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# COMPUTING THE NORMS OF ELEMENTARY OPERATORS

RICHARD M. TIMONEY

ABSTRACT. We provide a direct proof that the Haagerup estimate on the completely bounded norm of elementary operators is best possible in the case of $\mathcal{B}(H)$ via a generalisation of a theorem of Stampfli. We show that for an elementary operator $T$ of length $\ell$, the completely bounded norm is equal to the $k$-norm for $k = \ell$. A $C^*$-algebra $A$ has the property that the completely bounded norm of every elementary operator is the $k$-norm, if and only if $A$ is either $k$-subhomogeneous or a $k$-subhomogeneous extension of an antiliminal $C^*$-algebra.

## 1. Introduction

For $A$ a $C^*$-algebra, an operator $T\colon A \to A$ is called an *elementary operator* if $T$ can be expressed in the form

$$Tx = \sum_{i=1}^{\ell} a_i x b_i \tag{1}$$

with $a_i$ and $b_i$ ($1 \le i \le \ell$) in the multiplier algebra $M(A)$ of $A$ (see [17]). A well-known estimate due to Haagerup states that

$$\|T\| \le \|T\|_{cb} \le \sqrt{\left\| \sum_{j=1}^{\ell} a_j a_j^* \right\| \left\| \sum_{j=1}^{\ell} b_j^* b_j \right\|} \tag{2}$$

where $\|T\|_{cb}$ is the completely bounded (or CB) norm of $T$.

For $A = \mathcal{B}(H)$, our main result shows how to recognise equality in (2), in a way that generalises a result of Stampfli [22] dealing with special elementary operators $Tx = a_1 x 1 - 1 x b_2$. The bound on $\|T\|_{cb}$ in the estimate (2) is known to be sharp, at least in the case $A = \mathcal{B}(H)$, provided one considers all possible representations of $T$ as $Tx = \sum_{j=1}^{\ell} a_j x b_j$ (and takes the infimum of the upper bounds obtained). We first give a direct argument to characterise

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equality of $\|T\|$, $\|T\|_{cb}$ with the right hand side of (2) for $A = \mathcal{B}(H)$ and this involves a balance condition on certain numerical ranges of the $a_i$ and $b_i$ (Proposition 3.1). More accurately, the numerical ranges we consider are asymmetric and involve $a_j a_i^*$ on the left and $b_j^* b_i$ on the right. These numerical ranges are not convex in general but when we apply our condition to look for equality of $\|T\|_k$, $\|T\|_{cb}$ and the right hand side of (2) we end up considering convex combinations of $k$ elements of the numerical range we use for $k = 1$. We reach the convex hull by the time $k = \ell$ and for this $k$ the balance condition must be satisfied for some representation of $T$ (Theorem 3.3).

We can pass to general $C^*$-algebras $A$ by considering representations and we can then conclude that $\|T\|_k = \|T\|_{cb}$ for $k = \ell$. It seems to be new to have any bound on $k$ (except for $\ell = 1$) without conditions on $A$. A simple example (Example 3.5) shows that the result is optimal (that is, not true for $k = \ell - 1$ for any $\ell > 1$).

Our techniques allow us to embed the example in a continuous trace $C^*$-algebra as long as the algebra has an irreducible representation of large enough dimension (Theorem 4.3). This is the step we need to characterise those $C^*$-algebras $A$ where $\|T\|_k = \|T\|_{cb}$ for all elementary $T\colon A \to A$ (with $k$ independent of $T$). The remaining parts of the proof of this characterisation can be borrowed from [4] where the case $k = 1$ was settled.

We recall that there are somewhat similar results for complete positivity of elementary operators. In [23] it is shown that an elementary $T$ (as in (1)) must be completely positive (CP) if it is $k$-positive for any $k$ at least as big as the integer part of $\sqrt{\ell}$ (and again this is optimal). In [24] the class of $C^*$-algebras $A$ where $k$-positivity implies complete positivity of elementary operators $T\colon A \to A$ is characterised, leading to the same class of algebras as for the CB situation. Again the case $k = 1$ was settled earlier in [4].

This difference between the optimal $k$ in the CP and CB cases suggests looking at norms for the subclass of hermitian-preserving elementary operators. In Theorem 3.12 we establish a smaller $k$ (that is $k < \ell$ if $\ell > 1$) for which $\|T\|_k = \|T\|_{cb}$ holds in this subclass, but examples show that the optimal $k$ must be proportional to $\ell$ in general.

**Notation.** We are using $M_n$ for the $n \times n$ complex matrices, or the bounded linear operators on the standard $n$-dimensional Hilbert space $\mathbb{C}^n$. Our Hilbert spaces $H$ are all complex and $H^n$ means the orthogonal direct sum of $n$ copies of $H$, or the space of $n$-tuples of elements of $H$ with the natural inner product. $\mathcal{B}(H)$ denotes the bounded linear operators on $H$. $M_n(A)$ means the $n \times n$ matrices with entries in $A$.

The CB norm of a linear map $T\colon A \to A$ is defined as $\|T\|_{cb} = \sup_{k \geq 1} \|T\|_k$ where $\|T\|_k = \|T^{(k)}\|$ and $T^{(k)}\colon M_k(A) \to M_k(A)$ is defined via

$$T^{(k)}(x_{ij})_{i,j=1}^k = (T(x_{ij}))_{i,j=1}^k.$$

If $A \subset \mathcal{B}(H)$ then we can regard $M_k(A) = A \otimes M_k$ as a $C^*$-subalgebra of $\mathcal{B}(H) \otimes M_k = \mathcal{B}(H \otimes \mathbb{C}^k) = \mathcal{B}(H^k)$ and in this way there is a unique

$C^*$ norm on each $M_k(A)$ (compatible with the natural algebra structure and involution). There is an extensive literature relating to the CB norm and we cite [18], [10], [11] as general references.

We will use $\mathcal{E}\ell(A)$ for the elementary operators on $A$, $M_n^+$ for the positive semidefinite $n \times n$ matrices.

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## 2. Joint numerical ranges

Our terminology here is motivated by concepts of Stampfli [22] and does not follow standard terminology exactly (see [6, Chapter 7]).

Definition 2.1. *For a tuple $(c_1, c_2, \ldots, c_\ell)$ of operators $c_i \in \mathcal{B}(H)$, we denote by $W_m(c_1, c_2, \ldots, c_\ell)$ the 'matrix numerical range'*

$$W_m(c_1, c_2, \ldots, c_\ell) = \{(\langle c_j^* c_i \xi, \xi \rangle)_{i,j=1}^{\ell} : \xi \in H, \|\xi\| = 1\} \subset M_\ell.$$

*We will also consider a subset of the closure of $W_m$ which we call the 'extremal matrix numerical range' and denote by*

$$W_{m,e}(c_1, c_2, \ldots, c_\ell) = \left\{ \alpha \in \overline{W_m(c_1, c_2, \ldots, c_\ell)} : \mathrm{trace}(\alpha) = \left\| \sum_{i=1}^{\ell} c_i^* c_i \right\| \right\}.$$

Fixing any preferred linear order for the $\ell^2$ entries of an $\ell \times \ell$ matrix, our $W_m$ is the joint spatial numerical range $W$ of [6, p. 137] for the $\ell^2$-tuple $c_j^* c_i$. For future use, note that $\langle c_j^* c_i \xi, \xi \rangle = \langle c_i \xi, c_j \xi \rangle$.

We will sometimes abbreviate $W_m(c_1, c_2, \ldots, c_\ell)$ as $W_m(\mathbf{c})$ with $\mathbf{c}$ denoting the $\ell$-tuple (and usually viewed as a column).

Proposition 2.2. *For $c_1, c_2, \ldots, c_\ell \in \mathcal{B}(H)$, $W_m(c_1, c_2, \ldots, c_\ell)$ is contained in $M_\ell^+$ and $W_{m,e}(c_1, c_2, \ldots, c_\ell)$ is nonempty and consists of those elements of the closure $\overline{W_m}$ of maximal trace.*

*Proof.* To show the positivity, consider $(z_1, z_2, \ldots, z_n) \in \mathbb{C}^\ell$ and observe

$$\sum_{i,j} z_i \overline{z_j} \langle c_j^* c_i \xi, \xi \rangle = \left\| \sum_{i=1}^{\ell} z_i c_i \xi \right\|^2 \geq 0.$$

The fact that $W_{m,e}(c_1, c_2, \ldots, c_\ell) \neq \emptyset$ is easy to verify, as are the other assertions. $\square$

Remark 2.3. Arveson [5] gives another definition of (a sequence of) matrix-valued numerical ranges associated with a fixed operator. For $T \in \mathcal{B}(H)$, $\mathcal{W}_n(T)$ is the set of all possible values $\phi(T)$ where $\phi\colon C^*(T) \to M_n$ is a completely positive unital map on the $C^*$-algebra generated by $T$.

Our $W_m(c_1, c_2, \dots, c_\ell)$ is contained in $\mathcal{W}_\ell(T)$ when we take $T = (c_j^* c_i)_{i,j=1}^\ell$ in $M_\ell(\mathcal{B}(H)) = \mathcal{B}(H^\ell)$. To see this note that for $\xi \in H$ of norm one, $\phi_\xi\colon \mathcal{B}(H) \to \mathbb{C}$ given by $\phi_\xi(x) = \langle x\xi, \xi\rangle$ is a (pure) state on $\mathcal{B}(H)$ so that it is a completely positive unital map. We have $W_m(c_1, c_2, \dots, c_\ell) = \{\phi_\xi^{(\ell)}(T) : \xi \in H, \|\xi\| = 1\}$.

If we take $e_{ij} \in M_{\ell+1}$ to be the matrix with 1 in the $(i,j)$ place and zeros elsewhere, and $c_i = e_{1i}$ then $W_m(c_1, c_2, \dots, c_\ell)$ consists of all positive semidefinite rank one matrices of trace $\le 1$. In this case the convex hull of $W_m$ coincides with $\mathcal{W}_n(T)$, but because $\mathcal{W}_n(T)$ is invariant under conjugation by unitary matrices one can see that the convex hull of $W_m$ is in general smaller than $\mathcal{W}_n(T)$.

Proposition 2.4. *Let* $\mathbf{c} = (c_1, c_2, \dots, c_\ell)$ *with* $c_i \in \mathcal{B}(H)$. *Denote by* $c_i^{(k)} = c_i \otimes I_k \in M_k(\mathcal{B}(H)) = \mathcal{B}(H^k)$ *the block diagonal* $k \times k$ *matrix with* $c_i$ *in the diagonal blocks. Let* $\mathbf{c}^{(k)}$ *denote the corresponding* $\ell$*-tuple* $(c_i^{(k)})_{i=1}^\ell$. *Then*

$$W_m(\mathbf{c}^{(k)}) = \left\{ \sum_{j=1}^k t_j \alpha_j : \alpha_j \in W_m(\mathbf{c}), t_j \ge 0, \sum_j t_j = 1 \right\}$$

*(the set of convex combinations of* $k$ *elements of* $W_m(\mathbf{c})$*). A similar statement holds for* $W_{m,e}$.

*Moreover, for* $k = \min(\ell, \dim(H))$, $W_m(\mathbf{c}^{(k)})$ *is convex, and* $W_{m,e}(\mathbf{c}^{(k)})$ *is convex and closed.*

*Proof.* A simple calculation shows that if $\xi = (\xi_1, \xi_2, \dots, \xi_k) \in H^k$ is a unit vector, then

$$\langle (c_i^{(k)})\xi, (c_j^{(k)})\xi \rangle = \sum_r t_r \langle c_i \xi_r', c_j \xi_r' \rangle$$

where $t_r = \|\xi_r\|^2$ and $\xi_r'$ is the unit vector in the direction of $\xi_r$.

Alternatively if we denote by $\xi_i^* \otimes \xi_i$ the rank one operator on $H$ given by $\theta \mapsto \langle \theta, \xi_i\rangle \xi_i$ we can see that $y = \sum_{i=1}^k \xi_i^* \otimes \xi_i$ is a positive operator of trace $\sum_{i=1}^k \|\xi_i\|^2 = 1$ and of rank at most $k$. Every such $y$ can be written in the form $\sum_{i=1}^k \xi_i^* \otimes \xi_i$. Moreover

$$\left( \left\langle (c_i^{(k)})\xi, (c_j^{(k)})\xi \right\rangle \right)_{i,j=1}^\ell = \left( \operatorname{trace}(c_j^* c_i y) \right)_{i,j=1}^\ell .$$

To show that $W_m(\mathbf{c}^{(k)})$ is convex we need only show that $W_m(\mathbf{c}^{(k+1)}) = W_m(\mathbf{c}^{(k)})$ and if $k = \dim H$ that is clearly true. For $k = \ell < \dim H$, start with

$\alpha = \left(\operatorname{trace}(c_j^* c_i y_0)\right)_{i,j=1}^{\ell} \in W_m(\mathbf{c}^{(k+1)})$ where $y_0 = \sum_{i=1}^{k+1} \xi_i^* \otimes \xi_i$ is positive, of trace 1 and rank at most $k+1$. If the rank of $y_0$ is $< k+1$ we are done and so we assume that the rank is $k+1$. We will work within the span of the $\xi_i$, by taking $P$ to be the orthogonal projection onto the span, temporarily restricting $H$ to $PH$ and considering $c_{ij} = P c_j^* c_i P \in \mathcal{B}(PH)$ in place of $c_j^* c_i$. Note that $c_{ij}^* = c_{ji}$.

Consider

$$S_{k+1} = \{y \in \mathcal{B}(PH) : y > 0, \operatorname{trace} y = 1, \operatorname{trace}(c_{ij} y) = \alpha_{ij} \text{ for } 1 \leq i, j \leq \ell\}.$$

Note that this set is compact (a closed subset of the trace one and positive definite matrices). The total number of real linear equations to be satisfied by $y \in S_{k+1}$ is $1 + \ell^2$ and we are working inside the hermitian elements of $\mathcal{B}(PH)$, a space of dimension $(\dim PH)^2 = (\ell+1)^2 > 1 + \ell^2$. More precisely we have $S_{k+1} \subset \{y = y^* \in \mathcal{B}(PH), \operatorname{trace} y = 1\} = \Pi_{k+1}$, an affine space of dimension $(\ell+1)^2 - 1$. $S_{k+1}$ is the intersection of the convex set $\Sigma_{k+1}$ of positive elements of $\Pi_{k+1}$ with an affine subspace of $\Pi_{k+1}$ of codimension $\ell^2$. $S_{k+1} \neq \emptyset$ because of $y_0$. Thus $S_{k+1}$ must contain some point $y$ which is not a relative interior point of $\Sigma_{k+1} \subset \Pi_{k+1}$. Such a $y$ must have rank $\leq k$ and so $\alpha = \left(\operatorname{trace}(c_j^* c_i y)\right)_{i,j=1}^{\ell} \in W_m(\mathbf{c}^{(k)})$.

The statement about $W_{m,e}$ now follows. □

Remark 2.5. The argument above is a proof of a remnant of convexity for the joint (spatial) numerical range of the finite list of operators on $\mathcal{B}(H)$. The Toeplitz-Hausdorff theorem asserts that the numerical range of a single operator is convex. That is known to be false in general for the joint numerical range of two operators $\{(\langle x_1\xi, \xi\rangle, \langle x_2\xi, \xi\rangle) : \xi \in H, \|\xi\| = 1\}$, though it is true for two hermitian operators $x_1, x_2$. The argument above shows that the set of all convex combinations of $k$ elements of the joint numerical range of $n$ operators $x_1, x_2, \ldots, x_n \in \mathcal{B}(H)$ is convex provided $(k+1)^2 > 1 + d$ for $d$ the dimension of the real span of the real and imaginary parts of the $x_i$ (or $k = \dim H$).

There is a case where the joint numerical range is known to be convex, that is for a commuting $n$-tuple of normal operators $(x_1, x_2, \ldots, x_n)$ (see [6, p. 137]). It follows that if $c_j^* c_i$ are commuting operators, then $W_m(\mathbf{c})$ is convex.

## 3. Norms of elementary operators on $\mathcal{B}(H)$

The Haagerup estimate (2) can be derived from the following matrix formulation of the representation (1)

$$Tx = [a_1, a_2, \ldots, a_\ell](x \otimes I_\ell) \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_\ell \end{bmatrix} = \mathbf{a}(x \otimes I_\ell)\mathbf{b}$$

where $x \otimes I_\ell$ is the block diagonal element of $M_\ell(A) = A \otimes M_\ell$ with $x$'s along the diagonal. We will use this row ($\mathbf{a}$) and column ($\mathbf{b}$) notation often. From $Tx = \mathbf{a}(x \otimes I_\ell)\mathbf{b}$ ($x \in A$), $T^{(k)}(X) = \mathbf{a}^{(k)}(X \otimes I_\ell)\mathbf{b}^{(k)}$ ($X \in M_k(A)$) where

$$\mathbf{a}^{(k)} = [a_1 \otimes I_k, a_2 \otimes I_k, \ldots, a_\ell \otimes I_k]$$

and $\mathbf{b}^{(k)}$ is similarly related to $\mathbf{b}$. We get the estimate (2) from $\|T\|_k \leq \|\mathbf{a}^{(k)}\| \, \|\mathbf{b}^{(k)}\| = \|\mathbf{a}\| \, \|\mathbf{b}\|$. From (2) we get

$$\|T\|_{cb} \leq \frac{1}{2}\left(\left\|\sum_{j=1}^{\ell} a_j a_j^*\right\| + \left\|\sum_{j=1}^{\ell} b_j^* b_j\right\|\right). \tag{3}$$

As a simple argument shows, this estimate is essentially equivalent to (2) because of the ambiguity in the choice of $a_i$ and $b_i$ in (1).

This ambiguity extends at least to the bilinearity of $x \mapsto axb$ in $a$ and $b$ and we can say that every $T \in \mathcal{E}\ell(A)$ can be represented in the form (1) with linearly independent $(a_i)_{i=1}^\ell$ and $(b_i)_{i=1}^\ell$. For general $A$, further ambiguity can arise (for example from the centre of the multiplier algebra $M(A)$—see [1], [8]) but if we simplify to the case of $A = \mathcal{B}(H)$ then no further ambiguity can arise.

An argument using polar decompositions given in [11, Lemma 9.2.3] shows that the infimum of the right hand side of (3) over all possible representations (1) of $T$ is the same as the infimum with $(a_i)_{i=1}^\ell$ and $(b_i)_{i=1}^\ell$ assumed linearly independent. In the case of $A = \mathcal{B}(H)$ we can relate all such linearly independent representations of $T$ to one another via an invertible matrix $\alpha = (\alpha_{ij})_{i,j=1}^\ell$ of scalars:

$$Tx = \sum_{i=1}^{\ell} a_i' x b_j' = \mathbf{a}'(x \otimes I_\ell)\mathbf{b}' = \mathbf{a}\alpha^{-1}(x \otimes I_\ell)\alpha\mathbf{b}.$$

We have

$$W_m(\mathbf{b}') = \alpha W_m(\mathbf{b})\alpha^*, \tag{4}$$

$$W_m((\mathbf{a}')^*) = (\alpha^{-1})^* W_m(\mathbf{a}^*)\alpha^{-1} \tag{5}$$

by simple calculations. If we assume that $\alpha$ is unitary, then the trace is invariant and we have similar relations for $W_{m,e}$, the elements of $\overline{W_m}$ of maximal trace:

$$W_{m,e}((\mathbf{a}')^*) = \alpha W_{m,e}(\mathbf{a}^*)\alpha^*, \; W_{m,e}(\mathbf{b}') = \alpha W_{m,e}(\mathbf{b})\alpha^* \quad (\alpha^* = \alpha^{-1}). \tag{6}$$

An important fact we will use is that there is a representation of $T$ with the right hand side of (3) attaining the minimum possible. A more general statement is shown in [11, Lemma 9.2.7], but the fact we use can be shown by elementary means.

Proposition 3.1. *Let $A = \mathcal{B}(H)$ and let $T \in \mathcal{E}\ell(\mathcal{B}(H))$ be given by* (1). *Then we have equality in*

$$\|T\| \leq \|T\|_{cb} \leq \frac{1}{2}\left(\left\|\sum_{j=1}^{\ell} a_j a_j^*\right\| + \left\|\sum_{j=1}^{\ell} b_j^* b_j\right\|\right)$$

*if and only if the intersection*

$$W_{m,e}(a_1^*, a_2^*, \ldots, a_\ell^*) \cap W_{m,e}(b_1, b_2, \ldots, b_\ell)$$

*is nonempty.*

*Proof.* Consider first the case when $H$ is finite-dimensional and the intersection is non-empty. Thus there exist unit vectors $\xi, \eta \in H$ with $\langle a_j a_i^* \xi, \xi\rangle = \langle b_j^* b_i \eta, \eta\rangle$ for $1 \leq i, j \leq \ell$ and

$$\sum_i \langle a_i a_i^* \xi, \xi\rangle = \left\|\sum_i a_i a_i^*\right\| = \left\|\sum_i b_i^* b_i\right\|.$$

Then $u(b_j\eta) = a_j^*\xi$ specifies a unique unitary map $u$ from the span of $b_j\eta$ to the span of $a_j\xi$. (To make the argument more easy to follow we can assume that $(\langle b_j^* b_i \eta, \eta\rangle)_{i,j}$ is a diagonal matrix by using a unitary matrix $\alpha$ and replacing $\mathbf{a} = (a_1, a_2, \ldots, a_\ell)$ by $\mathbf{a}\alpha^*$ and $\mathbf{b} = [b_1, b_2, \ldots, b_\ell]^t$ by $\alpha\mathbf{b}$.)

We can then extend $u$ to a unitary (or unitary times orthogonal projection) map on $H$ and compute that

$$\langle T(u)\eta, \xi\rangle = \sum_{i=1}^{\ell} \langle u b_i \eta, a_i^* \xi\rangle = \sum_{i=1}^{\ell} \langle a_i a_i^* \xi, \xi\rangle = \|\mathbf{a}\| = \|\mathbf{b}\|.$$

Thus we have

$$\|T\| \geq (1/2)(\|\mathbf{a}\| + \|\mathbf{b}\|) \geq \|T\|_{cb} \geq \|T\|_1 = \|T\|,$$

forcing equality all around in this case.

When $H$ is infinite dimensional we have to modify the argument only slightly to take account of that fact that we can only find unit $\xi$ and $\eta$ so as to get arbitrarily close approximations $\langle a_j a_i^* \xi, \xi\rangle \cong \langle b_j^* b_i \eta, \eta\rangle$ for $1 \leq i, j \leq \ell$ and

$$\sum_i \langle a_i a_i^* \xi, \xi\rangle \cong \left\|\sum_i a_i a_i^*\right\| = \left\|\sum_i b_i^* b_i\right\|.$$

We can then say that our $u$ will have norm approximately 1.

For the converse, if $\|\sum_i a_i a_i^*\| \neq \|\sum_i b_i^* b_i\|$, then we have strict inequality between the right hand sides of (2) and (3). So we may suppose equality and normalise $\|\sum_i a_i a_i^*\| = \|\sum_i b_i^* b_i\| = 1$.

We know that $\|T\| = \sup \|T(u)\|$ over $u$ unitary (by the Russo-Dye theorem [21], [12], or the more elementary fact that the each element of the open unit ball of $\mathcal{B}(H)$ is an average of unitaries [15, p. 253]). Now $\|T(u)\| = \sup \Re\langle T(u)\eta, \xi\rangle$ over unit vectors $\xi, \eta \in H$ and we note that

$$\Re\langle T(u)\eta, \xi\rangle = \sum_{i=1}^{\ell} \Re\langle ub_i\eta, a_i^*\xi\rangle.$$

Let $\zeta_i = ub_i\eta$ and $\theta_i = a_i^*\xi$. Now

$$(\langle \zeta_i, \zeta_j\rangle)_{ij} = (\langle b_j^* b_i\eta, \eta\rangle)_{ij} \in W_m(\mathbf{b})$$

while $(\langle \theta_i, \theta_j\rangle)_{ij} \in W_m(\mathbf{a}^*)$.

Clearly

$$\begin{aligned} \Re\langle T(u)\eta, \xi\rangle &= \sum_{i=1}^{\ell} \Re\langle \zeta_i, \theta_i\rangle \leq \sum_{i=1}^{\ell} \|\zeta_i\|\,\|\theta_i\| \\ &\leq \sqrt{\sum_{i=1}^{\ell} \|\zeta_i\|^2}\sqrt{\sum_{i=1}^{\ell} \|\theta_i\|^2} \leq 1 \end{aligned}$$

and we have strict inequality unless $\zeta_i = \theta_i$ for all $i$ and $\sum_{i=1}^{\ell} \|\zeta_i\|^2 = \sum_{i=1}^{\ell} \|\theta_i\|^2 = 1$, which forces the desired condition

$$(\langle \zeta_i, \zeta_j\rangle)_{ij} = (\langle \theta_i, \theta_j\rangle)_{ij} \in W_{m,e}(\mathbf{a}^*) \cap W_{m,e}(\mathbf{b}) \neq \emptyset.$$

Our aim is to quantify the inequality when the intersection is empty and show $\sum_{i=1}^{\ell} \Re\langle \zeta_i, \theta_i\rangle < 1 - \varepsilon$ where $\varepsilon > 0$ depends on $W_m(\mathbf{a}^*)$ and $W_m(\mathbf{b})$. The following argument is essentially a proof of the Cauchy-Schwarz estimate just above. With an eye to reusing this argument later, we prove a little more than we need just now. Applying the lemma to the closures $W_\theta = \overline{W_m}(\mathbf{a}^*)$ and $W_\zeta = \overline{W_m}(\mathbf{b})$ gives the desired inequality. □

Lemma 3.2. *Let $W_\theta$ and $W_\zeta$ be two closed subsets of $M_\ell^+$, where the maximum value of the trace on each set is 1 and $W_\theta \cap W_\zeta$ has no elements of trace 1. Then there are $\varepsilon > 0$ and open subsets $U_\theta$ and $U_\zeta$ of the positive definite $\ell \times \ell$ matrices with $W_\theta \subset U_\theta$ and $W_\zeta \subset U_\zeta$ so that for any vectors $\theta_i, \zeta_i$ in any Hilbert space $H$ such that $(\langle \theta_i, \theta_j\rangle)_{i,j=1}^{\ell} \in U_\theta$, $(\langle \zeta_i, \zeta_j\rangle)_{i,j=1}^{\ell} \in U_\zeta$ we always have*

$$\Re\sum_{i=1}^{\ell} \langle \theta_i, \zeta_i\rangle < 1 - \varepsilon.$$

*Proof.* Let $W_{\theta,e}$ be the intersection of $W_\theta$ with the matrices of trace 1, and similarly for $W_{\zeta,e}$. There is a positive shortest distance $\delta_0 > 0$ between points of the sets of $W_{\theta,e}$ and $W_{\zeta,e}$. (We measure the distance in the $L^2$ or Hilbert-Schmidt norm $\|\cdot\|_2$ on $M_\ell$.) We can find $r_0 < 1$ so that

$$\alpha \in W_\theta, \operatorname{trace}(\alpha) \geq r_0 \Rightarrow \operatorname{dist}(\alpha, W_{\theta,e}) < \delta_0/4.$$

(If not, a compactness argument produces extra points in $W_{\theta,e}$.) We can make a similar claim for $W_{\zeta,e}$ and we choose $r_0$ to work for both. Of course

$$\alpha, \beta \in M_\ell^+, \mathrm{dist}(\alpha, W_{\theta,e}) < \frac{\delta_0}{4}, \mathrm{dist}(\beta, W_{\zeta,e}) < \frac{\delta_0}{4} \Rightarrow \|\alpha - \beta\|_2 > \delta_0/2.$$

We can further find $r_1 < 1$ so that $r_1 < t \leq 1$ implies

$$\min(\|\alpha - t^2\beta\|_2, \|t^2\alpha - \beta\|_2) > \delta_1 = \delta_0/4$$

for all such $\alpha$ and $\beta$. Choose $\varepsilon_1 > 0$ with

$$1 + \varepsilon_1 < \min\left(\frac{1}{r_0}, \frac{1}{r_1}, 2\right) \text{ and } (1+\varepsilon_1)(1+\varepsilon_1 - \delta_1^2/8) < 1.$$

We take

$$\begin{aligned} U_\theta &= \{\alpha \in M_\ell^+ : \mathrm{trace}(\alpha) < r_0\} \\ &\quad \cup \{\alpha \in M_\ell^+ : \mathrm{trace}(\alpha) < 1 + \varepsilon_1 \text{ and } \mathrm{dist}(\alpha, W_{\theta,e}) < \frac{\delta_0}{4}\} \\ U_\zeta &= \{\beta \in M_\ell^+ : \mathrm{trace}(\beta) < r_0\} \\ &\quad \cup \{\beta \in M_\ell^+ : \mathrm{trace}(\beta) < 1 + \varepsilon_1 \text{ and } \mathrm{dist}(\beta, W_{\zeta,e}) < \frac{\delta_0}{4}\} \end{aligned}$$

and we claim these open sets have the desired properties.

By the choice of $r_0$, we have $W_\theta \subset U_\theta$ and $W_\zeta \subset U_\zeta$.

Consider now vectors $\theta_i, \zeta_i$ in any Hilbert space $H$ such that

$$\alpha = (\langle \theta_i, \theta_j \rangle)_{i,j=1}^\ell \in U_\theta \text{ and } \beta = (\langle \zeta_i, \zeta_j \rangle)_{i,j=1}^\ell \in U_\zeta.$$

By the symmetry of the situation so far, it is enough to verify the claim in the case $\mathrm{trace}(\alpha) = \sum_i \|\theta_i\|^2 \leq \mathrm{trace}(\beta) = \sum_i \|\zeta_i\|^2$. If $\mathrm{trace}(\alpha) \leq r_0$ we can use $\mathrm{trace}(\beta) < 1 + \varepsilon_1$ to get $\sum_{i=1}^\ell \Re\langle \theta_i, \zeta_i \rangle < \sqrt{r_0(1+\varepsilon_1)} < 1$.

Let

$$t = \frac{\sum_{i=1}^\ell \Re\langle \theta_i, \zeta_i \rangle}{\sum_i \|\zeta_i\|^2}.$$

From $\mathrm{trace}(\alpha) \leq \mathrm{trace}(\beta)$ we must have $t \leq 1$. Note that if $t \leq r_1$ we have $\sum_{i=1}^\ell \Re\langle \theta_i, \zeta_i \rangle \leq r_1 \sum_i \|\zeta_i\|^2 \leq r_1(1+\varepsilon_1) < 1$.

Finally for $t > r_1$, $\operatorname{trace}(\alpha) > r_0$ (and hence $\operatorname{trace}(\beta) > r_0$) we must have $\operatorname{dist}(\alpha, W_{\theta,e}) < \delta_0/4$ and $\operatorname{dist}(\beta, W_{\theta,e}) < \delta_0/4$ and hence

$$\begin{aligned}
\delta_1^2 &\leq \left\|(\langle\theta_i,\theta_j\rangle)_{ij} - t^2(\langle\zeta_i,\zeta_j\rangle)_{ij}\right\|_2^2 \\
&= \sum_{ij} |\langle\theta_i,\theta_j\rangle - t^2\langle\zeta_i,\zeta_j\rangle|^2 \\
&= \sum_{ij} |\langle\theta_i - t\zeta_i,\theta_j\rangle + \langle t\zeta_i,\theta_j - t\zeta_j\rangle|^2 \\
&\leq 2\left(\sum_{ij} |\langle\theta_i - t\zeta_i,\theta_j\rangle|^2 + |\langle t\zeta_i,\theta_j - t\zeta_j\rangle|^2\right) \\
&\leq 2\left(\sum_i \|\theta_i\|^2 + t^2\|\zeta_i\|^2\right)\sum_j \|\theta_j - t\zeta_j\|^2 \\
&\leq 4(1+\varepsilon_1)\left(\sum_j \|\theta_j\|^2 - t^2\sum_j \|\zeta_j\|^2\right)
\end{aligned}$$

(using our choice of $t$). Hence

$$\begin{aligned}
\left(\Re\sum_{i=1}^{\ell}\langle\theta_i,\zeta_i\rangle\right)^2 &< \left(1+\varepsilon_1 - \frac{\delta_1^2}{8}\right)\sum_i \|\zeta_i\|^2 \\
&\leq \left(1+\varepsilon_1 - \frac{\delta_1^2}{8}\right)(1+\varepsilon_1) \\
&= 1-\varepsilon_2 < 1
\end{aligned}$$

in this case. In all cases, we have

$$\Re\sum_{i=1}^{\ell}\langle\theta_i,\zeta_i\rangle \leq \max\left(\sqrt{1-\varepsilon_2}, r_1(1+\varepsilon_1), \sqrt{r_0(1+\varepsilon_1)}\right) = 1-\varepsilon,$$

as claimed. □

THEOREM 3.3. *Let $A = \mathcal{B}(H)$ and let $T \in \mathcal{E}\ell(\mathcal{B}(H))$ be given by* (1). *Then we have equality in* (3) *if and only if the intersection of the convex hulls of $W_{m,e}(a_1^*, a_2^*, \ldots, a_\ell^*)$ and $W_{m,e}(b_1, b_2, \ldots, b_\ell)$ is nonempty.*
*Moreover $\|T\|_{cb} = \|T\|_k$ with $k = \min(\ell, \dim(H))$.*

*Proof.* It follows from Propositions 2.4 and 3.1 that for $k = \min(\ell, \dim(H))$, $\|T\|_k = \|T\|_{cb} =$ the right hand side of (3) if the convex hulls intersect.

We know we can represent $T$ in such a way as to get the minimum possible on the right hand side of (3). Fix $k = \min(\ell, \dim(H))$. We claim that in that case $W_{m,e}((\mathbf{a}^{(k)})^*) \cap W_{m,e}(\mathbf{b}^{(k)}) \neq \emptyset$. Assume we have normalised $T$ so that

$$\left\| \sum_i a_i a_i^* \right\| = \left\| \sum_i b_i^* b_i \right\| = 1.$$

As the sets $W_{m,e}((\mathbf{a}^{(k)})^*)$ and $W_{m,e}(\mathbf{b}^{(k)})$ are convex and closed by Proposition 2.4, if they do not intersect they can be separated by an $\mathbb{R}$-linear functional $\rho$ on the hermitian matrices. That is,

$$\sup\{\rho(\alpha) : \alpha \in W_{m,e}((\mathbf{a}^{(k)})^*)\} < \inf\{\rho(\beta) : \beta \in W_{m,e}(\mathbf{b}^{(k)})\}.$$

As the trace is constant on these sets, we can subtract a multiple of the trace from $\rho$ and assume there is $\delta > 0$ with

$$\sup\{\rho(\alpha) : \alpha \in W_{m,e}(\mathbf{a}^*)\} \leq -\delta < \delta \leq \inf\{\rho(\beta) : \beta \in W_{m,e}(\mathbf{b})\}.$$

Such an $\mathbb{R}$-linear functional can be written as

$$\rho(\alpha) = \sum_{i,j=1}^{\ell} \gamma_{ji}\alpha_{ij} = \operatorname{trace}(\gamma\alpha)$$

with $\gamma^* = \gamma$. Arguing as in the proof of Lemma 3.2 we can find $r < 1$ so that

$$\alpha \in W_m(\mathbf{a}^*), \operatorname{trace}(\alpha) \geq r \Rightarrow \rho(\alpha) < -\frac{\delta}{2}$$

and

$$\beta \in W_m(\mathbf{b}), \operatorname{trace}(\beta) \geq r \Rightarrow \rho(\beta) > \frac{\delta}{2}.$$

Now consider a new representation of $T$ as $Tx = \sum_i a_i' x b_i'$ where

$$\mathbf{a}' = \mathbf{a}e^{t\gamma}, \qquad \mathbf{b}' = e^{-t\gamma}\mathbf{b}$$

and $t > 0$ is very small. From (4), elements of $W_m(\mathbf{b}'))$ have the form

$$\beta' = e^{-t\gamma}\beta e^{-t\gamma}$$

with $\beta \in W_m(\mathbf{b})$. For $\operatorname{trace}(\beta) \leq r$ we can assume $t$ is small enough that $\operatorname{trace}(\beta') \leq (1+r)/2 < 1$. For $\operatorname{trace}(\beta) \geq r$ we have

$$\frac{d}{dt}\Big|_{t=0} \operatorname{trace}(e^{-t\gamma}\beta e^{-t\gamma}) = -\operatorname{trace}(\gamma\beta + \beta\gamma) = -2\operatorname{trace}(\gamma\beta) = -2\rho(\beta) < -\delta.$$

Thus, by uniform continuity of the derivative as a function of $t$ at such $\beta \in W_m(\mathbf{b})$, for $t$ small enough $\operatorname{trace}(\beta') < 1 - (\delta/2)t$. Similarly for small $t$, $\operatorname{trace}(\alpha') < 1-(\delta/2)t$ if $\operatorname{trace}(\alpha) \geq r$ while $\operatorname{trace}(\alpha) \leq (1+r)/2$ for $\operatorname{trace}(\alpha) \leq r$. Thus, when $t > 0$ is small we have

$$\|(\mathbf{a}')^*\| < \|\mathbf{a}^*\| \text{ and } \|\mathbf{b}'\| < \|\mathbf{b}\|,$$

contradicting the choice of $\mathbf{a}$ and $\mathbf{b}$ to minimise the right hand side of (3). $\square$

Remark 3.4. For $T \in \mathcal{E}\ell(\mathcal{B}(H))$ the above gives a more constructive proof that $\|T\|_{cb}$ is the infimum of the estimates (3) or (2) than those in [20, Theorem 4.3], [7, Corollary 2] and (for the finite dimensional case) [10, p. 418]. The result is due to Haagerup [14] and his proof is published in [2, §5.4].

Example 3.5. Consider the map $T\colon M_n \to M_n$ where $Tx$ has its first column the same as the transpose $x^t/\sqrt{n}$ but zeros in all other columns. Then

$$Tx = \sum_{i=1}^{n} e_{i1} x (e_{i1}/\sqrt{n})$$

where $e_{ij}$ is as before (Remark 2.3). So in this case $a_i = \sqrt{n} b_i = e_{i1}$, $a_j a_i^* = e_{ji}$, $b_j b_i^* = \delta_{ij} e_{11}/n$ (where $\delta_{ij}$ is the Kronecker symbol). Thus the estimate (2) says $\|T\|_{cb} \leq 1$.

Taking the element of $M_n(M_n)$ with $e_{1i}$ in the $(i,1)$ block and zeros elsewhere, shows that $\|T\|_n = 1$. One can check that $W_m(\mathbf{a}^*)$ consists of rank one projections while $W_{m,e}(\mathbf{b})$ is exactly $\{I_n/n\}$ ($I_n =$ the $n \times n$ identity matrix). It is clear then that for $k < n$, $W_{m,e}((\mathbf{a}^{(k)})^*)$ contains only matrices of rank at most $k$ and does not intersect $W_{m,e}(\mathbf{b}^{(k)})$. Hence $\|T\|_k < \|T\|_{cb} = \|T\|_n$ for $k < n$.

This example shows that the value $k = \ell$ in the theorem cannot be reduced (that is, with large $\dim(H)$).

Example 3.6. We can relate our results to those of Stampfli [22] for the 'generalised derivations' $Tx = ax - xb$. To have a balance between the left and right, we prefer to have it expressed as

$$Tx = (a/\sqrt{\|a\|}) x \sqrt{\|a\|} + \sqrt{\|b\|} x (-b/\sqrt{\|b\|}) = a_1 x b_1 + a_2 x b_2.$$

Then the estimate (3) becomes $\|T\|_{cb} \leq \|a\| + \|b\|$. In this case the matrices in $W_{m,e}(\mathbf{a}^*)$ and $W_{m,e}(\mathbf{b})$ have diagonals $(\|a\|, \|b\|)$ and Stampfli shows that the off-diagonal entries form convex sets. The criterion that $W_{m,e}(\mathbf{a}^*) \cap W_{m,e}(\mathbf{b}) \neq \emptyset$ reduces to Stampfli's criterion [22, Theorem 7] for $\|T\| = \|a\| + \|b\|$. Stampfli shows that this equality is satisfied for some alternative representation of $T$ as $Tx = (a - \lambda)x - x(b - \lambda)$ with $\lambda \in \mathbb{C}$.

Corollary 3.7. *Let $A = \mathcal{B}(H)$ and let $T \in \mathcal{E}\ell(\mathcal{B}(H))$. Then there is a choice of $a_i, b_i \in \mathcal{B}(H)$ so that $T$ is given by (1), each of $(a_i)_{i=1}^{\ell}$ and $(b_i)_{i=1}^{\ell}$ is linearly independent and for $k = \min(\ell, \dim(H))$*

$$W_{m,e}((a_1^{(k)})^*, (a_2^{(k)})^*, \ldots, (a_\ell^{(k)})^*) \cap W_{m,e}(b_1^{(k)}, b_2^{(k)}, \ldots, b_\ell^{(k)}) \neq \emptyset.$$

*Proof.* We showed this in the course of the proof of the theorem. □

COROLLARY 3.8. *Let $A = \mathcal{B}(H)$ and let $T \in \mathcal{E}\ell(\mathcal{B}(H))$ be given by* (1). *Let $k = \min(\ell, \dim(H))$. Then*

$$\|T\|_{cb} = \|T\|_k = \sup\{\|T^{(k)}(u)\| : u \in \mathcal{B}(H^k), \|u\| \le 1, \operatorname{rank}(u) \le \ell\}.$$

*Proof.* Choose $a_i$ and $b_i$ so that the conclusions of the previous corollary hold. Recall $T^{(k)}(x) = \sum_i a_i^{(k)} x b_i^{(k)}$. In the proof of Proposition 3.1 we found that the norm of $T^{(k)}$ is the supremum in the statement. □

COROLLARY 3.9. *Let $A = \mathcal{B}(H)$ and let $T \in \mathcal{E}\ell(\mathcal{B}(H))$ be given by* (1). *Let $k \ge 1$. Then*

$$\|T\|_k = \sup \Re \sum_{i=1}^{\ell} \langle \zeta_i, \theta_i \rangle$$

*where the supremum is taken over all choices of vectors $\theta_i, \zeta_i \in H^k$ such that*

$$(\langle \theta_i, \theta_j \rangle)_{ij} \in W_m((\mathbf{a}^{(k)})^*), \quad (\langle \zeta_i, \zeta_j \rangle)_{ij} \in W_m(\mathbf{b}^{(k)}).$$

*Proof.* This is part of the proof of Proposition 3.1, when we apply it to $T^{(k)}(X) = \sum_{i=1}^{\ell} a_i^{(k)} X b_i^{(k)}$. We had $\theta_i = (a_i^{(k)})^* \xi$, $\zeta_i = u b_i^{(k)} \eta$. □

EXAMPLE 3.10. One may wonder whether the results can be improved if one restricts to $T \in \mathcal{E}\ell(\mathcal{B}(H))$ with the self-adjointness property $T^*(x) = T(x^*)^* = T(x)$, and indeed we present improved bounds on $k$ for this case below. Here are some examples with $T = T^*$.

The example of Choi [9] gives an elementary operator $T$ of length $n^2$ (on $M_n$, $n \ge 2$) which is $(n-1)$-positive but not $n$-positive (and is unital up to scaling: $T(x) = (n-1)(\operatorname{trace} x)I - x$, $T(I) = (n(n-1)-1)I$). Thus for $m < n$ we have $\|T\|_m = \|T(I)\| < \|T\|_n$. One may check that in this case $T$ can be written $Tx = \sum_{j=1}^{n^2-1} b_j^* x b_j - b_{n^2}^* x b_{n^2}$.

Modifying Example 3.5 consider $T\colon M_{n+1} \to M_{n+1}$ where

$$Tx = \sum_{j=2}^{n+1} e_{j1} x (e_{j1}/\sqrt{n}) + \sum_{j=2}^{n+1} (e_{1j}/\sqrt{n}) x e_{1j}.$$

One may check that $\|T\|_{cb} \le 1$ by the Haagerup estimate and $\|T\|_n \ge 1$ by taking the element of $M_n(M_{n+1})$ with $e_{1,i+1}$ in the $(i,1)$ block and zeros elsewhere. A calculation with $W_{m,e}$ shows that $\|T\|_{n-1} < 1$. One can check that in this case we can rewrite $T$ in the form $\sum_{j=1}^{n} b_j^* x b_j - \sum_{j=n+1}^{2n} b_j^* x b_j$.

LEMMA 3.11. *If $T \in \mathcal{E}\ell(\mathcal{B}(H))$ has $T^* = T$ then $T$ can be written as $Tx = \sum_{j=1}^{m} b_j^* x b_j - \sum_{j=m+1}^{\ell} b_j^* x b_j$ (for $0 \le m \le \ell =$ the length of $T$) with $\|T\|_{cb} = \left\| \sum_{j=1}^{\ell} b_j^* b_j \right\|$.*

*Proof.* We begin by expressing $Tx = \sum_{j=1}^{\ell} \tilde{a}_j x \tilde{b}_j$ so as to have equality in the Haagerup estimate and linearly independent sets $\{\tilde{a}_j\}$ and $\{\tilde{b}_j\}$. In [17, 4.9] it is shown that we can use a unitary rewriting (so it leaves the Haagerup estimate unchanged) to get a representation of $T$ with $\tilde{a}_j = \lambda_j \tilde{b}_j^*$ for some real scalars $\lambda_j$. We may assume that the terms are ordered so that the positive $\lambda_j$ (if any) come first and the negative ones later. We then take $b_j = \sqrt{|\lambda_j|}\tilde{b}_j$. With $\varepsilon_j = \lambda_j/|\lambda_j|$ we then have the desired form of the representation $Tx = \sum_{j=1}^{\ell} \varepsilon_j b_j^* x b_j$ and it remains to establish that the Haagerup bound is sharp in this representation.

$$\begin{aligned}\|T\|_{cb} &\leq \left\| \sum_{j=1}^{\ell} b_j^* b_j \right\| = \sup_{\xi \in H, \|\xi\|=1} \sum_{j=1}^{\ell} |\lambda_j| \langle \tilde{b}_j \xi, \tilde{b}_j \xi \rangle \\ &\leq \sup_{\xi \in H, \|\xi\|=1} \sqrt{ \sum_{i=1}^{\ell} |\lambda_i|^2 \langle \tilde{b}_i \xi, \tilde{b}_i \xi \rangle \sum_{j=1}^{\ell} \langle \tilde{b}_j \xi, \tilde{b}_j \xi \rangle } \\ &\leq \sqrt{ \left\| \sum_{i=1}^{\ell} \tilde{a}_i \tilde{a}_i^* \right\| \left\| \sum_{j=1}^{\ell} \tilde{b}_j^* \tilde{b}_j \right\| } = \|T\|_{cb} \qquad \square\end{aligned}$$

THEOREM 3.12. *Suppose $T \in \mathcal{E}\ell(\mathcal{B}(H))$ has $T^* = T$ and length $\ell$. Let $k = [\sqrt{1 + 2m(\ell - m)}]$ where $m$ is as in Lemma 3.11. Then $\|T\|_k = \|T\|_{cb}$.*

*Proof.* If we represent $T$ as in Lemma 3.11 we have $Tx = \mathbf{a}(x \otimes I_\ell)\mathbf{b}$ with $\mathbf{a} = [\varepsilon_1 b_1^*, \varepsilon_2 b_2^*, \ldots, \varepsilon_\ell b_\ell^*]$, $\mathbf{b} = [b_1, b_2, \ldots, b_\ell]^t$ and $\varepsilon_j = 1$ $(1 \leq j \leq m)$, $\varepsilon_j = -1$ $(m < j \leq \ell)$. If $m = \ell$ then $T$ is completely positive and $\|T\| = \|T\|_{cb}$. Similarly if $m = 0$, $-T$ is completely positive.

Consider the finite dimensional case $\dim H < \infty$ first. Then we know that the extremal numerical ranges $W_{m,e}(\mathbf{a})$ and $W_{m,e}(\mathbf{b})$ correspond to the joint numerical ranges of the compressions $pa_j^* a_i p$ and $pb_j^* b_i p$ to the subspace $pH$ where $pH = \left\{\xi \in H : \sum_{j=1}^{\ell} b_j^* b_j \xi = \left\| \sum_{j=1}^{\ell} b_j^* b_j \right\| \xi \right\}$ is the eigenspace of the maximal eigenvalue (and $p$ is the orthogonal projection). We can also see a simple relationship between $W_{m,e}(\mathbf{a})$ and $W_{m,e}(\mathbf{b})$—to get from a matrix $\alpha = (\alpha_{ij})_{i.j=1}^{\ell} \in W_{m,e}(\mathbf{a})$ change $\alpha_{ij}$ to $-\alpha_{ij}$ in the blocks $\{(i,j) : i \leq m, j > m\} \cup \{(i,j) : i > m, j \leq m\}$. As the convex hulls of $W_{m,e}(\mathbf{a})$ and $W_{m,e}(\mathbf{b})$ intersect (by Theorem 3.3) it follows that there is an $\alpha$ in the intersection of the convex hulls with $(\alpha_{ij})_{i=1}^{m}{}_{j=m+1}^{\ell} = 0$.

By Remark 2.5, if $(k+1)^2 > 1 + d$ with

$$d = \dim \operatorname{span}_{\mathbb{R}} \{(pb_j^* b_i p + pb_i^* b_j p)/2, (pb_j^* b_i p - pb_i^* b_j p)/(2\imath)\}$$

(and here $1 \leq i \leq m, m < j \leq \ell$), then there is such an $\alpha$ which is a convex combination of at most $k$ elements of the joint numerical ranges of the $pb_j^* b_i p$. As $d \leq 2m(\ell - m)$, the result follows.

Now consider $\dim(H) = \infty$, $Tx = T^*x = \sum_{j=1}^{\ell} a_j x c_j$. We can see fairly easily that $\|T\|_k = \sup_p \|T_p\|_k$ where the supremum is over all finite dimensional projections $p$ on $H$ and $T_p(x) = pT(pxp)p = \sum_{j=1}^{\ell}(pa_jp)x(pc_jp)$. Given unit vectors $\xi = (\xi_i)_{i=1}^k \in H^k$, $\eta = (\eta_i)_{i=1}^k \in H^k$ and a unitary $u \in \mathcal{B}(H^k)$ choose $p$ so that $\langle T^{(k)}(u)\eta, \xi\rangle$ is not changed when $T$ is replaced by $T_p$. This means the range of $p$ should contain all $\xi_i$, $\eta_i$, $b_j\eta_i$, $c_j^*\xi_i$ and all components of $ub_j^{(k)}\eta$.

If we further assume that $p$ is large enough to ensure that $\{pa_jp : 1 \leq j \leq \ell\}$ and $\{pc_jp : 1 \leq j \leq \ell\}$ are each linearly independent, then we can show as follows that $(m, \ell - m)$ must be the same for $T_p$ as for $T$. Given any two representations of $T$ as $Tx = \sum_{j=1}^{\ell} \varepsilon_j b_j^* x b_j = \sum_{j=1}^{\ell} \tilde{\varepsilon_j} \tilde{b}_j^* x \tilde{b}_j$ ($\varepsilon_j, \tilde{\varepsilon_j} \in \{\pm 1\}$) there must be an invertible $\ell \times \ell$ matrix $\beta$ with

$$[\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_\ell]^t = \tilde{\mathbf{b}} = \beta^{-1}[b_1, b_2, \ldots, b_\ell]^t = \beta^{-1}\mathbf{b}$$

and

$$[\tilde{\varepsilon_1}\tilde{b}_1^*, \tilde{\varepsilon_2}\tilde{b}_2^*, \ldots, \tilde{\varepsilon_\ell}\tilde{b}_\ell^*] = \tilde{\mathbf{b}}^* \operatorname{diag}(\tilde{\varepsilon_1}, \tilde{\varepsilon_2}, \ldots, \tilde{\varepsilon_\ell}) = \mathbf{b}^* \operatorname{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\ell)\beta.$$

It follows that

$$\operatorname{diag}(\tilde{\varepsilon_1}, \tilde{\varepsilon_2}, \ldots, \tilde{\varepsilon_\ell}) = \beta^* \operatorname{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\ell)\beta$$

and so the number of $j$ with $\varepsilon_j = 1$ must be the same as the number where $\tilde{\varepsilon_j} = 1$.

From the finite dimensional case ($T_p$ is essentially an operator on $\mathcal{B}(pH)$) we get

$$\|T\|_{cb} = \|T\|_\ell = \sup_p \|T_p\|_\ell = \sup_p \|T_p\|_k = \|T\|_k. \qquad \square$$

Remark 3.13. The examples 3.10 suggest that the optimal $k$ for Theorem 3.12 must be at least proportional to $\sqrt{m(\ell - m)}$, or about $\ell/2$ in the worst case. But the Theorem requires $k$ to be about $\sqrt{2}$ times what the examples indicate.

One may check that for $\ell = 3$, $m = 1$ it is necessary to have $k = 2$ in some cases. For example $T \in \mathcal{E}\ell(M_2)$, $Tx = b_1^* x b_1 - (b_2^* x b_2 + b_3^* x b_3)$ where

$$b_1 = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}, b_2 = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & -1 \end{pmatrix}, b_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

In this case $\sum_{j=1}^{3} b_j^* b_j = 2I$ and the joint numerical range of $(b_1^* b_2, b_1^* b_3)$ consists of $\{(|\xi_1|^2/2 - |\xi_2|^2, \xi_1\overline{\xi_2}) : |\xi_1|^2 + |\xi_2|^2 = 1\}$. This does not contain $(0, 0)$ but its convex hull does. Hence $\|T\|_{cb} = 2$ by Theorem 3.3, but $W_m(b_1, -b_2, -b_3) \cap W_m(b_1, b_2, b_3) = \emptyset$ and so $\|T\| < 2$ by Proposition 3.1.

In a recent preprint [16] it is shown that for $T \in \mathcal{E}\ell(\mathcal{B}(H))$ of the form $Tx = a^*xb + b^*xa$, $\|T\| = \|T\|_{cb}$ (which also follows from Theorem 3.12 for $\ell = 2$, $m = 1$).

## 4. Elementary operators on $C^*$-algebras

To transfer our methods from the case $A = \mathcal{B}(H)$ to general $C^*$-algebras $A$ we can rely on the irreducible representations $\pi\colon A \to \mathcal{B}(H_\pi)$ of $A$. As is customary we take $\hat{A}$ to denote the unitary equivalence classes of irreducible representations of $A$, $P(A)$ to be the pure states of $A$, $S(A)$ all the states. We denote the unitary equivalence class of an irreducible representation $\pi$ by $[\pi]$. For $\phi \in P(A)$, there is an associated (irreducible) cyclic representation $\pi_\phi$. We call the equivalence class $[\pi_\phi]$ the 'support' of $\phi$ in $\hat{A}$. We let $F_k(A)$ $(k = 1, 2, \ldots)$ denote the $k$-factorial states of $A$, which are finite convex combinations $\phi = \sum_{j=1}^k t_j\phi_j$ of $\phi_j \in P(A)$, all with the same support.

It is well known and easy to verify that for $T \in \mathcal{E}\ell(A)$ given as in (1), we have $\|T\| = \sup_{\pi\in\hat{A}} \|T^\pi\|$ where $T^\pi\colon \mathcal{B}(H_\pi) \to \mathcal{B}(H_\pi)$ is given by

$$T^\pi(x) = \sum_{i=1}^{\ell} \pi(a_i)x\pi(b_i).$$

(There is a technicality involved here when $a_i, b_i \in M(A)$ and then we must know that $\pi$ can be extended to a representation of $M(A)$.) It is also well known that $\|T\|_k = \sup_\pi \|T^\pi\|_k$ and $\|T\|_{cb} = \sup_\pi \|T^\pi\|_{cb}$.

From this and Corollary 3.8 we can deduce immediately that $\|T\|_\ell = \|T\|_{cb}$ for $T \in \mathcal{E}\ell(A)$ of length $\ell$. Using Remark 2.5 we can also assert that if each of the sets $\{a_j^*a_i : 1 \le i, j \le \ell\}$ and $\{b_j^*b_i : 1 \le i, j \le \ell\}$ is commutative, then $\|T\| = \|T\|_{cb}$. (For this one must observe that the commutativity assumption is preserved when passing to $\pi(a_i)$, $\pi(b_j)$ and still preserved when passing to a representation of $T^\pi$ minimising the Haagerup estimate.)

For an $\ell$-tuple $\mathbf{c} = (c_1, c_2, \ldots, c_\ell)$ of elements $c_i \in M(A)$ and $\pi \in \hat{A}$ we define

$$W_m^\pi(\mathbf{c}) = W_m(\pi(\mathbf{c})), \qquad W_{m,e}^\pi(\mathbf{c}) = W_{m,e}(\pi(\mathbf{c}))$$

(where by $\pi(\mathbf{c})$ we mean $(\pi(c_1), \pi(c_2), \ldots, \pi(c_\ell))$. From Proposition 2.4, we know that $W_m^\pi(\mathbf{c}^{(k)})$ (strictly we should use $\pi^{(k)}$ here) is the set of convex combinations of $k$ elements of $W_m^\pi(\mathbf{c})$, and it is convex for $k \ge \ell$. Similarly for $W_{m,e}^\pi(\mathbf{c}^{(k)})$ and $k \ge \ell$.

We also define

$$V_m^\pi(\mathbf{c}) = \{t\alpha : \alpha \in W_m^\pi(\mathbf{c}), 0 \le t \le 1\}.$$

Lemma 4.1. *For an $\ell$-tuple $\mathbf{c}$ of elements of $M(A)$ and $\pi \in \hat{A}$*

$$W_m^\pi(\mathbf{c}) = \{(\phi(c_j^*c_i))_{i,j=1}^\ell : \phi \in P(A), [\pi_\phi] = [\pi]\}.$$

*The convex combinations of k elements of $W_m^\pi(\mathbf{c})$ are representable as the set of all $(\phi(c_j^* c_i))_{i,j=1}^\ell \in M_\ell$ where $\phi \in F_k(A)$ and $\phi$ is a convex combination of pure states supported at $[\pi]$.*

*The convex combinations of k elements of $V_m^\pi(\mathbf{c})$ form*

$$V_m^\pi(\mathbf{c}^{(k)}) = \{t(\phi(c_j^* c_i))_{i,j=1}^\ell : 0 \le t \le 1, \phi \in F_k(A) \text{ supported by } [\pi]\}.$$

*Proof.* Observe that those $\phi \in P(A)$ with $[\pi_\phi] = [\pi]$ take the form $\phi(x) = \langle \pi(x)\xi, \xi\rangle$ with $\xi \in H_\pi$ a unit vector. The result follows. □

On $\hat{A}$ we can take the usual topology obtained via the hull-kernel topology on the primitive ideal space $\mathrm{Prim}(A)$ (see [19, 4.1.2] for example). In the case we deal with continuous trace algebras there is a bijection between $\hat{A}$ and $\mathrm{Prim}(A)$ since elements of $\hat{A}$ are characterised by their kernels (see [19, 6.1.5]).

LEMMA 4.2. *If $A$ is a continuous trace $C^*$-algebra, and $\mathbf{c}$ is an $\ell$-tuple of elements of $A$, then the map*

$$[\pi] \mapsto V_m^\pi(\mathbf{c}^{(k)})$$

*is an upper semicontinuous set-valued map on $\hat{A}$ with values in the compact subsets of $M_\ell^+$.*

*Proof.* When $A$ has continuous trace and $\pi\colon A \to \mathcal{B}(H_\pi)$ is an irreducible representation, then $\pi(A) = \mathcal{K}(H_\pi) =$ the compact operators [19, 6.1.11, 6.1.6]. The pure states of $A$ supported at $[\pi] \in \hat{A}$ are then vector states $\phi(x) = \langle \pi(x)\xi, \xi\rangle$ (with $\xi \in H_\pi$ a unit vector). As the closed unit ball of $H_\pi$ is weakly compact, any net of unit vectors has a subnet $(\xi_\gamma)_\gamma$ that converges weakly to a vector $\theta \in H_\pi$ of norm at most 1. It follows that $\langle \pi(x)\xi_\gamma, \xi_\gamma\rangle$ also converges to $\langle \pi(x)\theta, \theta\rangle$ when $\pi(x)$ has finite rank. The same conclusion for all $\pi(x) \in \mathcal{K}(H_\pi)$ follows by norm density of the finite ranks in $\mathcal{K}(H_\pi)$. This allows us to show that $V_m^\pi(\mathbf{c})$ is compact. It follows that $V_m^\pi(\mathbf{c}^{(k)})$ is compact by considering it as made up of convex combinations of $k$ matrices in $V_m^\pi$.

By upper semicontinuity we mean that for any open subset $U$ of $M_\ell$ the set of $\pi \in \hat{A}$ where $V_m^\pi(\mathbf{c}^{(k)}) \subset U$ is an open subset of $\hat{A}$. Fix $\pi = \pi_0$ with the corresponding $V_m^\pi(\mathbf{c}^{(k)}) \subset U$. If $[\pi_0]$ fails to be an interior point of such $[\pi] \in \hat{A}$, we can find a net $(\phi_\gamma)_\gamma$ of elements of $F_k(A)$, a net $(t_\gamma)_\gamma$ in the unit interval $[0,1]$ so that the supports of $\phi_\gamma$ in $\hat{A}$ converge to $[\pi_0]$ but the matrices $(t_\gamma \phi_\gamma(c_j^* c_i))_{i,j=1}^\ell$ all lie outside $U$.

When $A$ has continuous trace, $\hat{A}$ is Hausdorff (see [19, 6.1.11]). The weak*-closure of $P(A)$ is contained in the multiples of $P(A)$ by numbers $t \in [0,1]$ [13, Theorem 6], and this set of multiples of pure states is weak*-compact. If a net of pure states $(\psi_\gamma)_\gamma$ converges weak* to a nonzero multiple $t\psi$ of a pure state $\psi$ $(0 < t \le 1)$, then the supports of $\psi_\gamma$ converge to the support of $\psi$ in

$\hat{A}$ (see [19, 4.3.3] for an argument). Using these facts it is easy to see that we can extract a subnet from $(\phi_\gamma)_\gamma$ which converges weak* to a multiple of some $\phi \in F_k(A)$ supported at $[\pi_0]$. (A similar argument is given in [3, Lemma 4.2].) Passing to a further subnet ensures $t_\gamma$ converges, and then the limit of the above matrices is an element of $V_m^{\pi_0}(\mathbf{c}^{(k)})$ outside $U$—a contradiction. □

THEOREM 4.3. *If $k \geq 1$ and $A$ is a continuous trace $C^*$-algebra which is not $k$-subhomogeneous, then there exists an elementary operator $T \in \mathcal{E}\ell(A)$,*

$$T(x) = \sum_{i=1}^{k+1} a_i x b_i \qquad (a_i, b_i \in A \text{ for } 1 \leq i \leq k+1)$$

*with $\|T\|_k < \|T\|_{cb}$.*

*Proof.* If $A$ is not $k$-subhomogeneous, then there exists an irreducible representation $\pi$ of $A$ on a Hilbert space $H_\pi$ of dimension at least $k+1$. The basic idea of the proof is to construct $T$ so that $T^\pi$ looks like Example 3.5.

Fix $k+1$ orthonormal vectors $\xi_1, \xi_2, \ldots, \xi_{k+1}$ in $H_\pi$. We use the notation $\xi^* \otimes \eta$ for the operator $\langle \cdot, \xi \rangle \eta$ in $\mathcal{B}(H_\pi)$ (when $\xi, \eta \in H_\pi$). Let $e_{ij}$ denote the operator $\xi_j^* \otimes \xi_i$. Out aim is to construct $T$ so that $T^\pi(x) = \sum_{i=1}^{k+1} e_{i1} x (e_{i1}/\sqrt{k+1})$ and $\|T\|_k < 1 = \|T^\pi\|_{k+1} \leq \|T\|_{cb}$.

Since $\pi(A) = \mathcal{K}(H_\pi)$ we can find $a_i' \in A$ with $\pi(a_i') = e_{i1}$. Let $b_i' = a_i'/\sqrt{k+1}$. Apply Lemma 3.2 to $W_\theta = V_m^\pi(((\mathbf{a}')^*)^{(k)})$ and $W_\zeta = V_m^\pi((\mathbf{b}')^{(k)})$. We then find open neighbourhoods $U_\theta$ and $U_\zeta$ of these sets, to which we can apply the upper semicontinuity Lemma 4.2 to produce an open neighbourhood $\mathcal{N}$ of $[\pi]$ in $\hat{A}$ so that for $s \in \mathcal{N}$ and $\pi_s$ a representative of $s$ we have

$$V_m^{\pi_s}(((\mathbf{a}')^*)^{(k)}) \subset U_\theta, \quad V_m^{\pi_s}((\mathbf{b}')^{(k)}) \subset U_\zeta.$$

By Urysohn's lemma, we can find a continuous functions $f\colon \hat{A} \to [0,1]$ supported in $\mathcal{N}$ so that $f([\pi]) = 1$. From the Dauns Hofmann theorem we can multiply $a_i'$ and $b_i'$ by $f$ to get $a_i$ and $b_i$ in $A$. That is $\pi_s(a_i) = f(s)\pi_s(a_i')$ and $\pi_s(b_i) = f(s)\pi_s(b_i')$. Thus

$$\begin{aligned} V_m^{\pi_s}((\mathbf{a}^*)^{(k)}) = f(s)^2 V_m^{\pi_s}(((\mathbf{a}')^*)^{(k)}) &\subseteq \qquad V_m^{\pi_s}(((\mathbf{a}')^*)^{(k)}) \subset U_\theta, \\ V_m^{\pi_s}(\mathbf{b}^{(k)}) = f(s)^2 V_m^{\pi_s}((\mathbf{b}')^{(k)}) &\subseteq \qquad V_m^{\pi_s}((\mathbf{b}')^{(k)}) \subset U_\zeta \end{aligned}$$

for $s \in \mathcal{N}$. For other $s \in \hat{A}$ we have

$$V_m^{\pi_s}((\mathbf{a}^*)^{(k)}) = V_m^{\pi_s}(\mathbf{b}^{(k)}) = 0.$$

Taking $T$ as in the statement, for all $s$ we have $\|T^{\pi_s}\|_k < 1 - \varepsilon$ by the method of proof for Proposition 3.1. On the other hand $\|T\|_{cb} \geq \|T^\pi\|_{k+1} = 1$. □

THEOREM 4.4. *Suppose a $C^*$-algebra $A$ has the property (for some $k \geq 1$) that $\|T\|_{cb} = \|T\|_k$ for each $T \in \mathcal{E}\ell(A)$ as in (1) with $a_i, b_i \in A$. Then $A$ is* *either $k$-subhomogeneous or a $k$-subhomogeneous extension of an antiliminal $C^*$-algebra.*

*Proof.* As shown in [4], the assumption on $A$ implies that the same is true of any ideal of $A$, including the maximal postliminal ideal $J$ of $A$. $J$ has an essential continuous trace ideal $J_c$ [19, 2.2.11] and by Theorem 4.3, $J_c$ must be $k$-subhomogeneous. The set ${}_k\hat{J}$ of those $s \in \hat{J}$ where the corresponding representation acts on a Hilbert space of dimension $\leq k$ is closed in $\hat{J}$ [19, 4.4.10, 6.1.5]. It is also dense because it contains $\hat{J}_c$. Hence $J$ is $k$-subhomogeneous. □

Corollary 4.5. *Let $A$ be a $C^*$-algebra. Then the following are equivalent properties for $A$:*

(i) $\|T\|_{cb} = \|T\|_k$ *for each* $T \in \mathcal{E}\ell(A)$*;*
(ii) $\|T\|_{cb} = \|T\|_k$ *for each* $T \in \mathcal{E}\ell(A)$ *as in* (1) *with* $a_i, b_i \in A$*;*
(iii) *$A$ is either $k$-subhomogeneous or an antiliminal extension of a $k$-subhomogeneous $C^*$-algebra.*

*Proof.* (i) clearly implies (ii) and we have proved that (ii) implies (iii) in Theorem 4.4 above. If $A$ is $k$-subhomogeneous, then it is easy to see that (i) holds by using representations and [18, Proposition 7.9]. See [4] for the remaining details of a proof that (iii) implies (i). □

In [4], this result is proved for $k = 1$. See [3] for an early reference to this class of $C^*$-algebras and see [24] for a further list of equivalent conditions including some dealing with $k$-positivity implying complete positivity of elementary operators.

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School of Mathematics, Trinity College, Dublin 2, Ireland
*E-mail address*: richardt@maths.tcd.ie