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BMO RESULTS FOR OPERATORS ASSOCIATED TO HERMITE EXPANSIONS

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Abstract. We prove BMO and L^∞ results for operators associated to the heat-diffusion and Poisson semigroups in the multi-dimensional Hermite function expansions setting. These include maximal functions and square function operators. In the proof a technique of vector valued Calderón-Zygmund operators is used.

1. Introduction

In the d-dimensional Euclidean space $\mathbb{R}^d$, $d \geq 1$, consider the system of multi-dimensional Hermite functions

$$h_\alpha(x) = h_{\alpha_1}(x_1) \cdot \ldots \cdot h_{\alpha_d}(x_d),$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \{0, 1, \ldots\}$, $x = (x_1, \ldots, x_d)$, and

$$h_k(s) = (\pi^{1/2} 2^k k!)^{-1/2} H_k(s) \exp(-s^2/2), \qquad k = 0, 1, \ldots,$$

are the one-dimensional Hermite functions, and $H_k(s)$ denotes the kth Hermite polynomial. The system $\{h_\alpha\}$ is complete and orthonormal in $L^2 = L^2(\mathbb{R}^d)$; it consists of eigenfunctions of the d-dimensional harmonic oscillator (Hermite operator)

$$L = -\Delta + |x|^2, \qquad \Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}.$$

More specifically, one has

$$L h_\alpha = (2|\alpha| + d) h_\alpha,$$

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where $|\alpha| = \alpha_1 + \cdots + \alpha_d$. The operator L is positive and symmetric in L^2 on the domain $C_c^\infty(\mathbb{R}^d)$. It may be easily shown that the operator $\mathcal{L}$ given by

$$\mathcal{L}\left(\sum \langle f, h_\alpha\rangle h_\alpha\right) = \sum (2|\alpha| + d)\langle f, h_\alpha\rangle h_\alpha$$

on the domain

$$\mathrm{Dom}(\mathcal{L}) = \{f \in L^2 : \sum |(2|\alpha| + d)\langle f, h_\alpha\rangle|^2 < \infty\},$$

is a self-adjoint extension of L, has the discrete spectrum $\{2n + d : n = 0, 1, \dots\}$ and admits the spectral decomposition

$$\mathcal{L}f = \sum_{n=0}^{\infty} (2n + d)\mathcal{P}_n f, \qquad f \in \mathrm{Dom}(\mathcal{L}),$$

where the spectral projections $\mathcal{P}_n$ are $\mathcal{P}_n f = \sum_{|\alpha|=n} \langle f, h_\alpha\rangle h_\alpha$.

The heat-diffusion semigroup $\{T_t\}_{t>0}$, associated to $\mathcal{L}$, is defined by

$$T_t f(x) = e^{-t\mathcal{L}} f(x) = \sum_{n=0}^{\infty} e^{-t(2n+d)} \mathcal{P}_n f(x), \qquad f \in L^2. \tag{1.1}$$

The Poisson semigroup $\{P_t\}_{t>0}$, associated to $\mathcal{L}$, is given by

$$P_t f(x) = e^{-t\mathcal{L}^{1/2}} f(x) = \sum_{n=0}^{\infty} e^{-t(2n+d)^{1/2}} \mathcal{P}_n f(x), \qquad f \in L^2. \tag{1.2}$$

In [7] the action of T_t and P_t was extended onto $L^q(w)$ spaces, $1 \le q < \infty$, $w \in A_q$, and L^∞ by using the series in (1.1) and (1.2) (they are pointwise convergent for every $x \in \mathbb{R}^d$). It was then shown that for any $f \in L^q(w)$, $1 \le q < \infty$, $w \in A_q$, or $f \in L^\infty$, $T_t f(x)$ and $P_t f(x)$ are, respectively, equal to the *heat-diffusion* and the *Poisson integral* of f, defined by

$$g(t, x) = \int_{\mathbb{R}^d} G_t(x, y) f(y) dy, \qquad f(t, x) = \int_{\mathbb{R}^d} P_t(x, y) f(y) dy, \tag{1.3}$$

where

$$G_t(x, y) = \big(2\pi \sinh(2t)\big)^{-d/2} \exp\left(-\frac{1}{4}\big(\tanh(t)|x + y|^2 + \coth(t)|x - y|^2\big)\right) \tag{1.4}$$

and

$$P_t(x, y) = \frac{t}{\sqrt{4\pi}} \int_0^\infty G_s(x, y) s^{-3/2} e^{-t^2/4s} ds \tag{1.5}$$

are the *heat-diffusion* and *Poisson kernels*. Apart from investigating the operators $f \to g(t, x)$ and $f \to f(t, x)$ for fixed $t > 0$ we also proved results for the maximal operators

$$T^* f(x) = \sup_{t>0} |g(t, x)|, \qquad P^* f(x) = \sup_{t>0} |f(t, x)|; \tag{1.6}$$

cf. [7, Theorems 2.6 and 2.8].

In [8] we investigated L^p behaviour of the square functions

(1.7)
$$\tilde{g}(f)(x) = \left(\int_0^\infty \Big|\frac{\partial}{\partial t} g(t,x)\Big|^2 t dt\right)^{1/2}, \quad g(f)(x) = \left(\int_0^\infty \Big|\frac{\partial}{\partial t} f(t,x)\Big|^2 t dt\right)^{1/2}$$

and

$$g_\nabla(f)(x) = \left(\int_0^\infty |\nabla f(t,x)|^2 t dt\right)^{1/2}, \tag{1.8}$$

where

$$\nabla = \nabla_H = (\delta_1^-, \dots, \delta_d^-, \frac{\partial}{\partial t}, \delta_1^+, \dots, \delta_d^+)$$

denotes the Hermite-type full gradient, $\delta_j^\pm = \partial_{x_j} \pm x_j$ are the Hermite-type derivatives and

$$\Big|\nabla P_t f(x)\Big| = \left(\Big|\frac{\partial}{\partial t} P_t f(x)\Big|^2 + \sum_{j=1}^d \left(|\delta_j^- P_t f(x)|^2 + |\delta_j^+ P_t f(x)|^2\right)\right)^{1/2}$$

is the Euclidean norm of the vector $\nabla P_t f(x)$ in $\mathbb{R}^{2d+1}$.

The essential aim of the present paper is to investigate the action of the aforementioned operators: T_t and P_t, $t>0$ fixed, the maximal operators T^* and P^*, the square functions $\tilde{g}$, g and g_∇, on the spaces L^∞ and BMO. It will be shown that (1.3) extends to BMO functions; hence for $f \in BMO$, (1.6), (1.7) and (1.8) make sense. It is, however, far from being clear that for any $f \in BMO$ the objects in (1.6), (1.7) and (1.8) are finite x-a.e.; we shall prove that this is the case.

A characteristic feature of these operators is that they do not map BMO into BMO with a control of the BMO seminorm. A reason for this is that the image of 1 (1 represents the function on $\mathbb{R}^d$ identically equal one) either under the action of these operators or under the action of their vector valued linearizations is not a constant function. This feature is a major difference between the situation we consider here and the classic (Euclidean) situation.

Apart from this feature, in the classic case a dichotomy similar to that from [1, Theorem 4.2 (b)] takes place also for some maximal functions (different from the Hardy-Littlewood maximal function) and some square functions. Recall that Bennet, DeVore and Sharpley proved for the Hardy-Littlewood maximal operator M on $\mathbb{R}^d$ that, given $f \in BMO$, either $Mf(x) = \infty$ x-a.e. or $Mf(x) < \infty$ x-a.e., and, in the latter case, $\|Mf\|_{BMO} \le C\|f\|_{BMO}$. The same statement is certainly true for the maximal operator $M_\varphi f(x) = \sup_{t>0} |f * \varphi_t(x)|$ with φ satisfying some mild regularity conditions (which includes the cases $\varphi = W_1$ and $\varphi = P_1$), since for such a φ, $C^{-1} Mf(x) \le M_\varphi f(x) \le CMf(x)$; cf. the end of Section 4 for an example.

A similar situation occurs for the square function operators. Wang [10] proved that for the classic full gradient g-function g_∇ based on the Euclidean

Poisson kernel $\{P_t(x)\}_{t>0}$, given $f \in BMO$ either $g_\nabla(f)(x) = \infty$ x-a.e., or $g_\nabla(f)(x) < \infty$ x-a.e., and, in the latter case, $\|g_\nabla(f)\|_{BMO} \le C\|f\|_{BMO}$. Even though not explicitely stated by Wang, the same result holds for the g-function based on the Gauss-Weierstrass kernel $\{W_t(x)\}_{t>0}$. Moreover, it was observed by Kurtz [5] that the same statement is true for other Littlewood-Paley operators, namely the area integral $S(f)$ and the g_λ^* function, $\lambda > 1$.

In the situation we consider such a dichotomy does not take place. In Section 4 we prove that for every $f \in BMO$, $T^*f(x) < \infty$ x-a.e. In Proposition 5.1 we prove that the square functions given in (1.7) and (1.8) are finite x-a.e. for every $f \in BMO$.

This paper constitutes the final version of research started by both authors a couple of years ago. Meanwhile in [3] a BMO space related to Schrödinger operators was defined and investigated. This new BMO space can be seen as a good substitute of the classic BMO space, especially when boundedness of operators connected with Schrödinger operators is treated. In the case of the potential $V(x) = |x|^2$, our present paper and [3] should be seen as complementary papers treating similar boundedness problems by different techniques.

The letter B will be frequently used to denote a ball $B = B(x_o, r)$ in $\mathbb{R}^d$ with center x_o and radius r. If B is a ball, $B = B(x_o, r)$, and $k > 0$, then kB will mean $B(x_o, kr)$. Given a locally integrable function g, we shall define $g_B = \frac{1}{|B|}\int_B g(z)dz$. Given a subset $A \subset \mathbb{R}^d$, A^c will denote the complement $A^c = \mathbb{R}^d \setminus A$. By $\{W_t(x)\}_{t>0}$ and $\{P_t(x)\}_{t>0}$, $x \in \mathbb{R}^d$, we shall denote, respectively, the (Euclidean) Gauss-Weierstrass and Poisson kernels, defined by

$$W_t(x) = (4\pi t)^{-d/2} \exp(-|x|^2/4t), \qquad P_t(x) = c_d t\big(t^2 + |x|^2\big)^{-(d+1)/2},$$

For any other unexplained symbol or notion we refer the reader to [7], [8], [9].

2. General results

Let $(E, \|\cdot\|_E)$ be a Banach space. We shall integrate E-valued functions defined on $\mathbb{R}^d$ or on a subset of $\mathbb{R}^d$ by using the notion of the Bochner integral. For a short discussion of Bochner's integral and its basic properties we kindly refer the reader to [11]. The symbol $\mathcal{M}_E = \mathcal{M}_E(\mathbb{R}^d)$ will mean the linear space of all (equivalence classes of) measurable in the strong sense E-valued functions on $\mathbb{R}^d$. Given $1 \le p \le \infty$, by $L^p_E = L^p_E(\mathbb{R}^d)$ we mean the Lebesgue space of all functions $f \in \mathcal{M}_E$ for which the quantity $\int_{\mathbb{R}^d} \|f(x)\|^p\,dx$ (with the usual interpretation when $p = \infty$) is finite. If Lebesgue measure dx is replaced by $w(x)dx$, where $w(x)$ denotes a non-negative weight on $\mathbb{R}^d$, then we consider the weighted Lebesgue spaces $L^p_E(w)$. By $BMO_E = BMO_E(\mathbb{R}^d)$

we denote the linear space of all $f \in L^1_{\mathrm{loc},E}$ for which the seminorm

$$\|f\|_{BMO_E} = \sup_B \frac{1}{|B|} \int_B \|f(y) - f_B\|_E \, dy \tag{2.1}$$

is finite. We will use the fact that the above seminorm is equivalent to the seminorm

$$f \mapsto \sup_B \inf_{a \in E} \frac{1}{|B|} \int_B \|f(y) - a\|_E \, dy.$$

See [2] for this and other properties of the space BMO (the vector-valued case can be developed analogously). We will also use the fact that

$$BMO(\mathbb{R}^d) \subset L^1(w), \qquad w(x) = (1 + |x|)^{-d-1}. \tag{2.2}$$

Identifying functions that differ by a constant, i.e., considering the quotient BMO_E/E, a Banach space is obtained with norm given by

$$\|[f]\|_{\mathbf{BMO}_E} = \|f\|_{BMO_E}$$

($[f]$ denotes the quotient class $f + E$). In what follows, to distinguish between BMO_E and the quotient BMO_E/E we write $\mathbf{BMO}_E$ for the latter Banach space. Also, when $E = \mathbb{C}$, we drop the symbol $\mathbb{C}$ and simply write $\mathcal{M}$, L^p, $L^p(w)$, BMO, $\mathbf{BMO}$, $\|f\|_{BMO}$ and $\|[f]\|_{\mathbf{BMO}}$. Since all functions from the considered function spaces live on $\mathbb{R}^d$, when denoting these spaces we consequently drop the symbol $\mathbb{R}^d$.

It is clear that, given a linear operator $T : BMO \to BMO_E$ such that $T1 = \mathbf{b} \in E$, we may define the operator $\mathbf{T} : \mathbf{BMO} \to \mathbf{BMO}_E$ by the rule $\mathbf{T}([f]) = Tf$. Moreover, if T satisfies $\|Tf\|_{BMO_E} \le C\|f\|_{BMO}$, then, necessarily, $T1 = \mathbf{b}$ and $\mathbf{T}$ satisfies $\|\mathbf{T}([f])\|_{\mathbf{BMO}_E} \le C\|[f]\|_{\mathbf{BMO}}$ (with the same C). The condition $T1 = \mathbf{b} \in E$ is the *conditio sine qua non* for factoring T.

In the classic (Euclidean) setting maximal and square function operators map 1 onto 1, so there is no real reason to distinguish between BMO and $\mathbf{BMO}$; in fact, without any comment in relevant places BMO is always tacitly treated as the quotient $BMO/\mathbb{C}$. In the setting we consider the property "1 is mapped onto a constant function" is no longer valid; thus the distinction we suggest (between BMO and $\mathbf{BMO}$) seems to be justified.

Given a Banach space E, we shall consider kernels $\mathcal{U}(x, y)$ defined in $\mathbb{R}^d \times \mathbb{R}^d \setminus \Delta$ with values in E, where Δ denotes the diagonal

$$\Delta = \{(x, x) : x \in \mathbb{R}^d\}.$$

We assume that for every $x \in \mathbb{R}^d$ the function $y \to \|\mathcal{U}(x, y)\|_E$ is locally bounded away from x, and therefore the quantity $\int_{\mathbb{R}^d} \mathcal{U}(x, y) f(y) dy$ is well defined for every compactly supported function $f \in L^1(\mathbb{R}^d)$ and $x \notin \operatorname{supp} f$ (we agree to multiply vectors by scalars from the right).

We say that a bounded operator $U : L^2 \to L^2_E$ has $\mathcal{U}(x,y)$ as the associated kernel if

$$Uf(x) = \int_{\mathbb{R}^d} \mathcal{U}(x,y)f(y)dy, \qquad x - \text{a.e. on } (\operatorname{supp} f)^c, \tag{2.3}$$

for every $f \in L^2$ with compact support. If, in addition, the associated kernel $\mathcal{U}$ satisfies

$$\|\mathcal{U}(x,y)\|_E \leq C \frac{1}{|x-y|^d} \tag{2.4}$$

and

$$\|\mathcal{U}(x,y) - \mathcal{U}(z,y)\|_E \leq C \frac{|x-z|}{|x-y|^{d+1}}, \qquad |x-y| \geq 2|x-z|, \tag{2.5}$$

$$\|\mathcal{U}(x,y) - \mathcal{U}(x,z)\|_E \leq C \frac{|y-z|}{|x-y|^{d+1}}, \qquad |x-y| \geq 2|y-z|, \tag{2.6}$$

then U is called a Calderón-Zygmund operator. It is well known that such an operator uniquely extends to a bounded operator from $L^q(w)$ into $L^q_E(w(x)dx)$ for every $1 < q < \infty$, $w \in A_q$, and to a bounded operator from $L^1(w)$ into $L^{1,\infty}_E(w(x)dx)$ for every $w \in A_1$; cf. [2], for instance. To avoid cumbersome notation we use the same letter U to denote these extensions.

It is also well known (cf. [2, Theorem 6.6] with a proof that mimics that one of the scalar case version) that if E is a Banach space and $U: L^2 \to L^2_E$ is a Calderón-Zygmund operator, then $Uf \in BMO_E$ whenever f is a bounded function of compact support and

$$\|Uf\|_{BMO_E} \leq C\|f\|_{L^\infty}.$$

An interesting feature of the Calderón-Zygmund theory is that $p = 2$ can be replaced in the definition of the Calderón-Zygmund operator by any p, $1 \leq p \leq \infty$. This is particularly helpful when considering maximal operators, cf. Section 4, since then we have a natural $L^\infty - L^\infty$ boundedness instead of the usual $L^2 - L^2$ one. We should also add that in Propositions 2.1–2.3, $p = 2$ can be replaced by any p, $1 \leq p \leq \infty$, as well.

Observe that Uf may not be a priori defined for some bounded functions f, and even $U1$ may not be defined. In the scalar case different extensions can be given in order to have a satisfactory action of the operator U on L^∞; see [6, IV.4.1] and [2, VI.2]. Some of these procedures (for example, the one developed in [2]) can be reproduced in the vector valued setting with a corresponding extension of U mapping L^∞ into BMO_E and, what is more important, they also give a way of defining the action of U on BMO functions.

Here we take the opportunity to present a refined version of such a procedure in the case when the kernel of the involved Calderón-Zygmund operator has better than usual decay outside the diagonal Δ. This will be applied in the next sections since the operators we investigate possess such a feature.

Proposition 2.1. *Let E be a Banach space and $U : L^2 \to L^2_E$ be a Calderón-Zygmund operator associated with the kernel $\mathcal{U}(x,y)$ that, in addition to* (2.4), (2.5) *and* (2.6), *satisfies*

$$\|\mathcal{U}(x,y)\|_E \le C|x-y|^{-d-1}, \qquad |x-y| \ge 1. \tag{2.7}$$

Given a ball B, for $f \in BMO$ and $n = 0,1,\dots,$ define $(\hat U_B f)_n(x)$ to be

$$(\hat U_B f)_n(x) = U(f\chi_{2^{n+1}B})(x) + \int_{(2^{n+1}B)^c} \mathcal{U}(x,y) f(y)\, dy \tag{2.8}$$

for $x \in 2^n B$, and 0 otherwise. Then the formula

$$\hat U f(x) = \lim_{n\to\infty} (\hat U_B f)_n(x), \tag{2.9}$$

defines a linear operator $\hat U : BMO \to L^1_{\mathrm{loc},E}$, independent of the choice of B, that coincides with the (unique) extensions of U onto the spaces $L^q(w)$, $1 \le q < \infty$, $w \in A_q$. In addition, $\hat U$ restricted to L^∞ maps L^∞ into BMO_E and satisfies

$$\|\hat U f\|_{BMO_E} \le C\|f\|_\infty. \tag{2.10}$$

Proof. We start by noting that for $f \in BMO$, $(\hat U_B f)_n$ is well defined. Indeed, $f\chi_{2B} \in L^2$, hence $U(f\chi_{2B})(x)$ is well defined a.e. on $\mathbb{R}^d$. On the other hand, the integral in (2.8) converges due to (2.2) and (2.7). The limit in (2.9) exists since, for every x, the sequence $(\hat U_B f)_n(x)$ stabilizes: $(\hat U_B f)_n$ and $(\hat U_B f)_{n+1}$ agree on $2^n B$. This is because for $x \in 2^n B$

$$\begin{aligned}
U(f\chi_{2^{n+1}B})(x) &- U(f\chi_{2^{n+2}B})(x) \\
&+ \int_{(2^{n+1}B)^c} \mathcal{U}(x,y) f(y)\, dy - \int_{(2^{n+2}B)^c} \mathcal{U}(x,y) f(y)\, dy \\
&= -U(f\chi_{2^{n+2}B\setminus 2^{n+1}B})(x) + \int_{2^{n+2}B\setminus 2^{n+1}B} \mathcal{U}(x,y) f(y)\, dy = 0.
\end{aligned}$$

The same argument shows that the limit in (2.9) is independent of the choice of B. Indeed, given balls B_1 and B_2, find m such that $B_1 \subset 2^m B_2$. Then $(\hat U_{B_1} f)_n$ and $(\hat U_{B_2} f)_{n+m+1}$ agree on $2^n B_1$.

To check that $\hat U f \in L^1_{\mathrm{loc},E}$ we show that

$$\int_{2^{n+1}B} \|\hat U f\|_E\, dx < \infty, \qquad n = 0,1,\dots .$$

Indeed, for the first term in (2.8) we have $U(f\chi_{2^{n+1}B}) \in L^2_E \subset L^1_{\mathrm{loc},E}$ and for the second term we have

$$\begin{aligned}\int_{2^nB}\int_{(2^{n+1}B)^c}\|\mathcal{U}(x,y)\|_E|f(y)|\,dydx &\le \int_{(2^{n+1}B)^c}|f(y)|\int_{2^nB}\|\mathcal{U}(x,y)\|_E\,dx\,dy\\ &\le C_n\int_{(2^{n+1}B)^c}|f(y)|(1+|y|)^{-d-1}\,dy\\ &< \infty.\end{aligned}$$

To verify that $\hat{U}$ is consistent with the action of U on $L^q(w)$ spaces assume $f \in BMO \cap L^q(w)$, $1 \le q < \infty$, $w \in A_q$. It is then easily seen that the integral in (2.8) converges to 0 when $n \to \infty$. Indeed, for $x \in B$,

$$\begin{aligned}&\left\|\int_{(2^{n+1}B)^c}\mathcal{U}(x,y)f(y)\,dy\right\|_E\\ &\qquad\le C\left(\int_{(2^{n+1}B)^c}|x-y|^{-Dq'}w(y)^{-q'/q}\,dy\right)^{1/q'}\|f\|_{L^q(w)}\end{aligned}$$

if $1 < q < \infty$, or

$$\left\|\int_{(2^{n+1}B)^c}\mathcal{U}(x,y)f(y)\,dy\right\|_E \le C\sup_{y\in(2^{n+1}B)^c}\left\{|x-y|^{-D}w(y)^{-1}\right\}\|f\|_{L^1(w)}$$

if $q = 1$, and the quantities on the right of the above inequalities tend to 0 as $n \to \infty$ (we use the fact that $w(y)^{-q'/q} \in A_{q'}$ if $1 < q < \infty$). The above shows that

$$\hat{U}f(x) = \lim_{n\to\infty} U(f\chi_{2^{n+1}B})(x)$$

a.e. on B. But $f\chi_{2^{n+1}B} \in L^q(w)\cap L^2$ and $f\chi_{2^{n+1}B} \to f$ in $L^q(w)$ as $n \to \infty$. Therefore, $U(f\chi_{2^{n+1}B}) \to Uf$ in $L^q(w)$ if $1 < q < \infty$ or in $L^{1,\infty}(w)$ if $q = 1$, where the last U denotes the extension of the operator U acting on $L^q(w)\cap L^2$ onto $L^q(w)$ (with appropriate modification when $q = 1$).

To show (2.10) take a ball $B = B(x_o, r)$ and $f \in L^\infty$, set

$$a = \int_{(2B)^c}\mathcal{U}(x_o,y)f(y)\,dy$$

and write

$$\begin{aligned}&\frac{1}{|B|}\int_B\|\hat{U}f(x)-a\|_E\,dx\\ &\qquad\le \frac{1}{|B|}\int_B\|\hat{U}(f\chi_{2B})(x)\|_E dx\\ &\qquad\qquad+\frac{1}{|B|}\int_B\left\|\int_{(2B)^c}(\mathcal{U}(x,y)-\mathcal{U}(x_o,y))f(y)\,dy\right\|dx\end{aligned}$$

$$\leq \left(\frac{1}{|B|} \int_B \|U(f\chi_{2B})(x)\|_E^2 \, dx \right)^{1/2}$$
$$+ C \frac{1}{|B|} \int_B \int_{\{y:|x-y|\geq r\}^c} \frac{|x-x_o|}{|x-y|^{-d-1}} \, dy \, dx \cdot \|f\|_\infty$$
$$\leq C \left(\frac{1}{|B|} \int_{2B} |f(x)|^2 \, dx \right)^{1/2} + C\|f\|_\infty$$
$$\leq C\|f\|_\infty.$$

This proves that $\hat{U}f \in BMO_E$ and, at the same time, shows (2.10). □

To make the picture complete, we also decided to present here a short proof of the fact that once a Calderón-Zygmund operator $U : L^2 \to L^2_E$ is *a priori* defined on a wider domain that includes BMO, then the necessary condition $U1 = \mathbf{b} \in E$ is also sufficient for U to map BMO into BMO_E with a control of seminorms.

Proposition 2.2. *Let E be a Banach space and $U : L^2 \to L^2_E$ be a Calderón-Zygmund operator with a domain a priori wider than L^2 and including BMO (and thus the constant functions). Assume also that $Uf \in L^1_{\mathrm{loc},E}$ whenever $f \in BMO$. If $U1 = \mathbf{b} \in E$, then*

$$\|Uf\|_{BMO_E} \leq C\|f\|_{BMO}, \qquad f \in BMO. \tag{2.11}$$

Consequently, U may be factorized to a bounded operator from **BMO** *into* $\mathbf{BMO}_E$.

Proof. Let $\mathcal{U}$ be the associated kernel of U. Take $f \in BMO$ and a ball $B = B(x_0, r)$. By assumption $(Uf)_B$ is well defined and, since $BMO \subset L^2_{\mathrm{loc}}$, $(f - f_B)\chi_{5B}$ (hence also $(f - f_B)\chi_{(5B)^c}$) belongs to the domain of U. Therefore, for $x \in B$ we can write

$$\begin{aligned} Uf(x) - (Uf)_B &= U((f - f_B)\chi_{5B})(x) + U((f - f_B)\chi_{(5B)^c})(x) + U(f_B)(x) \\ &\quad - \frac{1}{|B|} \int_B \bigg(U((f - f_B)\chi_{5B})(z) \\ &\quad + U((f - f_B)\chi_{(5B)^c})(z) + U(f_B)(z) \bigg) dz \\ &= \sigma_1(x) - \frac{1}{|B|} \int_B \sigma_1(z) \, dz + \frac{1}{|B|} \int_B \sigma_2(x, z) \, dz, \end{aligned} \tag{2.12}$$

(we used the fact that $f_B(U1(x) - (U1)_B) = 0$ since $U1$ is a constant function), where

$$\sigma_1(x) = U((f - f_B)\chi_{5B})(x)$$

and

$$\sigma_2(x,z) = U((f-f_B)\chi_{(5B)^c})(x) - U((f-f_B)\chi_{(5B)^c})(z).$$

Using the triangle inequality in (2.12) and then integrating over B produces

$$\begin{aligned}\frac{1}{|B|}\int_B \|Uf(x)-(Uf)_B\|_E dx \le 2\frac{1}{|B|}\int_B \|\sigma_1(z)\|_E\, dz \\ + \frac{1}{|B|}\int_B \frac{1}{|B|}\int_B \|\,\sigma_2(x,z)\|_E\, dz dx.\end{aligned}$$

Next, since the operator U maps L^2 into L^2_E, we obtain

$$\begin{aligned}\frac{1}{|B|}\int_B \|\sigma_1(z)\|_E\, dz &= \frac{1}{|B|}\int_B \|U((f-f_B)\chi_{5B})(z)\|_E dz \\ &\le \left(\frac{1}{|B|}\int_B \|U((f-f_B)\chi_{5B})(z)\|_E^2 dz\right)^{1/2} \\ &\le C\left(\frac{1}{|B|}\int_{5B} |f(z)-f_B|^2\, dz\right)^{1/2} \\ &\le C\|f\|_{BMO}.\end{aligned}$$

On the other hand, if $x,z\in B$, then $|x-z|<2r$. Therefore,

$$\begin{aligned}\|\,\sigma_2(x,z)\|_E &\le \int_{(5B)^c} \|\mathcal{U}(x,y)-\mathcal{U}(z,y)\|_E|f(y)-f_B|\, dy \\ &\le \int_{|x-y|\ge 4r} \|\mathcal{U}(x,y)-\mathcal{U}(z,y)\|_E|f(y)-f_B|\, dy \\ &\le C\sum_{j=2}^{\infty}\int_{2^jr\le|x-y|<2^{j+1}r} \frac{|x-z|}{|x-y|^{d+1}}|f(y)-f_B|\, dy \\ &\le C\sum_{j=2}^{\infty}\frac{1}{2^j(2^jr)^d}\int_{|x-y|<2^{j+1}r}|f(y)-f_B|\, dy \\ &\le C\sum_{j=2}^{\infty}\frac{1}{2^j}\|f\|_{BMO} \\ &\le C\|f\|_{BMO},\end{aligned}$$

where we used (2.5) in the third inequality. Combining the last three estimates gives (2.11). $\square$

Finally we state and prove a result that will be used in Section 4.

PROPOSITION 2.3. *Let E be a Banach space and $U : L^2 \to L^2_E$ be a Calderón-Zygmund operator with a domain a priori wider than L^2 and including BMO. Assume also that $Uf \in L^1_{\mathrm{loc},E}$ whenever $f\in BMO$. Let V be defined as $Vf(x) = \|Uf(x)\|_E$, $f\in \mathrm{Dom}(U)$. Then, if U maps BMO*

into BMO_E and satisfies $\|Uf\|_{BMO_E} \le C\|f\|_{BMO}$, $f \in BMO$, then V maps BMO into BMO and satisfies $\|Vf\|_{BMO} \le C\|f\|_{BMO}$, $f \in BMO$.

Proof. Due to the basic inequality $\big|\|a\|_E - \|b\|_E\big| \le \|a-b\|_E$, we have

$$\frac{1}{|B|}\int_B \big|\|Uf(x)\|_E - \|(Uf)_B\|_E\big|\,dx \le \frac{1}{|B|}\int_B \|Uf(x)-(Uf)_B\|_E\,dx,$$

and using the seminorm property mentioned after (2.1) gives the claim. □

3. The operators T_t and P_t

Since the weight function $(1+|x|)^{-d-1}$ does not belong to the Muckenhoupt class A_1, we cannot use (2.2) to apply the results of [7, Section 2] directly to BMO functions. There are, however, some arguments that may be applied.

LEMMA 3.1. *Let $f \in BMO$ and $w(x) = (1+|x|)^{-d-1}$. Then the Fourier-Hermite coefficients $a_\alpha = a_\alpha(f)$ exist and, moreover, there is an $\epsilon \ge 0$ and $C > 0$, such that*

$$|a_\alpha| = |\langle f, h_\alpha\rangle| \le C(|\alpha|+1)^\epsilon \|f\|_{L^1(w)}.$$

Proof. From the pointwise estimates of the Hermite functions (a simplified version of estimates proved by Askey and Wainger with a modification furnished by Muckenhoupt), cf. [7, p. 448] for the proper citation, we infer that

$$\sup_{x\in\mathbb{R}^d}\big[(1+|x|)^{d+1}|h_\alpha(x)|\big] \le C(|\alpha|+1)^\epsilon,$$

where $\epsilon = \epsilon(d)$ do not depend on α. Thus, the required estimate easily follows. □

Let $t > 0$ be fixed. We extend the action of the operators T_t and P_t on BMO by using the pointwise versions of (1.1) and (1.2) (note that the series are convergent for every $t > 0$ and $x \in \mathbb{R}^d$). The justification of the fact that $T_tf(x)$ and $P_tf(x)$ are equal, for a given $f \in BMO$, to the heat-diffusion and Poisson integrals $g(t,x)$ and $f(t,x)$ given by (1.3) is completely analogous to the justification of [7, (2.8)] and the identity preceding (2.12) in [7]. Note that the integrals in (1.3) are indeed convergent since outside the diagonal Δ the kernels $G_t(x,y)$ and $P_t(x,y)$ satisfy

$$G_t(x,y) \le C|x-y|^{-d-1}, \qquad P_t(x,y) \le C|x-y|^{-d-1}, \qquad |x-y| \ge 1. \tag{3.1}$$

The first estimate above is a consequence of $G_t(x,y) \le W_t(x-y)$, see [7, (2.9)] for an explanation, while the second one follows from the first by using the subordination identity (1.5).

LEMMA 3.2. *Let $f \in BMO$. Then the heat-diffusion and Poisson integrals of f, $g(t,x)$ and $f(t,x)$, are C^∞ functions on $\mathbb{R}_+ \times \mathbb{R}^d$ satisfying the differential equations*

$$(L_x + \frac{\partial}{\partial t})g(t,x) = 0, \qquad (-L_x + \frac{\partial^2}{\partial t^2})f(t,x) = 0.$$

Proof. To prove that $g(t,x)$ is C^∞, we repeat the argument from the proof of [7, Proposition 2.5]. To show that $f(t,x)$ is C^∞, we slightly simplify the argument from the proof of [7, Proposition 2.7] by observing that $-L_x + \partial_t^2$ is hypoelliptic. This property together with the simply proved fact that $f(t,x)$ is a C^2 function shows that this function is also C^∞. □

Since T_t and P_t are contractions on L^∞, cf. [7, Remark 2.10], we also have $\|T_t f\|_{BMO} \le \|f\|_\infty$ and $\|P_t f\|_{BMO} \le \|f\|_\infty$. It is however hopeless to expect extending these inequalities onto BMO as the following result shows.

PROPOSITION 3.3. *Given $t > 0$ we have*

$$T_t 1(x) = (2\pi \cosh(2t))^{-d/2} \exp\left(-\frac{1}{2}\tanh(2t)|x|^2\right)$$

and

$$P_t 1(x) = \frac{t}{\sqrt{4\pi}} \int_0^\infty (2\pi\cosh(2u))^{-d/2} \times \\ \times \exp\left(-\frac{1}{2}\tanh(2u)|x|^2\right) u^{-3/2} e^{-t^2/(4u)}\, du.$$

Thus $T_t 1(x)$ and $P_t 1(x)$ are not constant functions of the x-variable. Consequently, the inequalities $\|T_t f\|_{BMO} \le C\|f\|_{BMO}$ and $\|P_t f\|_{BMO} \le C\|f\|_{BMO}$ do not hold.

Proof. Using (1.4) we obtain

$$\begin{aligned}(2\pi\sinh(2t))^{d/2} T_t 1(x) &\\ &= \int_{\mathbb{R}^d} \exp\left(-\frac{1}{4}\left(\tanh(t)|x+y|^2 + \coth(t)|x-y|^2\right)\right) dy \\ &= \int_{\mathbb{R}^d} \exp\left(-\frac{1}{2}\left(\coth(2t)(|x|^2+|y|^2) - \frac{2}{\sinh(2t)}\langle x,y\rangle\right)\right) dy.\end{aligned}$$

But

$$\begin{aligned}\coth(2t)|y|^2 - \frac{2}{\sinh(2t)}\langle x,y\rangle &\\ &= \frac{1}{\coth(2t)}\left|\coth(2t)y - \frac{1}{\sinh(2t)}x\right|^2 - \frac{1}{\sinh(2t)\cosh(2t)}|x|^2.\end{aligned}$$

Hence the last integral equals

$$\begin{aligned}
&\exp\left(-\frac{1}{2}\left(\coth(2t)-\frac{1}{\sinh(2t)\cosh(2t)}\right)|x|^2\right)\times\\
&\qquad\times\int_{\mathbb{R}^d}\exp\left(-\frac{1}{2\coth(2t)}\left|\coth(2t)y-\frac{1}{\sinh(2t)}x\right|^2\right)dy\\
&=\exp\left(-\frac{|x|^2}{2\coth(2t)}\right)\int_{\mathbb{R}^d}\exp\left(-\frac{1}{2}|(\coth(2t))^{1/2}y|^2\right)dy\\
&=\left(\frac{4\pi}{2\coth(2t)}\right)^{d/2}\exp\left(-\frac{|x|^2}{2\coth(2t)}\right).
\end{aligned}$$

Taking into account the factor $(2\pi\sinh(2t))^{-d/2}$ gives the first identity. The second follows from the first by using the subordination identity (1.5). □

Note that the concluding sentence of Proposition 3.3 shows an essential difference between the BMO behaviour of the heat-diffusion and Poisson semigroups in the Hermite function expansion setting and the classic (Euclidean) setting. This is because $W_t * 1 = 1$ and $\|W_t * f\|_{BMO} \le C\|f\|_{BMO}$, and thus $\|W_t * f\|_{\mathbf{BMO}} \le C\|f\|_{\mathbf{BMO}}$, and the same remains valid for the convolution with the (Euclidean) Poisson kernel $P_t(x)$.

4. Maximal functions

The maximal operators T^* and P^* given by (1.6) are well defined for BMO functions since the heat diffusion and Poisson integrals of any $f \in BMO$ are well defined by means of (1.3). However, in order to say something on the action of the non-linear operators T^* and P^* on the BMO space, it is necessary to linearize the situation by considering instead of T^* and P^* the vector valued linear operators

$$\mathcal{T}f(x) = \Big\{\int_{\mathbb{R}^d} G_t(x,y)f(y)dy\Big\}_{t\in\mathbb{Q}^+} \tag{4.1}$$

and

$$\mathcal{P}f(x) = \Big\{\int_{\mathbb{R}^d} P_t(x,y)f(y)dy\Big\}_{t\in\mathbb{Q}^+}. \tag{4.2}$$

The expressions on the right of (4.1) and (4.2) are considered as functions of $t \in \mathbb{Q}^+$. (4.1) and (4.2) make sense for any $f \in BMO$ as well as for any $f \in L^q(w)$, $1 \le q < \infty$, $w \in A_q$. Restricted to functions $f \in L^\infty$, the formulas (4.1) and (4.2) define operators acting boundedly from L^∞ into $L^\infty_{\ell^\infty}$ (coordinates in ℓ^∞ are indexed by $\mathbb{Q}^+$).

We now prove that $\mathcal{T}$ and $\mathcal{P}$ are vector valued Calderón-Zygmund operators with the associated vector valued kernels

$$\mathcal{T}(x,y) = \Big\{G_t(x,y)\Big\}_{t\in\mathbb{Q}^+}, \qquad \mathcal{P}(x,y) = \Big\{P_t(x,y)\Big\}_{t\in\mathbb{Q}^+}.$$

THEOREM 4.1. *The operator $\mathcal{T}$, initially considered as a bounded operator from L^∞ into $L^\infty_{\ell^\infty}$, is a Calderón-Zygmund operator with the associated kernel $\mathcal{T}(x,y)$ that satisfies*

$$\|\mathcal{T}(x,y)\|_{\ell^\infty} \le \frac{C}{|x-y|^d}, \qquad x \ne y, \tag{4.3}$$

and

$$\|\nabla_x \mathcal{T}(x,y)\|_{\ell^\infty} + \|\nabla_y \mathcal{T}(x,y)\|_{\ell^\infty} \le \frac{C}{|x-y|^{d+1}}, \qquad x \ne y. \tag{4.4}$$

The analogous conclusions and estimates hold for $\mathcal{P}$ and $\mathcal{P}(x,y)$.

Proof. The size estimate (4.3) follows since

$$\begin{aligned}
&\|\mathcal{T}(x,y)\|_{\ell^\infty} \\
&\qquad = \sup_{t\in\mathbb{Q}^+} |G_t(x,y)| \\
&\qquad \le C \sup_{t\in\mathbb{Q}^+} \Bigg[(\sinh(t)\cosh(t))^{-d/2} \times \\
&\qquad\qquad \times \exp\left(-\frac{1}{4}\coth(t)|x-y|^2\right)\Bigg] \\
&\qquad \le \frac{C}{|x-y|^d} \sup_{t>0} \Bigg[(\cosh(t))^{-d} (\coth(t)|x-y|^2)^{d/2} \times \\
&\qquad\qquad \times \exp\left(-\frac{1}{4}\coth(t)|x-y|^2\right)\Bigg] \\
&\qquad \le \frac{C}{|x-y|^d} \sup_{t>0} \left[(\cosh(t))^{-d}\right] \\
&\qquad \le \frac{C}{|x-y|^d}.
\end{aligned}$$

Note that for $t>0$ and $D>d$ we have $(\cosh(t))^{-d} \le (\cosh(t))^{-D}$. Therefore we also have for every $D>d$,

$$\|\mathcal{T}(x,y)\| \le C_D |x-y|^{-D}, \qquad |x-y| \ge 1.$$

For the smoothness estimate (4.4) (note that (4.4) implies (2.5) and (2.6)) it is sufficient to obtain the estimate

$$\left|\partial_{x_i} G_t(x,y)\right| \le C|x-y|^{-d-1}, \qquad i=1,\dots,d,$$

with C independent of $t>0$, since the corresponding bound with ∂_{x_i} replaced by ∂_{y_i} follows by the symmetry of $G_t(x,y)$ in x and y. The task is accomplished by writing down the explicit form of $\partial_{x_i} G_t(x,y)$ and applying arguments similar to those from the proof of (4.4); see also the proof of [7, Proposition 3.1].

To verify that the kernel $\mathcal{T}(x,y)$ is associated to $\mathcal{T}$ it is sufficient to check that for a given $f \in L^\infty$ with compact support and for a.e. $x \notin \operatorname{supp} f$,

$$\Big\{ \int_{\mathbb{R}^d} G_t(x,y) f(y)\,dy \Big\}_{t\in\mathbb{Q}^+} = \int_{\mathbb{R}^d} \big\{ G_t(x,y) \big\}_{t\in\mathbb{Q}^+} f(y)\,dy. \tag{4.5}$$

(Note that (4.3) together with the assumptions on f and x guarantee that the integral on the right does define an element of ℓ^∞.) (4.5) is understood as an equality of two elements from ℓ^∞. Hence it should hold for any $t \in \mathbb{Q}^+$. For a given $t_o \in \mathbb{Q}^+$ the right side of (4.5) at $t = t_o$ equals the value of the functional $\delta_{t_o} \in (\ell^\infty)^*$ (δ_{t_o} is understood as an element of ℓ^1) applied to the right side of (4.5). By the well know property of the Bochner integral we have

$$\begin{aligned} \Big\langle \delta_{t_o}, \int_{\mathbb{R}^d} \big\{ G_t(x,y) \big\}_{t\in\mathbb{Q}^+} f(y)\,dy \Big\rangle &= \int_{\mathbb{R}^d} \langle \delta_{t_o}, \big\{ G_t(x,y) f(y) \big\}_{t\in\mathbb{Q}^+} \rangle\,dy \\ &= \int_{\mathbb{R}^d} G_{t_o}(x,y) f(y)\,dy, \end{aligned}$$

which is the left side of (4.5) at $t = t_o$. Finally we conclude that the estimates for $\mathcal{P}$ are rather straightforward consequences of those for $\mathcal{T}$ and the subordination principle given by (1.5). □

Since T_t and P_t are contractions on L^∞, we also have $\|\mathcal{T} f\|_{L^\infty_{\ell^\infty}} \le \|f\|_\infty$ which implies $\|\mathcal{T} f\|_{BMO_{\ell^\infty}} \le \|f\|_\infty$ and similarly for $\mathcal{P}$. On the other hand we have:

Proposition 4.2. *$\mathcal{T}1$ is not a constant function. Consequently, the inequality*

$$\|\mathcal{T} f\|_{BMO_{\ell^\infty}} \le C\|f\|_{BMO}$$

does not hold. The analogous statements are true for $\mathcal{P}$ replacing $\mathcal{T}$.

Proof. It follows from Proposition 3.3 that $\{T_t 1(x)\}_{t\in\mathbb{Q}^+}$ and $\{P_t 1(x)\}_{t\in\mathbb{Q}^+}$ treated as elements of ℓ^∞ depend on $x \in \mathbb{R}$. □

Theorem 4.3. *For every $f \in BMO$, $T^* f(x)$ is finite x-a.e. The analogous statement is true for P^* replacing T^*.*

Proof. By (1.4),

$$\begin{aligned} G_t(x,y) &\le \big(2\pi \sinh(2t)\big)^{-d/2} \exp\left(-\frac{1}{4} \coth(t) |x-y|^2 \right) \\ &= \big(\cosh(t)\big)^{-d/2} W_{\tanh(t)}(x-y). \end{aligned}$$

It is therefore clear that

$$T^* f(x) = \sup_{t>0} |T_t f(x)| \le \sup_{0<s<1} W_s * |f|(x). \tag{4.6}$$

However, if $f \in BMO$, then, in particular, $f \in L^1_{\mathrm{loc}}$; hence

$$\lim_{s\to 0^+} W_s * |f|(x) = |f(x)|, \qquad x - \text{a.e.}$$

This and the fact that $s \mapsto W_s * |f|(x)$ is continuous on $(0,1]$ shows that $\sup_{0<s<1} W_s * |f|(x) < \infty$, x-a.e. The fact that $P^* f(x)$ is finite x-a.e. for any $f \in BMO$ is an immediate consequence of the same fact for T^* and the subordination principle represented by (1.5). $\square$

To indicate that the situation described above greatly differs from the classic (Euclidean) setting consider the maximal operator $\Phi^* f(x) = \sup_{t>0} |f * \varphi_t(x)|$, where φ is a function on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} \varphi(y)dy \neq 0$, $|\varphi(x)| \leq C(1+|x|)^{-d-1}$ (then $f * \varphi_t(x)$ is well defined for every $f \in BMO$ and every $x \in \mathbb{R}^d$) and $\varphi_t(x) = t^{-d}\varphi(x/t)$; for instance, one can take $\varphi(x) = W_1(x)$ or $\varphi(x) = P_1(x)$. Taking $f(x) = \log|x|$ produces $\Phi^* f(x) = \infty$ for every x. Indeed,

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{1}{t^d}\varphi\left(\frac{x-y}{t}\right)\log|y|\,dy &= \int_{\mathbb{R}^d} \varphi\left(\frac{x}{t}-u\right)\log|tu|\,du \\ &= \log t \int_{\mathbb{R}^d} \varphi\left(\frac{x}{t}-u\right)du + \int_{\mathbb{R}^d} \varphi\left(\frac{x}{t}-u\right)\log|u|\,du.\end{aligned}$$

It is now clear that $\lim_{t\to\infty} \varphi_t * f(x) = \infty$.

Finally, it is perhaps interesting to note that T^*1 is a constant function. This is because

$$\sup_{t>0} |T_t 1(x)| = \sup_{t>0}\left[(2\pi\cosh(2t))^{-d/2}\exp\left(-\frac{1}{2}\tanh(2t)|x|^2\right)\right] = (2\pi)^{-d/2},$$

since, as a calculation shows, the expression in brackets is, as a function of $t>0$, decreasing on $(0,\infty)$.

5. g-functions

The square function operators $\tilde{g}$, g and g_∇ given by (1.7) and (1.8) are well defined for BMO functions since the heat diffusion and Poisson integrals are well defined for any $f \in BMO$ by means of (1.3) and are smooth by Lemma 3.2. As we mentioned is not clear, however, whether, for instance, $\tilde{g}(f)(x)$ is finite x-a.e. for every $f \in BMO$ or even for every $f \in L^\infty$. In order to answer this question and to say something more on the action of these non-linear operators on the L^∞ space we linearize the situation considering the vector valued linear operators $f \to \tilde{G}(f)$, $f \to G(f)$ and $f \to G_\nabla(f)$, cf. [8, (2.1), (3.1), (4.1)].

To focus the attention we consider the case of $\tilde{G}$ only but formulate the result for G and G_∇ as well. Recall, [8], that

$$\tilde{G}(f)(x) = \left\{\frac{\partial}{\partial t} g(t,x)\right\}_{t>0}, \qquad x \in \mathbb{R}^d, \tag{5.1}$$

where $g(t,x)$ is the heat diffusion integral of f given by (1.3). The expression on the right of (5.1) is considered as a function of $t>0$; thus $\tilde{G}$ is a linear vector valued operator. (5.1) makes sense for any $f \in BMO$ as well as for any $f \in L^q(w)$, $1 \le q < \infty$, $w \in A_q$.

Specialized to functions $f \in L^2$, the formula (5.1) defines an operator acting boundedly from L^2 into $L^2_{L^2(tdt)}$. It was shown in [8, Proposition 3.1] that $\tilde{G}$ is a vector valued Calderón-Zygmund operator with the associated kernel

$$\Big\{\frac{\partial}{\partial t} G_t(x,y)\Big\}_{t>0}$$

that, apart from satisfying the standard Calderón-Zygmund conditions (2.4), (2.5) and (2.6), also satisfies the additional condition (2.7) of better decay outside the diagonal; cf. [8, Proposition 2.1] and a remark at the end of Section 3 of [8] (see also [8, Propositions 3.1 and 4.1] concerning G and G_∇). By the general theory, for every given $1 \le q < \infty$ and $w \in A_q$, $\tilde{G}$ then extends to a bounded operator acting on $L^q(w)$, and, as may be easily shown, this extension agrees with (5.1). On the other hand, the same operator $\tilde{G}$, still treated as a bounded operator from L^2 into $L^2_{L^2(tdt)}$, gives rise to an operator $(\tilde{G})\hat{}$ acting on BMO by means of Proposition 2.1. We now show that the action of $(\tilde{G})\hat{}$ on BMO functions also agrees with (5.1).

Proposition 5.1. *Let $(\tilde{G})\hat{}$ be the operator defined on BMO from the operator $\tilde{G} : L^2 \to L^2_{L^2(tdt)}$ by means of Proposition 2.1. Then, for every $f \in BMO$, $(\tilde{G})\hat{}(f) = \tilde{G}(f)$, where the latter $\tilde{G}(f)$ is given by (5.1). The analogous statements are true for G and G_∇ (in the case of G_∇ the space $L^2(tdt)$ has to be replaced by $\prod_{j=1}^{2d+1} L^2(tdt)$).*

Proof. Given $f \in BMO$ and the ball $B = 2^n B(0,1)$ it is sufficient to check that

$$\tilde{G}(f\chi_{2B})(x) + \int_{(2B)^c} \Big\{\frac{\partial}{\partial t} G_t(x,y)\Big\}_{t>0} f(y)\,dy \tag{5.2}$$

agrees x-a.e. on B with

$$\Big\{\frac{\partial}{\partial t}\int_{\mathbb{R}^d} G_t(x,y) f(y)\,dy\Big\}_{t>0}. \tag{5.3}$$

Since $f\chi_{2B} \in L^2$, the first term in (5.2) equals

$$\Big\{\frac{\partial}{\partial t}\int_{2B} G_t(x,y) f(y)\,dy\Big\}_{t>0}.$$

The integral in (5.3) may be split onto $2B$ and $(2B)^c$. Hence our task reduces to proving that

$$\int_{(2B)^c} \Big\{\frac{\partial}{\partial t} G_t(x,y)\Big\}_{t>0} f(y)\,dy = \Big\{\frac{\partial}{\partial t}\int_{(2B)^c} G_t(x,y) f(y)\,dy\Big\}_{t>0} \tag{5.4}$$

x-a.e. on B. We first explain that the left side of (5.4) equals

$$\Big\{ \int_{(2B)^c} \frac{\partial}{\partial t} G_t(x,y) f(y)\, dy \Big\}_{t>0}.$$

Indeed, given $x \in B$, to simplify the notation we let $F(t,y) = \frac{\partial}{\partial t} G_t(x,y) f(y)$ and $A = (2B)^c$. We encounter the following situation: (a) $F(t,y)$ is measurable on the product $(0,\infty) \times A$; (b) for a.e. $t \in (0,\infty)$, $\int_A |F(t,y)|\, dy < \infty$; (c) $\int_A \|F(t,y)\|_{L^2(tdt)} dy < \infty$. We now claim that the Bochner integral $\int_A F(t,y)\, dy$, as an element of $L^2(tdt)$, agrees with the function $t \mapsto \int_A F(t,y)\, dy$, where the last integral is the Lebesgue integral. To prove the claim take an arbitrary $g \in L^2(tdt)$ and by using properties of the Bochner integral write

$$\Big\langle \int_A F(\cdot, y)\, dy, g \Big\rangle_{L^2(tdt)} = \int_A \int_0^\infty F(t,y) \overline{g(t)}\, tdt\, dy.$$

On the other hand,

$$\Big\langle t \mapsto \int_A F(t,y)\, dy, g(t) \Big\rangle_{L^2(tdt)} = \int_0^\infty \int_A F(t,y)\, dy\, \overline{g(t)}\, tdt.$$

Since

$$\int_A \int_0^\infty |F(t,y) g(t)|\, tdt\, dy \le \int_A \|F(\cdot,y)\|_{L^2(tdt)} \|g\|_{L^2(tdt)}\, dy < \infty,$$

Fubini's theorem applies and our claim finally follows.

It now remains to verify that

$$\Big\{ \int_{(2B)^c} \frac{\partial}{\partial t} G_t(x,y) f(y)\, dy \Big\}_{t>0} = \Big\{ \frac{\partial}{\partial t} \int_{(2B)^c} G_t(x,y) f(y)\, dy \Big\}_{t>0} \tag{5.5}$$

x-a.e. on B. In fact, we will prove that (5.5) holds for every fixed $x \in B$ and $t_o > 0$. This will be achieved by showing that the function

$$F(y) = F_{x,t_o,\varepsilon}(y) = \left(\sup_{|t-t_o|<\varepsilon} \left| \frac{\partial}{\partial t} G_t(x,y) \right| \right) f(y)$$

is integrable on $(2B)^c$ (we choose ε to be sufficiently small, say $\varepsilon \le t_o/2$). Then (5.5) easily follows by using the dominated convergence theorem.

A simple differentiation performed in (1.4) yields

$$\begin{aligned} \frac{\partial}{\partial t} G_t(x,y) = (\sinh(2t))^{-d/2} \times \\ \times \exp\left(-\frac{1}{4} \left(\tanh(t)|x+y|^2 + \coth(t)|x-y|^2 \right) \right) \times \\ \times \left(-d \coth(2t) - \frac{1}{4\cosh^2 t} |x+y|^2 - \frac{1}{4 \sinh^2 t} |x-y|^2 \right). \end{aligned} \tag{5.6}$$

Using this we will show that for $x \in B$ and $t_o > 0$ fixed and $\varepsilon = t_o/2$,

$$\sup_{|t-t_o|<t_o/2} \Big|\frac{\partial}{\partial t} G_t(x,y)\Big| \le C_{B,t_o}(|y|+1)^{-d-1}, \qquad y \in (2B)^c, \tag{5.7}$$

which is sufficient for our purposes since $\int_{(2B)^c} |f(y)|(|y|+1)^{-d-1}dy < \infty$.

Proving (5.7) we split the right side of (5.6) into three summands (according to the three terms in the last factor in (5.6)) and denote them by I_1, I_2 and I_3. Then we estimate each of them separately. For $x \in B$ and $|t-t_o| < t_o/2$ we have

$$|I_1| \le C(t_o)\exp(-D(t_o)|x-y|^2),$$

with $C(t_o) > 0$ and $D(t_o) > 0$, which is sufficient to get (5.7). The same estimate follows for $|I_2|$ by taking into account the fact that

$$\big(\tanh(t)|x+y|^2\big)\exp\left(-\frac{1}{4}\tanh(t)|x+y|^2\right) \le C.$$

The analogous estimate of $|I_3|$ also easily follows, which finishes the proof of (5.7).

To prove the analogous result for G and G_∇ a quite similar argument is used. The proof of Proposition 5.1 is completed. □

Proposition 5.2. *Let $\tilde{G}$ be defined by* (5.1). *Then*

$$\|\tilde{G}(f)\|_{BMO_{L^2(tdt)}} \le C\|f\|_\infty, \qquad f \in L^\infty.$$

The analogous statements are true for G and G_∇ (in the case of G_∇ the space $L^2(tdt)$ has to be replaced by $\prod_{j=1}^{2d+1} L^2(tdt)$).

Proof. This is a direct consequence of Proposition 5.1 and the conclusions of Proposition 2.1. A similar argument applies for G and G_∇. □

Proposition 5.3. *$\tilde{G}(1)$ given by* (5.1) *is not a constant function. In consequence, the inequality*

$$\|\tilde{G}(f)\|_{BMO_{L^2(tdt)}} \le C\|f\|_{BMO}$$

does not hold. The analogous statements are true for G and G_∇ replacing $\tilde{G}$.

Proof. A simple differentiation based on the identities from Lemma 3.3 shows that $\frac{\partial}{\partial t}T_t1(x)$ and $\frac{\partial}{\partial t}P_t1(x)$ are, respectively, equal to

$$-(2\pi\cosh(2t))^{-d/2}\left(d\tanh(2t) + \frac{|x|^2}{\cosh^2(2t)}\right)\exp\left(-\frac{1}{2}\tanh(2t)|x|^2\right)$$

and

$$\frac{1}{t}P_t1(x) - \frac{t^2}{4\sqrt{\pi}}\int_0^\infty (2\pi\cosh(2u))^{-d/2}\exp\left(-\frac{1}{2}\tanh(2u)|x|^2\right)u^{-5/2}e^{-t^2/4u}\,du.$$

Consequently, neither

$$\mathbb{R}^d \ni x \longmapsto \Big\{\frac{\partial}{\partial t} T_t 1(x)\Big\}_{t>0} \in BMO_{L^2(tdt)}$$

nor

$$\mathbb{R}^d \ni x \longmapsto \Big\{\frac{\partial}{\partial t} P_t 1(x)\Big\}_{t>0} \in BMO_{L^2(tdt)}$$

is a constant function. Since the last mapping is one of the coordinates of G_∇, the same conclusion is valid for $G_\nabla(1)$. □

THEOREM 5.4. *For every $f \in BMO$, $\tilde{g}(f)(x)$ is finite x-a.e. Moreover, the* (*nonlinear*) *square function operator $\tilde{g}$ defined by* (1.7) *maps L^∞ into BMO and satisfies*

$$\|\tilde{g}(f)\|_{BMO} \le C\|f\|_\infty.$$

However, the inequality

$$\|\tilde{g}(f)\|_{BMO} \le C\|f\|_{BMO} \tag{5.8}$$

does not hold. The analogous statements are true for g and g_∇ replacing $\tilde{g}$.

Proof. For the first statement note that $\tilde{g}(f)(x) = \|\tilde{G}(f)(x)\|_{L^2(tdt)}$ and, by Proposition 2.1, $(\tilde{G})\hat{}(f)(x)$ is an x-a.e. well defined element of $L^2(tdt)$. Using the just proved identity $(\tilde{G})\hat{}(f)(x) = \tilde{G}(f)(x)$, which holds x-a.e., shows that $\tilde{g}(f)(x) < \infty$ for almost every x. For the second statement note that the identity $\tilde{g}(f)(x) = \|\tilde{G}(f)(x)\|_{L^2(tdt)}$ and Proposition 5.2 prove the claim. (The argument from the proof of Proposition 2.3 is also helpful here.) On the other hand, a careful analysis based on the closed expression on $\partial_t T_t 1(x)$ (see the proof of Proposition 4.2) shows that

$$\int_0^\infty \big|\partial_t T_t 1(x)\big|^2\, tdt$$

is a decreasing function of $|x| \to \infty$. Thus $\tilde{g}(1)$ is not a constant function. Hence (5.8) does not hold.

Similar arguments prove the result for g and g_∇. □

Finally, we take the opportunity to provide a simple proof of the fact that for $h(x) = \log|x| \in BMO$ we have $g(h) \equiv \infty$, where by g we mean the classic g-function based on the Gauss-Weierstrass (or the Poisson) kernel (Wang [10] proved that for the classic full gradient g-function g_∇ there is a function $f \in L^\infty$ such that $g_\nabla(f) \equiv \infty$). Actually, we prove something more. Take a C^1 function φ on $\mathbb{R}^d$ such that $|\varphi(x)| \le C(1+|x|)^{-d-1}$, $|\nabla\varphi(x)| \le C(1+|x|)^{-d-1}$, $\int_{\mathbb{R}^d} \varphi(x)\, dx = 1$; for instance one can take $\varphi(x) = W_1(x)$ or $\varphi(x) = P_1(x)$. A calculation performed at the end of the previous section then shows that we have

$$\frac{\partial}{\partial t}\big(\varphi_t * h(x)\big) = \frac{1}{t} - \frac{1}{t^2}\sum_{i=1}^d x_i \int_{\mathbb{R}^d} \partial_{x_i}\varphi\Big(\frac{x}{t} - u\Big)\log|u|\, du.$$

Hence, for fixed $x \in \mathbb{R}^d$,

$$\Big|\frac{\partial}{\partial t}\big(\varphi_t * h(x)\big)\Big| \geq \frac{C}{t}, \qquad t > 1.$$

Thus, for the square function operator defined by

$$g_\varphi(f)(x) = \left(\int_0^\infty \Big|\frac{\partial}{\partial t}\big(\varphi_t * f(x)\big)\Big|^2 t dt\right)^{1/2},$$

we obtain $g_\varphi(h)(x) = \infty$ for every x. Taking $\varphi = W_1$ or $\varphi = P_1$ the claim follows. Since we have $g(h)(x) \leq g_\nabla(h)(x)$, the same conclusion also follows for g_∇.

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