0) \neq 0$, then

$$f_0^{-1}(R_k) \subset \Big\{ z \in \mathcal{P} : \log(L/|R'(0)|) - (k+1) \leq |\operatorname{Re}(z)| \\ \leq -k - \log(|R'(0)|L) \Big\},$$

where L denotes the distortion of R^{-1} on $B(\xi_1, r)$. Since $F'(z) = f'(z) = R'(e^z)(e^z)$, we obtain

$$|F'(z)| \leq (|R'(0)|L)^{-1}(|R'(0)|L)e^{-k} = e^{-k} \tag{2.10}$$

and

$$|F'(z)| \geq e^{-(k+1)} |R'(0)|^{-1} L |R'(0)| L^{-1} = e^{-(k+1)}. \tag{2.11}$$

If $R'(0) = 0$, there are constants $A > 0$ and $p \in \mathbb{N}$ such that

$$A^{-1}|z-0|^p \leq |R(z) - R(0)| \leq A|z-0|^p$$

and

$$A^{-1}|z-0|^{p-1} \le |R'(z)| \le A|z-0|^{p-1}\,.$$

In this case there exists a constant $A_1 > 0$, which depends on p and A, but is independent of k, such that

$$f_0^{-1}(R_k) \subset \left\{ z \in \mathcal{P} : \log A_1 - \frac{(k+1)}{p} \le |\mathrm{Re}(z)| \le -\frac{k}{p} - \log A_1 \right\}\,.$$

Moreover, for $z \in f_0^{-1}(R_k)$ we have

$$|F'(z)| \le A_1 \left(e^{-k/p}\right)^{p-1} e^{-k/p} = A_1 e^{-k} \tag{2.12}$$

and

$$|F'(z)| \ge (A_1)^{-1} \left(e^{-(k+1)/p}\right)^{p-1} e^{-(k+1)/p} = (A_1)^{-1} e^{-(k+1)}. \tag{2.13}$$

Next, we estimate $f'(z)$ for $z \in f_0^{-1}(Q_k)$. For simplicity we assume that $\pi(\xi_2) = \xi_2$. We know that $R(\infty) \ne \infty$ and suppose that $R'(\infty) = 0$. To count derivatives at $R'(\infty)$, we have to consider $R_1(w) = R(u)$, where $u = 1/w$, for w close to 0. Then $R_1'(0) = 0$. So there are constants $A_2 > 0$ and $p \in \mathbb{N}$ such that

$$A_2^{-1}|w|^p \le |R_1(w) - R_1(0)| \le A_2^{-1}|w|^p$$

and

$$A_2^{-1}|w|^{p-1} \le |R_1'(w)| \le A_2^{-1}|w|^{p-1}\,.$$

Substituting

$$R_1'(w) = R'\left(\frac{1}{w}\right)\left(-\frac{1}{w^2}\right),$$

we obtain

$$A_2^{-1}|w|^{p-1} \le \left| R'\left(\frac{1}{w}\right)\left(-\frac{1}{w^2}\right)\right| \le A_2|w|^{p-1},$$

or equivalently

$$A_2^{-1}|w|^{p+1} \le \left| R'\left(\frac{1}{w}\right)\right| \le A_2|w|^{p+1}\,.$$

Since $w = 1/u$ the above inequalities can be rewritten as

$$A_2^{-1}|u|^{-(p+1)} \le |R'(u)| \le A_2|u|^{-(p+1)}\,.$$

But $f(z) = R(e^z)$, so $f'(z) = R'(e^z)e^z$. Since $u = e^z$, we have

$$|F'(z)| = |R'(e^z)||e^z| \le A_2|e^z|^{-(p+1)}||e^z| = A_2 e^{-pz}$$

and

$$|F'(z)| \ge (A_2)^{-1}|e^z|^{-(p+1)}||e^z| = (A_2)^{-1}e^{-pz}\,.$$

Then

$$f_0^{-1}(Q_k) \subset \left\{ z \in \mathcal{P} : \frac{k}{p} + \log A_3 + \le |\mathrm{Re}(z)| \le \frac{(k+1)}{p} - \log A_3 \right\},$$

where A_3 is a constant, which depends on p and A_2, but is independent of k. So for $z \in f_0^{-1}(Q_k)$,

$$(A_3)^{-1}e^{-(k+1)} \leq |F'(z)| \leq A_3 e^{-k}. \tag{2.14}$$

Analogously, if $R'(\infty) \neq 0$, then

$$f_0^{-1}(Q_k) \subset \{z \in \mathcal{P} : k + \log A_4 + \leq |\mathrm{Re}(z)| \leq k + 1 - \log A_4\} ,$$

where A_4 depends on $R'(\infty)$ and the distortion of R in a neighbourhood of $R(\infty)$. Considering as before $R_1(w) = R_1(u)$ with $u = 1/w$ for w close to zero, we get

$$A_4^{-1}|u|^{-2} \leq |R'(u)| \leq A_4|u|^{-2} .$$

Since $F'(z) = R'(e^z)(e^z)$, the last inequalities we can rewritten as

$$(A_4)^{-1}e^{-z} \leq |F'(z)| \leq A_4 e^{-z} .$$

Thus for $z \in f_0^{-1}(Q_k)$ we have

$$(A_4)^{-1}e^{-(k+1)} \leq |F'(z)| \leq A_4 e^{-k}. \tag{2.15}$$

Combining (2.10), (2.12), (2.14), (2.15) and taking

$$D = \max\{1, A_2, A_3, A_4\},$$

we get the required estimate. □

Proof of Proposition 2.6. Let $\epsilon_0 = \min\{r_1, r_2\}$, where r_1, r_2 are defined by (2.8) and (2.9). Choose k_0 such that $\eta e^{-k_0} < \epsilon_0$ and set $A_0 := R_{k_0}$. Fix $j \geq 0$, and for all $l \in \mathbb{Z} \setminus \{0\}$ consider all holomorphic inverse branches $F_*^{-j} : B(\xi_1, \epsilon_0) \mapsto \mathcal{P}$ of F^j such that $f^j(F_*^{-j}(B(\xi_1, \epsilon_0))) = B(\xi_1 + 2l\pi i, \epsilon_0)$. Notice that $B(\xi_1 + 2l\pi i, \epsilon_0)$ is far from the singularity ξ_1, since we assumed that $\pi(\xi_1) = \xi_1$. So we can take inverse branches of f^j composed in the last step with π. To all of these inverse branches F_*^{-j} we can apply Koebe's distortion theorem. Thus for $k > k_0$ we have

$$\frac{m(F_*^{-j}(R_k))}{m(F_*^{-j}(A_0))} \leq K \frac{m(R_k)}{m(A_0)}, \tag{2.16}$$

where K is a distortion constant. Applying Lemma 2.7, we obtain

$$m(R_k) \leq C_3 e^{k[p_1-(p_1+1)h]}. \tag{2.17}$$

Combining this with (2.16), we get

$$\frac{m(F_*^{-j}(R_k))}{m(F_*^{-j}(A_0))} \leq K^h \frac{C_3 e^{k[p_1-(p_1+1)h]}}{m(A_0)}. \tag{2.18}$$

Let now $F_0^{-j} : B(\xi_1, \epsilon_0) \mapsto \mathcal{P}$ be a holomorphic inverse branch of F^j such that $f^j(F_0^{-j}(R_k)) = R_k$. Then there exists $k_1 > k_0$ such that

$$F^{j-1}\left(F_0^{-j}\left(\bigcup_{l=k_1}^{k} R_k\right)\right) \subset \{z \in \mathcal{P} : -(k+1) \leq \mathrm{Re}(z) \leq \log u - 2\}$$

where $u = \max\{1, |\xi_1|, |\xi_2|\}$. As in the proof of Proposition 2.4 we can write

$$\frac{m(F_0^{-j}(R_k))}{m(F_0^{-j}(A_0))} \leq C_4^h k^{3h} \frac{m(F^{j-1}(F_0^{-j}(R_k)))}{m(F^{j-1}(F^{-j}(A_0)))} \,.$$

Using now Lemmas 2.7 and 2.8, we get for k large enough

$$\begin{aligned}\frac{m(F_0^{-j}(R_k))}{m(F_0^{-j}(A_0))} &\leq C_4 k^{3h} \frac{m(F^{j-1}(F_0^{-j}(R_k))}{m(F^{j-1}(F^{-j}(A_0)))} \\ &\leq C_4 k^{3h} \frac{De^{-kh} C_3 e^{k[p_1-(p_1+1)h]}}{D^{-1}e^{-(k_0+1)h} m(A_0)} \\ &\leq C_5 k^{3h} e^{k[-2h+p_1(1-h)]},\end{aligned}$$

for some $C_5 > 0$. This, together with (2.18), implies that for every $j \geq 0$ and every $k > k_1$ we have

$$\frac{m((F^{-j}(R_k))}{m(F^{-j}(A_0))} \leq C_5 e^{k[p_1-(p_1+1)h]},$$

since $e^{k[-2h+p_1(1-h)]} < e^{k[p_1-(p_1+1)h]}$. Summing over $k \geq k_1$, we get

$$\frac{m(F^{-j}(B(\xi_1, \epsilon_1)))}{m(F^{-j}(A_0))} \leq C_5 \sum_{k=k_1}^{\infty} k^{3h} e^{k[p_1-(p_1+1)h]} < \infty \,,$$

where $\epsilon_1 := e^{-k_1}\eta$. Thus, for every $n \geq 0$,

$$\frac{\sum_{j=0}^{n} m(F^{-j}(B(\xi_1, \epsilon_1))}{\sum_{j=0}^{n} m(F^{-j}(F^{-j}(A_0)))} \leq C_5 \sum_{k=k_1}^{\infty} k^{3h} e^{k[p_1-(p_1+1)h]} < \infty \,.$$

Hence, applying Theorem 2.2,

$$\mu(B(\xi_1, \epsilon_1)) \leq C_5 \sum_{k=k_1}^{\infty} k^{3h} e^{k[p_1-(p_1+1)h]} < \infty \,.$$

To prove the second part of this proposition we define ϵ_0 as before. Let k_0 be such that $\eta e^{-k_0} < \epsilon_0$, and set $A_0 := Q_{k_0}$. Fix $j \geq 0$, and for all $\lambda \in \mathbb{Z} \setminus \{0\}$ consider all holomorphic inverse branches $F_*^{-j} : B(\xi_2, \epsilon_0) \mapsto \mathcal{P}$ of F^j such that $f^j(F_*^{-j}(B(\xi_2, \epsilon_0))) = B(\xi_2 + 2l\pi i, \epsilon_0)$. To all of these inverse branches

F_*^{-j} we can apply Koebe's distortion theorem. Thus for $k > k_0$ we have, analogously to (2.16),

$$\frac{m(F_*^{-j}(Q_k))}{m(F_*^{-j}(A_0))} \le K\frac{m(R_k)}{m(A_0)}, \tag{2.19}$$

where K is a distortion constant. By Lemma 2.7 we obtain

$$m(R_k) \le C_3 e^{k[p_2-(p_2+1)h]}. \tag{2.20}$$

This, together with (2.19), implies

$$\frac{m(F_*^{-j}(Q_k))}{m(F_*^{-j}(A_0))} \le K^h \frac{C_3 e^{k[p_2-(p_2+1)h]}}{m(A_0)}\,.$$

Let now $F_0^{-j} : B(\xi_2, \epsilon_0) \mapsto \mathcal{P}$ be a holomorphic inverse branch of F^j such that $f^j(F_0^{-j}(Q_k)) = Q_k$. Then there exists $k_1 > k_0$ such that

$$F^{j-1}\left(F_0^{-j}\left(\bigcup_{l=k_1}^{k} R_k\right)\right) \subset \{z \in \mathcal{P} : \log u + 2 \le \mathrm{Re}(z) \le\ k+1\}$$

The remaining part of the proof is analogous to the above argument. We therefore conclude that for some $C_6 > 0$ and $\epsilon_2 > 0$

$$\mu(B(\xi_2, \epsilon_2)) \le C_6 \sum_{k=k_1}^{\infty} k^{3h} e^{k[p_2-(p_2+1)h]} < \infty\,. \qquad \square$$

Proof of Theorem A. To complete the proof, we have to show that μ is finite at every point a of the forward trajectories of both asymptotic values ξ_1, ξ_2. We recall that both omitted values are eventually mapped onto ∞. In view of (1.2), $\mathrm{dist}_\chi(P^*(f), J_f) > 0$, so the critical points do not belong to the preimages of the forward trajectories of ξ_1, ξ_2. Thus, as in Proposition 2.6, we see that there exists $\epsilon > 0$ such that $\mu(B(a, \epsilon)) < \infty$ for every a. This, together with Proposition 2.4 and Proposition 2.6, finishes the proof. $\square$

3. Hausdorff and packing measures and dimensions

The results of this section provide, in some sense, a complete description of the geometrical structure of the sets J_F^r and J_f^r, and they also exhibit the geometrical meaning of the h-conformal measure m.

Theorem 3.1. *We have* $\mathrm{P}^h(J_f^r) = \mathrm{P}^h(J_F^r) = \infty$. *In fact,* $\mathrm{P}^h(G) = \infty$ *for every open nonempty subset* G *of* J_f^r.

Proof. Since $m(J_F^r \cap \mathcal{P}_M) > 0$ for every $M \in \mathbb{R}$, it follows from the ergodicity and conservativity of the measure m (see Remark 2.1) that there exists a set $E \subset J_F^r$ such that $m(E) = 1$ and

$$\limsup_{k\to\infty} \mathrm{Re}(F^k(z)) = +\infty$$

for every $z \in E$. Fix $z \in E$. The above relation means that there exists an unbounded increasing sequence $\{k_n\}_{k=1}^{\infty}$, depending on z, such that $\{F^{k_n}(z)\}_{k=1}^{\infty} \subset \mathcal{P}_M$ for some large $M > 0$ and

$$\lim_{n\to\infty} \mathrm{Re}(F^{k_n}(z)) = +\infty. \tag{3.1}$$

Fix $k_n \geq 1$ and consider the ball $B(z, K^{-1}|(F^{k_n})'(z)|^{-1})$. Then

$$B(z, K^{-1}|(F^{k_n})'|(z)|^{-1}) \subset F_z^{-k_n}(B(F^{k_n}(z), 1)),$$

where $F_z^{-k_n} : B(F_n^k(z), 1) \to \mathbb{C}$ is the analytic inverse branch of F^{k_n} mapping $F^{k_n}(z)$ to z. Applying Koebe's distortion theorem and using the conformality of the measure m, we obtain

$$\begin{aligned} m(B(z, K^{-1}|(F^{k_n})'(z)|^{-1}) &\leq K^h|(F^{k_n})'(z)|^{-h} m(B(F^{k_n}(z), 1)) \\ &\leq K^{2h}(K^{-1}|(F^{k_n})'(z)|^{-1})^h m(\mathcal{P}_{\mathrm{Re}F^{k_n}(z)+1}). \end{aligned}$$

Since, by (3.1), $\lim_{k\to\infty} m(\mathcal{P}_{\mathrm{Re}F^{k_n}(z)+1}) = 0$, we see that

$$\liminf_{r\to 0} \frac{m(B(z,r))}{r^h} = 0\,.$$

Since $m(G \cap J_F^r) > 0$ for every non-empty open subset of J_F^r, this implies that $\mathrm{P}^h(G) = \infty$. Since $J_f^r = \bigcup_{k\in\mathbb{Z}}(J_F^r + 2\pi ik)$, we are done. □

Theorem 3.2. *We have* $0 < \mathrm{H}^h(J_r(F)) < \infty$.

Proof. Let $n_0 > 0$ be the number defined in Theorem 1.3. Fix an integer $l \geq 1$ and a point $z \in J_F^r(n)$. Consider the holomorphic inverse branches $F_z^{-k_n(z)} : B(y(z), (2l)^{-1}) \to \mathcal{P}$ sending $F^{k_n(z)}(z)$ to z. By Koebe's (1/4)-distortion theorem and the standard version of Koebe's distortion theorem,

$$\begin{aligned} F_z^{-k_n(z)}\left(B\left(y(z), \frac{1}{2l}\right)\right) &\supset F_z^{-k_n(z)}\left(B\left(F^{k_n(z)}(z), \frac{1}{3l}\right)\right) \\ &\supset B\left(z, \frac{1}{12l}|(F^{k_n(z)})'(z)|^{-1}\right) \end{aligned}$$

and

$$\begin{aligned} F_z^{-k_z(z)}\left(B\left(y(z), \frac{1}{24Kl}\right)\right) &\subset F_z^{-k_n(z)}\left(B\left(F^{k_n(z)}(z), \frac{1}{12Kl}\right)\right) \\ &\subset B\left(z, \frac{1}{12l}|(F^{k_n(z)})'(z)|^{-1}\right). \end{aligned}$$

Using the conformality of the measure m along with the standard version of Koebe's distortion theorem, and the fact that $\inf\{m(B(w, (12Kl)^{-1}) : w \in W_{2l}\} > 0$, we deduce that

$$B_l^{-1} r_k(z)^h \leq m(B(z, r_k(z))) \leq B_l r_k(z)^h, \tag{3.2}$$

where $r_k(z) = (12l)^{-1}|(F^{n_k(z)})'(z)|^{-1}$ and B_l is independent of z and k. It follows from (3.2) that $\mathrm{H}^h|_{J_F^r(l)}$ is absolutely continuous with respect to m for every $l \geq 1$ and that $\mathrm{H}^h(J_F^r(n_0)) < \infty$. Since

$$m(J_F^r(l) \setminus J_F^r(n_0) \leq m(J_F^r \setminus J_F^r(n_0)) = 0$$

and $J_r(F) = \bigcup_{n=0}^{\infty} J_F^r(n_0+n)$, we conclude that $\mathrm{H}^h(J_F^r) = \mathrm{H}^h(J_F^r(n_0)) < \infty$.

We now prove that $\mathrm{H}^h(J_F^r) > 0$. Let ϵ be such that

$$0 < \epsilon < \frac{1}{De},$$

where D is the constant defined in Lemma 2.8. Fix $z \in J_F^r$. Take $r \in B(0, \epsilon(2^8K)^{-1})$. Since, by (1.7), $\limsup_{n\to\infty} |(f^n)'(z)| = +\infty$, there exists a minimal $n = n(z,r) \geq 1$ such that

$$r|(f^{n+1})'(z)| > \epsilon(2^8K)^{-1}. \tag{3.3}$$

Thus

$$r|(f^n)'(z)| \leq \epsilon(2^8K)^{-1}.$$

Assume the holomorphic inverse branch of f^n defined on $B(f^n(z), 32r|(f^n)'(z)|)$ and sending $f^n(z)$ to z, does not exist. Then $n \geq 1$. Let $1 \leq k \leq n$ be the largest integer such that the holomorphic inverse branch of $f^{n-(k-1)}$ defined on $B(f^n(z), 32r|(f^n)'(z)|)$ and sending $f^n(z)$ to $f^{k-1}(z)$ does not exist. This implies that at least one of the asymptotic values ξ_i, $i = 1,2$, satisfies

$$\xi_i \in f_k^{-(n-k)}(B(f^n(z), 32r|(f^n)'(z)|)),$$

where $f_k^{-(n-k)} : B(f^n(z), 32r|(f^n)'(z)|) \to \mathbb{C}$ is the holomorphic inverse branch of f^{n-k} sending $f^n(z)$ to $f^k(z)$. In addition, we have $n = k$ since $\xi_i \notin f^{-1}(\overline{\mathbb{C}})$, $i = 1,2$. Hence there is an i such that $|f^n(z) - \xi_i| < 32Kr|(f^n)'(z)| \leq \epsilon$. We assume that $i = 1$. So there exists $k \in \mathbb{N}$ such that

$$e^{-(k+1)} < |f^n(z) - \xi_1| < e^{-k}. \tag{3.4}$$

By Lemma 2.8 it follows that there exist constants $A_1 > 0$ and $p \in \mathbb{N}$ such that

$$-\frac{k+1}{p} - \log A_1 < \mathrm{Re}(f^{n-1}(z)) < -\frac{k}{p} - \log A_1$$

and

$$|f'(f^{n-1}(z))| \leq De^{-k}.$$

Combining this with (3.4), we get

$$|f'(f^{n-1}(z))| \leq De^{-k} \leq De|f^n(z) - \xi_1|. \tag{3.5}$$

Consequently, since $r|(f^{n-1})'(z)| < \epsilon(2^8K)^{-1}$, we conclude that

$$\begin{aligned}32Kr|(f^n)'(z)| &= 32Kr|(f^{n-1})'(z)| \cdot |f'(f^{n-1}(z))| \\ &\le 32Kr|(f^{n-1})'(z)| \cdot De|f^n(z) - \xi_1| \\ &\le \epsilon De|f^n(z) - \xi_1| \\ &< |f^n(z) - \xi_1|.\end{aligned}$$

This contradiction shows that the holomorphic inverse branch

$$f_z^{-n} : B(f^n(z), 32r|(f^n)'(z)|) \to \mathbb{C}$$

of f^n sending $f^n(z)$ to z is well-defined. Now, the map f restricted to $B(f^n(z), 32r|(f^n)'(z)|)$ is 1-to-1, and by Koebe's (1/4)-distortion theorem,

$$f(B(f^n(z), 32r|(f^n)'(z)|)) \supset B\left(f^{n+1}(z), 8r|(f^{n+1})'(z)|\right).$$

Hence there exists a unique holomorphic inverse branch

$$f_z^{-(n+1)} : B\left(f^{n+1}(z), 8r|(f^{n+1})'(z)|\right) \to \mathbb{C}$$

of f^{n+1} mapping $f^{n+1}(z)$ to z. Applying Koebe's (1/4)-distortion theorem again, we see that

$$f_z^{-(n+1)}\left(B\left(f^{n+1}(z), 4r|(f^{n+1})'(z)|\right)\right) \supset B(z, r). \tag{3.6}$$

Since the ball $B\left(f^{n+1}(z), 4r|(f^{n+1})'(z)|\right)$ intersects at most

$$\frac{1}{\pi}4r|(f^{n+1})'(z)| + 2 \preceq r|(f^{n+1})'(z)|$$

horizontal strips of the form $2\pi ij + (\mathbb{R} \times [0, 2\pi))$, $j \in \mathbb{Z}$, using (3.6), Koebe's distortion theorem, the h-conformality of the measure m and, in the final step, (3.3), we get

$$\begin{aligned}&r^{-h}(m(B(z,r))) \\ &\quad\preceq r^{-h}K^h\frac{(r|(f^{n+1})'(z)|)}{|(f^{n+1})'(z)|^h}m\left(\pi\left(B\left(f^{n+1}(z), 4r|(f^{n+1})'(z)|\right)\right)\right) \\ &\quad\le r^{-h}K^h|(f^{n+1})'(z)|^{-h}(r|(f^{n+1})'(z)|) \\ &\quad= K^h(r|(f^{n+1})'(z)|)^{1-h} \\ &\quad< K^h(2^8K)^{h-1}.\end{aligned}$$

The comparability sign $\preceq$ appearing in the above formulas means that the constants depend on z, but are independent of n. Thus we are done. □

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