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SOLUTION TO RUBEL'S QUESTION ABOUT DIFFERENTIALLY ALGEBRAIC DEPENDENCE ON INITIAL VALUES

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Abstract. We prove that, for generic systems of polynomial differential equations, the dependence of the solution on the initial conditions is not differentially algebraic. This answers, in the negative, a question posed by L.A. Rubel.

1. Introduction

This work answers a question posed by L.A. Rubel in [8], a paper which presents many research problems on differentially algebraic functions (see also [6]). We recall that an analytic function $f(z)$ (defined on some open subset of either $\mathbb{R}$ or $\mathbb{C}$) is called *Differentially Algebraic* (DA) if it satisfies some differential equation of the form

$$Q(z, f(z), f'(z), \ldots, f^{(n)}(z)) = 0, \tag{1}$$

for all z in its domain, where Q is a nonzero polynomial of $n + 2$ variables. A function of several variables is DA if it is DA in each variable separately when fixing the other variables. It is known that a function is DA precisely when it is computable by a general-purpose analog computer [4].

Problem 31 of [8] asks:

> *Given a 'nice' initial value problem for a system of algebraic equations in the dependent variables $y_1, \ldots, y_n$, must $y_1(x_0)$ be differentially algebraic as a function of the* **initial conditions***, for each x_0?*

Rubel adds: "we won't say more about what 'nice' means except that the problem should have a unique solution for each initial condition in a suitable open set".

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In other words, let

$$y_k' = p_k(y_1, \dots, y_m), \quad k = 1, \dots, m \tag{P}$$

be a system of differential equations, with p_k polynomials. Denote by $y_k(r_1, \dots, r_m; x)$ the solutions of (P) with initial conditions

$$y_k(0) = r_k, \quad k = 1, \dots, m. \tag{2}$$

The question is whether it is true that the dependence of y_i on r_j (fixing x and the other initial conditions r_k, $k \neq j$) is DA.

Here we first answer the question in the *negative* by constructing an initial value problem

$$y_1' = y_2 - y_1, \qquad y_1(0) = r_1, \tag{3}$$

$$y_2' = y_2(y_2 - y_1), \quad y_2(0) = r_2, \tag{4}$$

and proving that, for most $x \in \mathbb{R}$, both $y_1(0, r_2; x)$ and $y_2(0, r_2; x)$ are *not* DA with respect to r_2.

Moreover, we prove that this phenomenon is in fact *generic* in the class of nonlinear polynomial differential systems (P) (and thus the answer to Rubel's question is *very* negative). To state this precisely, let $\mathbf{S}(m, d)$ denote the set of all polynomial systems (P) of size m, where the polynomials p_i are of degree at most d. By identifying the coefficients of the various monomials occurring in the p_i's with coordinates, $\mathbf{S}(m, d)$ is a finite dimensional vector-space, so it inherits the standard topology and measure.

In all our statements the term *generic* will have the following meaning: when $U \subset \mathbb{R}^N$ is an open set and $V \subset U$, we will say that V is **generic** in U if its complement in U is *both of measure* 0 *and of first Baire category.*

For each system $P \in \mathbf{S}(m, d)$ and initial conditions $r = (r_1, \dots, r_m) \in \mathbb{R}^m$, let $I(P, r)$ denote the largest interval containing 0 for which the solutions of the initial-value problem (P),(2) are defined. Let

$$\Lambda(P) = \{(x, r) \mid r \in \mathbb{R}^m,\ x \in I(P, r)\}. \tag{5}$$

($\Lambda(P)$ is the domain of definition of the functions $y_i(r_1, \dots, r_m; x)$—we note that standard existence theory of ODE's implies that it is an open set; it is also easy to see that it is connected.) For each $1 \leq j \leq m$ let

$$\Lambda_j(P) = \big\{(x, r_1, \dots, r_{j-1}, r_{j+1}, \dots, r_m) \mid \exists r_j \text{ such that } (x, r_1, \dots, r_m) \in \Lambda(P)\big\}.$$

Theorem 1. *Assume $m \geq 2$ and $d \geq 2$. For generic $P \in \mathbf{S}(m, d)$ we have: For any $1 \leq i, j \leq m$, and for a generic choice of $(x, r_1, \dots, r_{j-1}, r_{j+1}, \dots, r_m) \in \Lambda_j(P)$, the function*

$$f_{ij}(z) = y_i(r_1, \dots, r_{j-1}, z, r_{j+1}, \dots, r_m; x) \tag{6}$$

is not DA.

We note that the construction of the specific example (3)–(4) is essential for our proof of the 'generic' result of Theorem 1.

The significance of the problem posed by Rubel is brought out when we note that the class of DA functions is a very "robust" one:

(i) The class is closed under many of the constructions of analysis: algebraic operations, composition of functions, inversion, differentiation and integration.
(ii) The components of an analytic solution of a *system* of algebraic differential equations are themselves DA [5].
(iii) A solution of a differential equation $R(z, f(z), f'(z), \ldots, f^{(n)}(z)) = 0$, where R is a DA function, is itself DA [9]. We note a special case $n = 0$ of this result, which can be called 'the DA implicit-function theorem', which we shall have occasion to use later:

LEMMA 2. *Assume $R(w, z)$ is a DA function, and $f(z)$ is a real-analytic function satisfying*

$$R(f(z), z) = 0$$

for all z in some interval. Then f is DA on this interval.

For the above reasons, most of the transcendental functions which are encountered in "daily life" are DA. (Notable exceptions are the Gamma function and Riemann Zeta function; see [7].) It is then natural to wonder whether functions obtained by the construction of looking at the dependence of the "final value" on the "initial value" (in other terms, the components of the "Poincaré map") of an algebraic differential equation are also DA. Our results show that this is almost always not so.

The following remarks show that the result of Theorem 1 is in some sense close to optimal:

(i) The condition $d \geq 2$ cannot be removed. Indeed, if $d = 0$ the solutions of the system are linear functions in x, and if $d = 1$ the solutions are linear combinations of exponentials in x, so the dependence on initial conditions is certainly DA.
(ii) The condition $m \geq 2$ cannot be removed. Indeed, if $m = 1$ then we are dealing with an equation of the form

$$y' = p(y),$$

which is solved in terms of elementary functions, which implies that the dependence on the initial condition is DA.
(iii) The restriction to generic systems cannot be removed, since, for example, when, for each k, p_k depends only on y_k, we are back to a decoupled system of equations of the same form as in (ii) above, so that again we have DA dependence on initial conditions. However, it might still be possible to obtain a stronger statement as to the size

of the set of 'exceptional' systems. For example, we do not know whether the set of systems $P \in \mathbf{S}(m,d)$ for which all the functions f_{ij} are DA is *nowhere dense* in $\mathbf{S}(m,d)$.

In Section 2 we prove a general result which underlies the proof of our 'genericity' results. In Section 3 we construct some explicit examples of functions which we prove are not DA. In Section 4 we show that the functions constructed in Section 3 arise in the solution of the system (3)–(4), and this is used to prove that the solutions of this system are not DA in the initial conditions. Finally, in Section 5 we use the specific example constructed in Section 4, together with the general result of Section 2, to prove Theorem 1.

2. The Alternative Lemma

In this section we present a result which we term the "Alternative Lemma", which is used several times in the arguments of the following sections. This result says that if we have a parameterized family of analytic functions (with the dependence on the parameters also analytic), then either: (I) all the functions in the family are DA, or (II) generic functions in the family are *not* DA. Thus, to show that a *generic* function in a certain family is *not* DA, it is sufficient to prove that *one* of the functions in the family is *not* DA.

The proof of the lemma is based on two facts:

- (i) The fact that an analytic function on a connected open set cannot vanish on a 'large' subset without vanishing identically.
- (ii) The Gourin-Ritt Theorem [1] which says that any DA function $f(z)$ in fact satisfies a differential equation of the form (1), where Q is a polynomial with *integer coefficients*.

We state the next lemma in both 'real' and 'complex' forms, since we shall have occasions to use both.

Lemma 3. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and let $F : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^p \times \mathbb{K}$ is open and connected, be an analytic function. For each $u \in \mathbb{R}^p$, define*

$$\Omega(u) = \{v \in \mathbb{K} \mid (u,v) \in \Omega\},$$

and let

$$\Omega' = \{u \in \mathbb{R}^p \mid \Omega(u) \neq \emptyset\ \}.$$

Then the following alternative holds. Either:

- (I) *For every $u \in \Omega'$, $F(u,.)$ is DA on $\Omega(u)$. In fact, all the functions $F(u,.)$, $u \in \Omega'$, satisfy the same differential equation.*

Or:

- (II) *For generic $u \in \Omega'$, $F(u,.)$ is not DA on $\Omega(u)$.*

Proof. Let $\mathbf{Q}$ be the set of all polynomials with integer coefficients. $\mathbf{Q}$ is a countable set. For each $Q \in \mathbf{Q}$, let $\Omega'(Q)$ be the set of $u \in \Omega'$, for which

$f = F(u,.)$ satisfies (1) on $\Omega(u)$. We claim that either $\Omega'(Q)$ is of measure 0 or $\Omega'(Q) = \Omega'$. Assume that $\Omega'(Q)$ has positive measure in $\mathbb{R}^p$. We choose open balls $B_1 \subset \Omega'$ and $B_2 \subset \mathbb{K}$ such that (i) $B_1 \times B_2 \subset \Omega$ and (ii) $\Omega'(Q) \cap B_1$ has positive measure. Then the real-analytic function

$$g(u,v) = Q(v, F(u,v), D_v F(u,v), \ldots, D_v^{(n)} F(u,v))$$

vanishes on the set $(\Omega'(Q) \cap B_1) \times B_2$, which is of positive measure. By analyticity and the fact that Ω is connected, this implies that g vanishes throughout Ω, which means that alternative (I) holds. So if alternative (I) does not hold, $\Omega'(Q)$ must be of measure 0. Since it is easy to see that $\Omega'(Q)$ is a relatively closed set in Ω', the fact that it has measure 0 implies that it is nowhere dense. Since this is true for any $Q \in \mathbf{Q}$, the countable union

$$K = \bigcup_{Q \in \mathbf{Q}} \Omega'(Q),$$

is a set of measure 0 and first category. By the Gourin-Ritt Theorem, for all u outside K, $F(u,.)$ is *not* DA. Thus (II) holds. □

3. Some functions which are not differentially algebraic

In this section we define some new functions and prove that they are not DA. These results will be used in Section 4 in our construction of an explicit differential equation with non-DA dependence on initial conditions.

We define the function $H(c)$ (of $c \in \mathbb{C}$) by

$$H(c) = \int_0^\infty \frac{du}{ce^u - u}. \tag{7}$$

For $c \in \mathbb{C}$ to belong to the domain of definition D_H of H we need to ensure that the denominator $ce^u - u$ does not vanish for any $u \geq 0$. It is then easy to check that

$$D_H = \mathbb{C} - [0, 1/e]. \tag{8}$$

Lemma 4. *The function $H(c)$ is not DA.*

Proof. We define $h(z) = H(1/z)$. We shall show that h is *not* DA, which implies that H is not DA. We expand h in a power series (which converges for $|z| < e$):

$$h(z) = \int_0^\infty \frac{ze^{-u} du}{1 - zue^{-u}} = \int_0^\infty ze^{-u} \sum_{k=0}^\infty z^k u^k e^{-ku} du$$
$$= \sum_{k=0}^\infty z^{k+1} \int_0^\infty u^k e^{-(k+1)u} du = \sum_{k=1}^\infty \frac{(k-1)!}{k^k} z^k.$$

We now use the theorem of Sibuya and Sperber [10], which gives the following necessary condition on a power series $\sum_{k=0}^{\infty} a_k z^k$ with rational coefficients in order for it to satisfy an algebraic differential equation:

$$|a_k|_p \leq e^{Ck} \text{ for all } k > 0, \tag{9}$$

where $|a|_p$ (p prime) is the p-adic valuation defined by writing $a = p^i \frac{m}{n}$ (m, n not divisible by p) and setting $|a|_p = p^{-i}$. Here we use this necessary condition with $p = 2$. We have $a_k = (k-1)!/(k^k)$. We now choose $k = 2^j$. The highest power of 2 dividing $(k-1)!$ is less than

$$\frac{k-1}{2} + \frac{k-1}{4} + \frac{k-1}{8} + \cdots \leq k - 1 = 2^j - 1.$$

On the other hand, the denominator of a_k is 2^{j2^j}. These two facts imply that

$$|a_{2^j}|_2 \geq 2^{(j-1)2^j+1},$$

which together with (9) implies

$$2^{(j-1)2^j+1} \leq |a_{2^j}|_2 \leq e^{C2^j},$$

or, taking logs,

$$((j-1)2^j + 1)\log(2) \leq C2^j,$$

which is obviously false for j sufficiently large. Hence the necessary condition of Sibuya and Sperber is not satisfied for our series, completing the proof. □

We note a few properties of the restriction of H to the real line (which is defined outside the interval $[0, 1/e]$), obtained by elementary arguments. See Figure 1 for the graph of H, plotted using MAPLE.

(i) H is decreasing on $(1/e, \infty)$, $\lim_{c\to 1/e+} H(c) = +\infty$, $\lim_{c\to\infty} H(c) = 0$.
(ii) H is decreasing on $(-\infty, 0)$, $\lim_{c\to-\infty} H(c) = 0$, $\lim_{c\to 0-} H(c) = -\infty$.

We now define the function $F(s, c)$, where $s \in \mathbb{R}$ is considered a parameter and $c \in \mathbb{C}$, by

$$F(s, c) = \int_0^s \frac{du}{ce^u - u}. \tag{10}$$

Since for $F(s, c)$ to be defined we need to ensure that the denominator does not vanish for $u \in [0, s]$, it can be checked that the domain of definition of F is

$$D_F = \{(s, c) \mid s \in \mathbb{R},\ c \in \mathbb{C} - J(s)\},$$

where

$$\begin{aligned} J(s) &= [se^{-s}, 0] && \text{for } s < 0, \\ J(s) &= [0, se^{-s}] && \text{for } 0 \leq s < 1, \\ J(s) &= [0, 1/e] && \text{for } s \geq 1. \end{aligned}$$

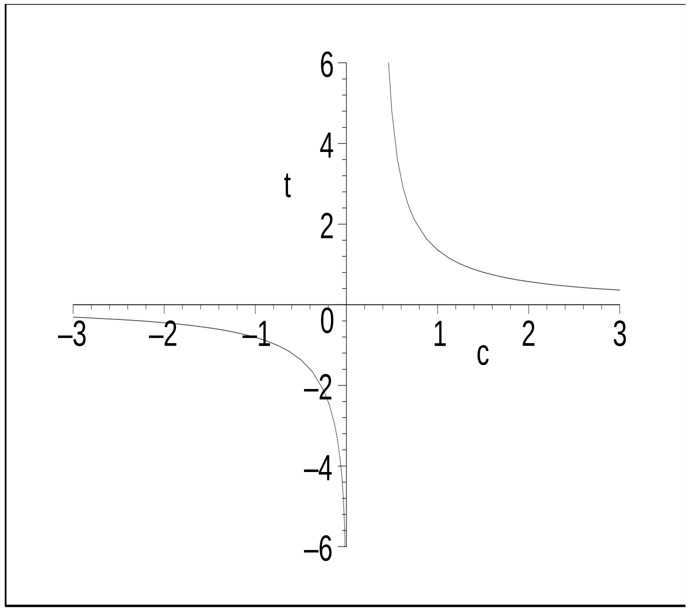

FIGURE 1. Graph of the function $H(c)$ defined by (7)

Using the fact that H is *not* DA (Lemma 4), we now prove that the function $F(s,.)$ is *not* DA for generic s. The argument is similar to the one in [2] showing that the function $z \to \int_a^b u^z e^{-u} du$ is not DA for generic a, b by using the fact that the Gamma function is not DA.

LEMMA 5. *For generic $s \in \mathbb{R}$, the function $F(s,.)$ is not DA.*

Proof. We apply Lemma 3 with $p = 1$, $\mathbb{K} = \mathbb{C}$, $\Omega = D_F \subset \mathbb{R} \times \mathbb{C}$ (which is indeed open and connected). We have $\Omega' = \mathbb{R}$. We want to show that alternative (II) of Lemma 3 holds. Let us assume by way of contradiction that (I) holds, so that $F(s,.)$ is DA for all $s \in \mathbb{R}$, and in fact there is a common polynomial differential equation satisfied by all $F(s,.)$:

$$Q(c, F(s,c), D_c F(s,c), \ldots, D_c^{(n)} F(s,c)) = 0, \quad \text{for all } (s,c) \in D_F. \tag{11}$$

We now note that

$$\lim_{s \to \infty} F(s,c) = H(c)$$

uniformly on compact subsets of D_H. Hence the derivatives of $D_c^{(k)} F(s,c)$ also converge to corresponding derivatives of H, and thus from (11) it follows that

$$Q(c, H(c), H'(c), \ldots, H^{(n)}(c)) = 0$$

for all $c \in D_H$. Hence H is DA, in contradiction with Lemma 4. This contradiction proves that alternative (II) holds, as we wanted to show. $\square$

One might guess that the 'exceptional' set in Lemma 5 is $\{0\}$, so that in fact $F(s,.)$ is *not* DA for *all* $s \neq 0$, but we do not know how to prove this.

We now restrict c to be a real nonzero number, and we define a new function $G(t,c)$ by the relation

$$G(F(s,c),c) = s \;\text{ for all } (s,c) \in D_F. \tag{12}$$

In other words, we now look at c as a parameter and define $G(.,c)$ as the inverse function of $F(.,c)$. To see that G is well-defined and to determine its domain of definition, we note the following properties of F, which are elementary to verify.

LEMMA 6. *For each $c \leq 1/e$ let $s^*(c)$ denote the solution (in the case $0 < c < 1/e$, the smaller solution) of the equation $se^{-s} = c$. We have:*

(i) *When $c < 0$, the function $F(.,c)$ is decreasing on $(s^*(c), \infty)$, with $\lim_{s\to s^*(c)+} F(s,c) = +\infty$ and $\lim_{s\to\infty} F(s,c) = H(c) < 0$. Hence $G(.,c)$ is defined on $(H(c), +\infty)$.*

(ii) *When $0 < c \leq 1/e$, the function $F(.,c)$ is increasing on $(-\infty, s^*(c))$, with $\lim_{s\to-\infty} F(s,c) = -\infty$ and $\lim_{s\to s^*(c)-} F(s,c) = +\infty$. Hence $G(.,c)$ is defined on $(-\infty,\infty)$.*

(iii) *When $c > 1/e$, the function $F(.,c)$ is increasing on $(-\infty,\infty)$, with $\lim_{s\to-\infty} F(s,c) = -\infty$ and $\lim_{s\to\infty} F(s,c) = H(c) > 0$. Hence $G(.,c)$ is defined on $(-\infty, H(c))$.*

Putting these facts together, we obtain that the domain of definition D_G of G is

$$D_G = D_G^- \cup D_G^+,$$

where

$$D_G^- = \{(t,c) \mid c < 0,\; H(c) < t < \infty\},$$
$$D_G^+ = \{(t,c) \mid 0 < c \leq 1/e\} \cup \{(t,c) \mid c > 1/e,\; -\infty < t < H(c)\}.$$

(In Figure 1, D_G^- is the domain bounded by the left part of the graph of H and the t-axis, and D_G^+ is the domain bounded by the right part of the graph and the t-axis.)

We also note a fact that seems hard to prove directly, but which follows indirectly from the results of the next section, as will be pointed out.

LEMMA 7. *The function G can be continued as a real-analytic function to the open connected domain*

$$D = D_G^- \cup D_G^+ \cup \{(t,0) \mid t \in \mathbb{R}\} \tag{13}$$

by setting

$$G(t,0)=0 \ \textit{ for all } t\in\mathbb{R}. \tag{14}$$

(In Figure 1, D is the domain bounded between the two parts of the curve representing the graph of H.)

LEMMA 8. *For generic $t\in\mathbb{R}$, the function $G(t,.)$ is not DA.*

Proof. Assume by way of contradiction that the conclusion of the lemma is not true, so that $G(t,.)$ is DA for a set of values of t which is either of positive measure or not of first category. Then, applying Lemma 3, with $\Omega=D$ (recall (13)), we conclude that $G(t,.)$ is DA for *all* $t\in\mathbb{R}$. We note also that $G(.,c)$ is DA for all values of c, since for $c\neq 0$ it is the inverse of $F(.,c)$ which is DA, and the inverse of a DA function is DA, while for $c=0$ it is identically 0 by (14). Thus G is DA with respect to both variables on Ω. Now fixing any $\overline{s}\in\mathbb{R}$, and using the defining relation (12) we have

$$G(F(\overline{s},c),c)-\overline{s}=0$$

for all c. We now use Lemma 2, with $R(w,z)=G(w,z)-\overline{s}$, to conclude that $F(\overline{s},.)$ is DA. Since $\overline{s}\in\mathbb{R}$ was arbitrary, we have that $F(s,.)$ is DA for all $s\in\mathbb{R}$, but this contradicts the result of Lemma 5. This contradiction concludes the proof. $\square$

4. Construction of a differential equation with non-DA dependence on initial conditions

THEOREM 9. *Consider the initial value problem*

$$y_1'=y_2-y_1, \qquad y_1(0)=r_1, \tag{15}$$

$$y_2'=y_2(y_2-y_1), \qquad y_2(0)=r_2. \tag{16}$$

Let the solutions be denoted by $y_1(r_1,r_2;x)$, $y_2(r_1,r_2;x)$, defined on the maximal interval $I(r_1,r_2)$. Then for generic $x\in\mathbb{R}$, the functions $y_1(0,r_2;x)$, $y_2(0,r_2;x)$ are not DA with respect to r_2.

To prove Theorem 9, we assume for the moment that $r_2\neq 0$, which implies that $y_2(x)\neq 0$ for all x (since by (16) and by the standard uniqueness theorem for ODE's, if y_2 vanishes at one point, it vanishes identically), and note that from (15) and (16) we have

$$\frac{y_2'}{y_2}=y_1',$$

which implies

$$y_2(x)=y_2(0)e^{(y_1(x)-y_1(0))},$$

and in the case $y_1(0)=r_1=0$, $y_2(0)=r_2$,

$$y_2(x)=r_2e^{y_1(x)}. \tag{17}$$

Substituting (17) back into (15), we have

$$y_1'(x) = r_2 e^{y_1(x)} - y_1(x),$$

or

$$\frac{y_1'(x)}{r_2 e^{y_1(x)} - y_1(x)} = 1.$$

Thus, integrating, and recalling the function $F(s, c)$ defined by (10), we have

$$F(y_1(0, r_2; x), r_2) = x$$

for all $x \in I(0, r_2)$. In other words, recalling the definition (12) of $G(t, s)$, we have

$$y_1(0, r_2; x) = G(x, r_2) \text{ for all } r_2 \neq 0,\ x \in I(0, r_2). \tag{18}$$

We note in passing that, since $y_1(0, r_2; x)$ is real-analytic in its domain of definition, and is defined also for $r_2 = 0$, with $y_1(0, 0; x) = 0$, (18) implies that G can be extended to D as Lemma 7 claimed.

In Lemma 8 we showed that $G(t, c)$ is *not* DA with respect to c for generic t. From (18) we then get, for generic x, that $y_1(0, r_2; x)$ is *not DA* with respect to r_2. Using this fact and (17), together with the fact that compositions and products of DA function are DA, we get the same conclusion for y_2, concluding the proof of Theorem 9.

5. Proof of the Main Theorem

We first set some notation. Define $\Sigma \subset \mathbf{S}(m, d) \times \mathbb{R} \times \mathbb{R}^m$ by

$$\Sigma = \{(P, x, r) \mid P \in \mathbf{S}(m, d),\ r \in \mathbb{R}^m,\ x \in I(P, r)\}.$$

For each $1 \leq j \leq m$ let

$$\Sigma_j' = \{(P, x, r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_m) \mid \exists r_j \text{ such that } (P, x, r) \in \Sigma\}.$$

The proof of Theorem 1 will be based on the following lemma.

Lemma 10. *Assume $m \geq 2$, $d \geq 2$. Fix $1 \leq j \leq m$. Then for generic $(P, r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_m, x) \in \Sigma_j'$, the functions f_{ij} $(1 \leq i \leq m)$ defined by* (6) *are not DA.*

Proof. We note first that we may assume, without loss of generality and for notational convenience, that $j = m$. Define $F_k : \Sigma \to \mathbb{R}$ $(1 \leq k \leq m)$ by

$$F_k(P, x, r_1, \ldots, r_m) = y_k(r_1, \ldots, r_m; x).$$

These are real analytic functions on Σ, and we may apply Lemma 3, with $\mathbb{K} = \mathbb{R}$, $\Omega = \Sigma$, u identified with $(P, x, r_1, \ldots, r_{m-1})$, v identified with r_m, and F being any of the F_k $(1 \leq k \leq m)$. We would like to rule out alternative

(I). We now fix some $1 \leq i \leq m-1$, and note that among the systems $P \in \mathbf{S}(m,d)$ is the system

$$\begin{aligned} y_i' &= y_m - y_i, \\ y_m' &= y_m(y_m - y_i), \\ y_k' &= 0 \text{ for } k \neq i, m. \end{aligned}$$

By Theorem 9, for this system, for $r_i = 0$ and for arbitrary values of r_k ($k \neq i, m$), the dependence of $y_i(r_1, \ldots, r_m; x)$ and of $y_m(r_1, \ldots, r_m; x)$ on r_m is not DA. This rules out alternative (I) of Lemma 3 both for F_i and for F_m, which implies that alternative (II) holds for both. Since $1 \leq i \leq m-1$ is arbitrary, this means that for each $1 \leq k \leq m$ there is a generic subset Σ_k^g of Σ_m' for which the function F_k is *not* DA with respect to r_m. Thus, setting

$$\Sigma^g = \bigcap_{1 \leq k \leq m} \Sigma_k^g$$

we get a generic set, and for each $(P, x, r_1, \ldots, r_{m-1}) \in \Sigma^g$ we have the conclusion of Lemma 10. □

Finally, to derive Theorem 1 from Lemma 10, we recall two classical results (see [3]). Let $C \subset \mathbb{R}^N \times \mathbb{R}^M$, and define for each $u \in \mathbb{R}^N$

$$C(u) = \{v \in \mathbb{R}^M \mid (u, v) \in C\}.$$

We have:

(i) If the set C is of measure 0, then $C(u)$ is of measure 0 in $\mathbb{R}^M$ for all $u \in \mathbb{R}^N$ except a set of measure 0. This is (a special case of) Fubini's theorem.

(ii) If the set C is of first category, then $C(u)$ is of first category in $\mathbb{R}^M$ for all $u \in \mathbb{R}^N$ except a set of first category. This is the Banach-Kuratowski theorem.

Proof of Theorem 1. We will prove the conclusion of Theorem 1 for all $P \in \mathbf{S}(m,d)$ outside a set of measure 0, making use of Fubini's theorem. To prove the same conclusion for P outside a set of first category, one only needs to replace Fubini's theorem with the Banach-Kuratowski theorem.

For $1 \leq i, j \leq m$ define C_{ij} to be the set of $(P, x, r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_m) \in \Sigma_j'$ for which the function f_{ij} defined by (6) is DA. By Lemma 10, each of the sets C_{ij} is of measure 0 and of first category. Hence by Fubini's theorem, for all $P \in \mathbf{S}(m,d)$ except a set of measure 0, the set

$$C_{ij}(P) = \{(x, r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_m) \mid (P, x, r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_m) \in C_{ij}\}$$

has measure 0. Below we show that the sets $C_{ij}(P)$ are also of first category for any P for which they are of measure 0. Since this conclusion is valid for any $1 \le i, j \le m$, we have that for all $P \in \mathbf{S}(m,d)$ except a set of measure 0, *all* the sets $C_{ij}(P)$ have measure 0 and are of first category. This is precisely the content of Theorem 1.

It remains to show that whenever $C_{ij}(P)$ is of measure 0, it is also of first category. Without loss of generality, and for notational convenience, we assume $j = m$. We apply Lemma 3 with $\mathbb{K} = \mathbb{R}$, $\Omega = \Lambda(P)$ (recall (5)), u identified with $(x, r_1, \ldots, r_{m-1})$, v identified with r_m, and $F = f_{im}$. Since we assume that $C_{im}(P)$ is of measure 0, so that F is *not* DA with respect to r_m for almost all $(x, r_1, \ldots, r_{m-1}) \in \Lambda_m(P)$, alternative (I) of Lemma 3 certainly does not hold, so (II) holds, which means that $C_{im}(P)$ is indeed also of first category, concluding the proof. □

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