

NEW EXAMPLES OF NONCOMMUTATIVE $\Lambda(p)$ SETS

WILLIAM D. BANKS AND ASMA HARCHARRAS

ABSTRACT. In this paper, we introduce a certain combinatorial property $Z^*(k)$, which is defined for every integer $k \geq 2$, and show that every set $E \subset \mathbb{Z}$ with the property $Z^*(k)$ is necessarily a noncommutative $\Lambda(2k)$ set. In particular, using number theoretic results about the number of solutions to so-called “ S -unit equations,” we show that for any finite set Q of prime numbers the set E_Q of natural numbers whose prime divisors all lie in the set Q is noncommutative $\Lambda(p)$ for every real number $2 < p < \infty$.

1. Introduction

For any finite set Q of prime numbers, let $E_Q \subset \mathbb{N}$ denote the set of all natural numbers n such that every prime divisor of n lies in Q . If Q contains only a single prime q , then $E_Q = \{q^j \mid j \geq 0\}$ is a Hadamard set and therefore also a Sidon set; consequently, for every real number $2 < p < \infty$, the bound

$$\|f\|_{L^p} \leq C \|f\|_{L^2}$$

holds for every function $f \in L^p$ whose Fourier coefficients are supported on the set E_Q , where $C > 0$ is a constant depending only on p ; in other words, the set E_Q is of type $\Lambda(p)$. When Q has cardinality $\#Q \geq 2$, the set E_Q is neither Hadamard nor Sidon; however, number theoretic results about solutions to so-called “ S -unit equations” imply that E_Q is again a $\Lambda(p)$ set for $2 < p < \infty$.

In this paper, we show that for any finite set Q of prime numbers and any real number $2 < p < \infty$, the set E_Q satisfies a much stronger analytic property, namely the noncommutative $\Lambda(p)$ property; that is, E_Q is of type $\Lambda(p)_{cb}$. More precisely, we show that the bound

$$\|f\|_{L^p(S^p)} \leq C \max \left\{ \left\| \left(\sum_n \widehat{f}(n)^* \widehat{f}(n) \right)^{1/2} \right\|_{S^p}, \left\| \left(\sum_n \widehat{f}(n) \widehat{f}(n)^* \right)^{1/2} \right\|_{S^p} \right\}$$

holds for every function $f \in L^p(S^p)$ whose Fourier coefficients are supported on the set E_Q , where the constant $C > 0$ depends only on p and on the cardinality $\#Q$ of the set Q . Here S^p denotes the Schatten p -class over the

Received October 14, 2002; received in final form July 21, 2003.

2000 *Mathematics Subject Classification*. Primary 43A46. Secondary 11N25, 47B10.

©2003 University of Illinois

Hilbert space ℓ_2 ; it is the Banach space of all compact operators $x : \ell_2 \rightarrow \ell_2$ with a finite norm given by

$$\|x\|_{S^p} = \left(\text{Tr}(x^*x)^{p/2} \right)^{1/p},$$

where $\text{Tr}(\cdot)$ denotes the usual trace. The Banach space $L^p(S^p)$ consists of all Bochner measurable S^p -valued functions defined on the unit circle, equipped with the norm

$$\|f\|_{L^p(S^p)} = \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(t)\|_{S^p}^p dt \right)^{1/p},$$

where dt is the Lebesgue measure.

To establish our results, we introduce a certain combinatorial property $Z^*(k)$, defined for every integer $k \geq 2$, and show that every set E with the property $Z^*(k)$ is necessarily of type $\Lambda(2k)_{cb}$. In particular, we observe that for any finite set Q of primes, the set E_Q satisfies $Z^*(k)$ for every $k \geq 2$; this follows from the number theoretic results mentioned earlier. Note that the sets E_Q with $\#Q \geq 2$, along with their translations, dilations, etc., provide the only currently known examples of sets that are of type $\Lambda(p)_{cb}$ for every $2 < p < \infty$ but are not Sidon sets.

The paper is organized as follows. Sections 2–7 are entirely expository in nature; there we review the definitions and results that are needed in the sequel. In Section 8, we show that the $Z^*(k)$ property implies the $\Lambda(2k)_{cb}$ property. In Section 9, we observe that every set E_Q satisfies $Z^*(k)$ for all $k \geq 2$, and that E_Q is not a Sidon set if $\#Q \geq 2$. In Section 10, we give some concluding remarks.

2. Khintchine inequalities

For every $n \in \mathbb{N}$, let $\varepsilon_n : \{\pm 1\}^{\mathbb{N}} \rightarrow \{\pm 1\}$ denote the n -th coordinate projection, let ν be the uniform probability measure on $\{\pm 1\}^{\mathbb{N}}$, and let p be an arbitrary real number with $2 < p < \infty$.

The classical *Khintchine inequalities* show that there exists a constant $C > 0$, depending only on p , such that for all $m \geq 1$ and any sequence x_1, x_2, \dots, x_m in \mathbb{C} , one has

$$(2.1) \quad \left\| \sum_{n=1}^m \varepsilon_n x_n \right\|_{L^p(\{\pm 1\}^{\mathbb{N}}, \nu, \mathbb{C})} \leq C \left(\sum_{n=1}^m |x_n|^2 \right)^{1/2};$$

see [6], for example, for a proof of Khintchine inequalities in the general case $1 \leq p < \infty$. The inequalities (2.1) were later generalized to the noncommutative setting by Lust-Piquard [7], who showed that there exists a constant $C > 0$, depending only on p , such that for all $m \geq 1$ and any sequence of

operators x_1, x_2, \dots, x_m in S^p , the following inequality holds:

$$(2.2) \quad \left\| \sum_{n=1}^m \varepsilon_n x_n \right\|_{L^p(\{\pm 1\}^N, \nu, S^p)} \leq C \max \left\{ \left\| \left(\sum_{n=1}^m x_n^* x_n \right)^{1/2} \right\|_{S^p}, \left\| \left(\sum_{n=1}^m x_n x_n^* \right)^{1/2} \right\|_{S^p} \right\};$$

see [7] for a proof of the noncommutative Khintchine inequalities in the more general case where $1 < p < \infty$; see also [8] for the case $p = 1$.

3. $\Lambda(p)$ sets

The notion of a $\Lambda(p)$ set was first introduced by Zygmund [20] and studied extensively by Rudin [16] and many others. In this paper, we restrict ourselves to the case where $2 < p < \infty$, for simplicity. For any set $E \subset \mathbb{Z}$, let

$$L_E^p = \{f \in L^p \mid \widehat{f} \text{ is supported on } E\},$$

where \widehat{f} denotes the Fourier transform of f . Then E is said to be of type $\Lambda(p)$, or E has the $\Lambda(p)$ property, if there exists a constant $C > 0$, depending only on p and E , such that for every function in L_E^p , the following bound holds:

$$\|f\|_{L^p} \leq C \left(\sum_{n \in E} |\widehat{f}(n)|^2 \right)^{1/2}.$$

We denote by $\lambda_p(E)$ the smallest constant C for which this inequality holds for all $f \in L_E^p$.

Using convexity, one sees that every $\Lambda(p)$ set is also a $\Lambda(q)$ set for any real number $2 < q < p$.

We also recall that, as shown in [16], there is a natural size limitation for the intersection of any $\Lambda(p)$ set with a fixed arithmetic progression. More precisely, if $2 < p < \infty$ is fixed, and E is a $\Lambda(p)$ set, then

$$(3.1) \quad \#(E \cap \{a + b, a + 2b, \dots, a + Nb\}) \leq 4 (\lambda_p(E))^2 N^{2/p}$$

for all integers a, b, N with $N \geq 1$. This result is optimal. Indeed, given $2 < p < \infty$, there is a subset E_N of $\{1, \dots, N\}$ for each integer N , satisfying $\#E_N \geq N^{2/p}$ and $\lambda_p(E_N) \leq C_p$, where the constant C_p depends only on p . This result was first shown by Rudin for even integers (see [16]), then later by Bourgain for arbitrary real numbers (see [2], and also [19]). It follows that for every $2 < p < \infty$, there exists a $\Lambda(p)$ set that is not a $\Lambda(q)$ set for any $q > p$.

In [16], a certain combinatorial property has been considered which is not only stronger but often easier to deal with than the analytic property $\Lambda(2k)$. Let $k \geq 1$ be a fixed integer. A set $E \subset \mathbb{Z}$ is called a $Z^+(k)$ set if there exists a constant $C > 0$, depending only on E , such that for all $\gamma \in \mathbb{Z}$,

$$\#\{(n_1, n_2, \dots, n_k) \in E^k \mid n_1 + n_2 + \dots + n_k = \gamma\} \leq C.$$

It has been shown by Rudin [16] that every $Z^+(k)$ set is necessarily of type $\Lambda(2k)$. In particular, for any finite set Q of primes, the set E_Q satisfies $Z^+(k)$ for all $k \geq 1$ (see Section 9); hence it follows that E_Q is of type $\Lambda(p)$ for every $2 < p < \infty$.

4. Noncommutative $\Lambda(p)$ sets

The notion of noncommutative $\Lambda(p)$ sets was first introduced and studied in [5]. For $2 < p < \infty$ and $E \subset \mathbb{Z}$, let

$$L_E^p(S^p) = \{f \in L^p(S^p) \mid \widehat{f} \text{ is supported on } E\}.$$

The set E is called a noncommutative $\Lambda(p)$ set (or simply, a $\Lambda(p)_{cb}$ set) if there exists a constant $C > 0$, depending only on p and E , such that for every function f in $L_E^p(S^p)$, the bound

$$(4.1) \quad \|f\|_{L^p(S^p)} \leq C \|f\|_p$$

holds, where the triple norm $\|\cdot\|_p$ is defined by

$$\|f\|_p = \max \left\{ \left\| \left(\sum_{n \in \mathbb{Z}} \widehat{f}(n)^* \widehat{f}(n) \right)^{1/2} \right\|_{S^p}, \left\| \left(\sum_{n \in \mathbb{Z}} \widehat{f}(n) \widehat{f}(n)^* \right)^{1/2} \right\|_{S^p} \right\}.$$

We denote by $\lambda_p^{cb}(E)$ the smallest constant C for which the inequality (4.1) holds for all $f \in L_E^p(S^p)$. Note that, by convexity, the opposite inequality

$$(4.2) \quad \|f\|_p \leq \|f\|_{L^p(S^p)}$$

always holds for every $f \in L^p(S^p)$. We remark that the notation cb is an abbreviation for the words ‘‘completely bounded.’’ Harcharras [5] showed that a given set E has the $\Lambda(p)_{cb}$ property if and only if every bounded sequence $(a_n)_{n \in E}$ can be extended to a completely bounded Fourier multiplier on the operator space L^p when the latter is endowed with its canonical operator space structure as defined by Pisier [13].

It is clear from the definition that every $\Lambda(p)_{cb}$ set is necessarily a $\Lambda(p)$ set; therefore the size restriction (3.1) applies. On the other hand, it has been shown in [5] that there exist sets with the $\Lambda(p)$ property for every p which do not have the $\Lambda(p)_{cb}$ property for any p ; thus, the $\Lambda(p)_{cb}$ property is much stronger than the $\Lambda(p)$ property in general.

Note that, by convexity, a $\Lambda(p)_{cb}$ set is also a $\Lambda(q)_{cb}$ set if $2 < q < p < \infty$. Building on the work of Rudin [16], it has been shown in [5] that for every even integer $p = 2k > 2$, there exists a $\Lambda(p)_{cb}$ set that does not have the $\Lambda(q)$ property for any $q > p$; the general case is still open.

In [5], a combinatorial property has been considered which is stronger and easier to deal with than the analytic property $\Lambda(2k)_{cb}$. Let $k \geq 1$ be a fixed

integer. A set $E \subset \mathbb{Z}$ is called a $Z(k)$ set if there exists a constant $C > 0$, depending only on k and E , such that for all $\gamma \in \mathbb{Z}$,

$$\#\left\{(n_1, n_2, \dots, n_k) \in E^k \mid n_i \neq n_j \text{ if } i \neq j, \text{ and } \sum_{j=1}^k (-1)^{j+1} n_j = \gamma\right\} \leq C.$$

It has been shown in [5] that an arbitrary $Z^+(k)$ set need not possess the $\Lambda(2k)_{cb}$ property even though it is a $\Lambda(2k)$ set as mentioned earlier. However, any set with the $Z(k)$ property is necessarily of type $\Lambda(2k)_{cb}$.

Our review of the combinatorial property $Z(k)$ has been intended primarily to motivate our consideration of the new property $Z^*(k)$ introduced in Section 8. In many situations, it is useful to have combinatorial criteria like $Z(k)$ and $Z^*(k)$ which imply the (albeit weaker) analytic property $\Lambda(2k)_{cb}$. For the purposes of this paper, the $Z(k)$ property alone is insufficient, since for an arbitrary finite set Q of primes, the set E_Q need not be of type $Z(k)$. For example, taking $Q = \{2, 3\}$, the relation

$$2^{i+3}3^j - 2^i3^{j+2} + 2^i3^j = 0, \quad \forall i, j \geq 0,$$

implies that E_Q is not of type $Z(3)$ even though it is of type $Z^*(k)$ for all $k \geq 2$ and therefore of type $\Lambda(p)_{cb}$ for every $2 < p < \infty$ (see Section 9).

5. Sidon sets

A set $E \subset \mathbb{Z}$ is called a Sidon set if there exists a constant $C > 0$, depending only on E , such that for all functions $f \in L_E^\infty$, the following bound holds:

$$(5.1) \quad \sum_{n \in E} |\widehat{f}(n)| \leq C \|f\|_{L^\infty}.$$

We denote by $\lambda_\infty(E)$ the smallest constant C for which this inequality holds for all $f \in L_E^\infty$.

It is well known that a Sidon set is a $\Lambda(p)$ set for every $2 < p < \infty$. In fact, it is a $\Lambda(p)_{cb}$ set for every $2 < p < \infty$ as shown in [5]. On the other hand, there is a natural size limitation for the intersections of any Sidon set with a fixed arithmetic progression. It has been shown in [16] that there exists an absolute constant $C > 0$ such that for every Sidon set E ,

$$\#(E \cap \{a + b, a + 2b, \dots, a + Nb\}) \leq C(\lambda_\infty(E))^2 \log N$$

for all integers a, b, N with $N \geq 1$.

6. Pisier’s Rademacherization principle

In this section, we describe a result of [14] that can be used to determine nontrivial upper bounds for the norm of certain sums of products of operators in which various repetitions of the indices occur.

Given two partitions $\mathcal{P} = \{\mathcal{P}_j\}$ and $\mathcal{Q} = \{\mathcal{Q}_i\}$ of the set $\{1, 2, \dots, k\}$, write $\mathcal{P} \leq \mathcal{Q}$ if for every j , $\mathcal{P}_j \subset \mathcal{Q}_i$ for some i , and write $\mathcal{P} < \mathcal{Q}$ whenever

$\mathcal{P} \leq \mathcal{Q}$ and $\mathcal{P} \neq \mathcal{Q}$. It is easily verified that the relation \leq provides a partial order on the set of all partitions of $\{1, 2, \dots, k\}$; the unique minimal element with respect to \leq is the partition $\mathcal{P}_{\min} = \{\{1\}, \{2\}, \dots, \{k\}\}$, while $\mathcal{P}_{\max} = \{\{1, 2, \dots, k\}\}$ is the unique maximal element.

Given a k -tuple $n = (n_1, n_2, \dots, n_k) \in E^k$, where E is an arbitrary set, let $\mathcal{P}_n = \{\mathcal{P}_{n,j}\}$ denote the canonical partition attached to n ; that is, for all $1 \leq i, \ell \leq k$, both i and ℓ belong to the same set $\mathcal{P}_{n,j}$ if and only if $n_i = n_\ell$.

PROPOSITION 1. *Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and $\{\varepsilon_n\}_{n \in E}$ a family of independent random variables with*

$$\mathbb{P}(\{\varepsilon_n = 1\}) = \mathbb{P}(\{\varepsilon_n = -1\}) = 1/2, \quad \forall n \in E.$$

Let $k \geq 2$ be an arbitrary integer. For $1 \leq j \leq k$, let X_j be a Banach space, and $f_j : E \rightarrow X_j$ a finitely supported function. Let

$$\varphi : X_1 \times X_2 \times \dots \times X_k \rightarrow X$$

be a k -linear map of norm at most 1, where X is a given Banach space. Finally, for any partition \mathcal{P} of the set $\{1, 2, \dots, k\}$, put

$$A_{\mathcal{P}} = \{j \in \{1, 2, \dots, k\} \mid \{j\} \in \mathcal{P}\}.$$

Then the following inequality holds:

$$\begin{aligned} & \left\| \sum_{\substack{n=(n_1, \dots, n_k) \in E^k \\ \mathcal{P}_n \geq \mathcal{P}}} \varphi(f_1(n_1), \dots, f_k(n_k)) \right\|_X \\ & \leq \prod_{j \in A_{\mathcal{P}}} \left\| \sum_{n \in E} f_j(n) \right\|_{X_j} \prod_{\substack{1 \leq j \leq k \\ j \notin A_{\mathcal{P}}}} \left(\int_{\Omega} \left\| \sum_{n \in E} \varepsilon_n f_j(n) \right\|_{X_j}^k d\mathbb{P} \right)^{1/k}. \end{aligned}$$

7. Some operator norm inequalities

PROPOSITION 2. *Let $1 \leq p \leq \infty$, $a, b > 1$ with $a^{-1} + b^{-1} = 1$, y a positive operator in S^{ap} , and x_1, x_2, \dots, x_m a sequence of operators, each in S^{2bp} . Then the following inequality holds:*

$$\left\| \sum_{n=1}^m x_n^* y x_n \right\|_{S^p} \leq \|y\|_{S^{ap}} \max \left\{ \left\| \sum_{n=1}^m x_n^* x_n \right\|_{S^{bp}}, \left\| \sum_{n=1}^m x_n x_n^* \right\|_{S^{bp}} \right\}.$$

This proposition first appears in [7] when x_1, x_2, \dots, x_n is a sequence of self-adjoint operators. The general case requires only the three line lemma and can be found in [15].

The following corollary follows from Proposition 2 by a simple inductive argument.

COROLLARY 3. Let $1 \leq p \leq \infty$ and $k \geq 1$ be fixed. For each $1 \leq j \leq k$, let E_j be a finite set of indices, let $a_j > 1$, and let $\{x_{j,n}\}_{n \in E_j}$ be a family of operators in $S^{2a_j p}$. Finally, suppose that $\sum_{j=1}^k a_j^{-1} = 1$. Then the following inequality holds:

$$\left\| \sum_{n_j \in E_j, 1 \leq j \leq k} x_{k,n_k}^* \cdots x_{2,n_2}^* x_{1,n_1}^* x_{1,n_1} x_{2,n_2} \cdots x_{k,n_k} \right\|_{S^p} \leq \prod_{j=1}^k \max \left\{ \left\| \sum_{n \in E_j} x_{j,n}^* x_{j,n} \right\|_{S^{a_j p}}, \left\| \sum_{n \in E_j} x_{j,n} x_{j,n}^* \right\|_{S^{a_j p}} \right\}.$$

8. Main results

Throughout this section, let k be a fixed integer with $k \geq 2$. Here we introduce a new combinatorial property for sets $E \subset \mathbb{Z}$, similar to the $Z(k)$ property described in Section 4.

We say that a set $E \subset \mathbb{Z}$ has the property $Z^*(k)$ if there is a constant $C > 0$, depending only on E and k , such that:

- (i) For every nonzero $\gamma \in \mathbb{Z}$, the conditions

$$(8.1) \quad n_1 - n_2 + \cdots + (-1)^{k+1} n_k = \gamma$$

and

$$(8.2) \quad \sum_{j \in \mathcal{J}} (-1)^{j+1} n_j \neq 0 \quad \text{for all } \emptyset \neq \mathcal{J} \subsetneq \{1, \dots, k\}$$

are satisfied for at most C elements $(n_1, n_2, \dots, n_k) \in E^k$.

- (ii) For every $\emptyset \neq \mathcal{J} \subset \{1, \dots, k\}$, there are at most C vectors $v_\ell \in \mathbb{Q}^{\#\mathcal{J}}$ such that if the vector $n = (n_j)_{j \in \mathcal{J}} \in E^{\#\mathcal{J}}$ satisfies the conditions

$$(8.3) \quad \sum_{j \in \mathcal{J}} (-1)^{j+1} n_j = 0$$

and

$$(8.4) \quad \sum_{j \in \mathcal{J}'} (-1)^{j+1} n_j \neq 0 \quad \text{for all } \emptyset \neq \mathcal{J}' \subsetneq \mathcal{J},$$

then $n = \eta v_\ell$ for some $\eta \in E$ and some $1 \leq \ell \leq C$.

THEOREM 4. If a set $E \subset \mathbb{Z}$ has the property $Z^*(k)$, then E is a $\Lambda(2k)_{cb}$ set.

Proof. Without loss of generality, one can assume that $E \subset \mathbb{N}$. Throughout the proof, the letter C is used to denote any positive constant that occurs and depends only on k and or E ; its precise meaning might change from line to line.

For every $n = (n_1, n_2, \dots, n_k) \in E^k$, let R_n denote the collection of all subsets $\emptyset \neq \mathcal{J} \subsetneq \{1, \dots, k\}$ such that (8.3) and (8.4) hold, and let \mathcal{R} be the set of all collections obtained in this way; that is,

$$\mathcal{R} = \{R \mid R = R_n \text{ for some } n \in E^k\}.$$

For $R, R' \in \mathcal{R}$, write $R' < R$ or $R > R'$ whenever $\emptyset \neq R' \subsetneq R$. Then the relation $<$ defines a partial order on \mathcal{R} . We also put

$$d_0 = \max\{\#R \mid R \in \mathcal{R}\},$$

and for $0 \leq d \leq d_0$, let

$$\mathcal{R}(d) = \{R \in \mathcal{R} \mid \#R = d\}.$$

Then \mathcal{R} is the disjoint union $\mathcal{R} = \bigcup_{d=0}^{d_0} \mathcal{R}(d)$.

Now let $f = \sum_{n \in E} x_n e^{int}$ be fixed; note that $x_n = \widehat{f}(n) \in S^{2k}$ for every $n \in E$. For simplicity, we assume that the Fourier transform \widehat{f} is finitely supported.

For every k -tuple $n = (n_1, n_2, \dots, n_k) \in E^k$, let

$$\tilde{x}_n = x_{n_1}^{\mu_1} x_{n_2}^{\mu_2} \dots x_{n_k}^{\mu_k} \in S^2,$$

where $\mu_j = 1$ if j is odd, and $\mu_j = *$ if j is even. Then

$$\|f\|_{L^{2k}(S^{2k})}^{2k} = \|f^{\mu_1} f^{\mu_2} \dots f^{\mu_k}\|_{L^2(S^2)}^2 = \left\| \sum_{\gamma \in \mathbb{Z}} e^{i\gamma t} \sum_{n \in E^k(\gamma)} \tilde{x}_n \right\|_{L^2(S^2)}^2,$$

where for each $\gamma \in \mathbb{Z}$,

$$E^k(\gamma) = \{n = (n_1, n_2, \dots, n_k) \in E^k \mid n_1 - n_2 + \dots + (-1)^{k+1} n_k = \gamma\}.$$

It follows that

$$\begin{aligned} \|f\|_{L^{2k}(S^{2k})}^{2k} &= \sum_{\gamma \in \mathbb{Z}} \left\| \sum_{n \in E^k(\gamma)} \tilde{x}_n \right\|_{S^2}^2 = \sum_{\gamma \in \mathbb{Z}} \left\| \sum_{\substack{0 \leq d \leq d_0 \\ R \in \mathcal{R}(d)}} \sum_{\substack{n \in E^k(\gamma) \\ R_n = R}} \tilde{x}_n \right\|_{S^2}^2, \\ \|f\|_{L^{2k}(S^{2k})}^{2k} &\leq C \sum_{\substack{0 \leq d \leq d_0 \\ R \in \mathcal{R}(d)}} \sum_{\gamma \in \mathbb{Z}} \left\| \sum_{\substack{n \in E^k(\gamma) \\ R_n = R}} \tilde{x}_n \right\|_{S^2}^2 = C \sum_{\substack{0 \leq d \leq d_0 \\ R \in \mathcal{R}(d)}} \mathcal{S}(R), \end{aligned}$$

where we have set

$$\mathcal{S}(R) = \sum_{\gamma \in \mathbb{Z}} \left\| \sum_{\substack{n \in E^k(\gamma) \\ R_n = R}} \tilde{x}_n \right\|_{S^2}^2.$$

For each collection R with $0 < \#R < d_0$, one has

$$\begin{aligned} \mathcal{S}(R) &\leq 2 \sum_{\gamma \in \mathbb{Z}} \left\| \sum_{\substack{n \in E^k(\gamma) \\ R_n \geq R}} \tilde{x}_n \right\|_{S^2}^2 + 2 \sum_{\gamma \in \mathbb{Z}} \left\| \sum_{\substack{n \in E^k(\gamma) \\ R_n > R}} \tilde{x}_n \right\|_{S^2}^2 \\ &\leq 2 \tilde{\mathcal{S}}(R) + C \sum_{\substack{d < d' \leq d_0 \\ R' \in \mathcal{R}(d')}} \mathcal{S}(R'), \end{aligned}$$

where we have set

$$\tilde{\mathcal{S}}(R) = \sum_{\gamma \in \mathbb{Z}} \left\| \sum_{\substack{n \in E^k(\gamma) \\ R_n \geq R}} \tilde{x}_n \right\|_{S^2}^2;$$

when $\#R = 0$ or $\#R = d_0$, it is clear that $\mathcal{S}(R) = \tilde{\mathcal{S}}(R)$. Consequently,

$$(8.5) \quad \|f\|_{L^{2k}(S^{2k})}^{2k} \leq C \sum_{\substack{0 \leq d \leq d_0 \\ R \in \mathcal{R}(d)}} \tilde{\mathcal{S}}(R).$$

Step 1. We start by showing that the inequality $\tilde{\mathcal{S}}(\emptyset) \leq C \|f\|_{2k}^{2k}$ holds for some constant $C > 0$. Indeed,

$$(8.6) \quad \tilde{\mathcal{S}}(\emptyset) = \mathcal{S}(\emptyset) = \sum_{\substack{\gamma \in \mathbb{Z} \\ \gamma \neq 0}} \left\| \sum_{\substack{n \in E^k(\gamma) \\ R_n = \emptyset}} \tilde{x}_n \right\|_{S^2}^2 + \left\| \sum_{\substack{n \in E^k(0) \\ R_n = \emptyset}} \tilde{x}_n \right\|_{S^2}^2.$$

Since E has the property $Z^*(k)$, for every $\gamma \neq 0$ the equation (8.1) has at most C solutions $n \in E^k$ such that (8.2) also holds. Thus,

$$\begin{aligned} \sum_{\substack{\gamma \in \mathbb{Z} \\ \gamma \neq 0}} \left\| \sum_{\substack{n \in E^k(\gamma) \\ R_n = \emptyset}} \tilde{x}_n \right\|_{S^2}^2 &\leq C \sum_{\substack{\gamma \in \mathbb{Z} \\ \gamma \neq 0}} \sum_{\substack{n \in E^k(\gamma) \\ R_n = \emptyset}} \|\tilde{x}_n\|_{S^2}^2 \\ &\leq C \sum_{n \in E^k} \|\tilde{x}_n^* \tilde{x}_n\|_{S^1} = C \left\| \sum_{n \in E^k} \tilde{x}_n^* \tilde{x}_n \right\|_{S^1} \\ &= C \left\| \sum_{n_1, n_2, \dots, n_k \in E} (x_{n_k}^{\mu_k})^* \dots (x_{n_2}^{\mu_2})^* (x_{n_1}^{\mu_1})^* x_{n_1}^{\mu_1} x_{n_2}^{\mu_2} \dots x_{n_k}^{\mu_k} \right\|_{S^1} \\ &\leq C \prod_{j=1}^k \max \left\{ \left\| \sum_{n_j \in E} x_{n_j}^* x_{n_j} \right\|_{S^k}, \left\| \sum_{n_j \in E} x_{n_j} x_{n_j}^* \right\|_{S^k} \right\}, \end{aligned}$$

where for the last inequality, we have applied Corollary 3. It follows that

$$\sum_{\substack{\gamma \in \mathbb{Z} \\ \gamma \neq 0}} \left\| \sum_{\substack{n \in E^k(\gamma) \\ R_n = \emptyset}} \tilde{x}_n \right\|_{S^2}^2 \leq C \|f\|_{2k}^{2k}.$$

For every n occurring in the second term of (8.6), since $\gamma = 0$, we see that the equation (8.3) holds with $\mathcal{J} = \{1, 2, \dots, k\}$; since $R_n = \emptyset$, the condition (8.4) also applies. Hence, since E has the property $Z^*(k)$, there are at most C vectors $v_\ell \in \mathbb{Q}^k$ such that for each n occurring in the second term of (8.6), $n = \eta v_\ell$ for some $\eta \in E$ and $1 \leq \ell \leq C$. Using Cauchy-Schwarz’s inequality, we see that

$$\left\| \sum_{\substack{n \in E^k(0) \\ R_n = \emptyset}} \tilde{x}_n \right\|_{S^2}^2 \leq C \sum_{1 \leq \ell \leq C} \left\| \sum_{\eta \in E} x_{\eta v_{\ell,1}}^{\mu_1} x_{\eta v_{\ell,2}}^{\mu_2} \cdots x_{\eta v_{\ell,k}}^{\mu_k} \right\|_{S^2}^2.$$

Note that here and elsewhere in the proof, we write $x_z = 0$ if $z \in \mathbb{Q}$, $z \notin E$.

At this point, fix $1 \leq \ell \leq C$. We apply Proposition 1 with the following choices: Ω is $\{\pm 1\}^{\mathbb{N}}$ equipped with the counting probability; $\{\varepsilon_\eta\}_{\eta \in E}$ is a family of coordinate projections, where ε_η is the m_η -th projection on Ω , for some enumeration $\{m_\eta \mid \eta \in E\}$ of the set \mathbb{N} ; \mathcal{P} is the maximal partition \mathcal{P}_{\max} ; φ is the k -linear contractive map that is simply the k -fold product from $S^{2k} \times S^{2k} \times \cdots \times S^{2k}$ into S^2 ; the functions $f_j : E \rightarrow S^{2k}$ are defined by mapping $\eta \in E$ to $f_j(\eta) = x_{\eta v_{\ell,j}}^{\mu_j}$ in S^{2k} , for each $1 \leq j \leq k$. By the proposition, it follows that

$$\left\| \sum_{\eta \in E} x_{\eta v_{\ell,1}}^{\mu_1} x_{\eta v_{\ell,2}}^{\mu_2} \cdots x_{\eta v_{\ell,k}}^{\mu_k} \right\|_{S^2} \leq \prod_{j=1}^k \left(\int_{\Omega} \left\| \sum_{\eta \in E} \varepsilon_\eta x_{\eta v_{\ell,j}} \right\|_{S^{2k}}^k d\mathbb{P} \right)^{1/k}.$$

Now, apply Jensen’s inequality followed by the noncommutative version of the Khintchine inequalities (2.2) as follows:

$$\begin{aligned} \left\| \sum_{\eta \in E} x_{\eta v_{\ell,1}}^{\mu_1} x_{\eta v_{\ell,2}}^{\mu_2} \cdots x_{\eta v_{\ell,k}}^{\mu_k} \right\|_{S^2} &\leq \prod_{j=1}^k \left(\int_{\Omega} \left\| \sum_{\eta \in E} \varepsilon_\eta x_{\eta v_{\ell,j}} \right\|_{S^{2k}}^{2k} d\mathbb{P} \right)^{1/(2k)} \\ &\leq C \prod_{j=1}^k \left\| \sum_{\eta \in E} x_{\eta v_{\ell,j}} e^{i\eta v_{\ell,j} t} \right\|_{2k} \leq C \|f\|_{2k}^k, \end{aligned}$$

since for each $1 \leq j \leq C$,

$$\left\| \sum_{\eta \in E} x_{\eta v_{\ell,j}} e^{i\eta v_{\ell,j} t} \right\|_{2k} \leq \|f\|_{2k}.$$

It follows that

$$\left\| \sum_{\substack{n \in E^k(0) \\ R_n = \emptyset}} \tilde{x}_n \right\|_{S^2}^2 \leq C \|f\|_{2k}^{2k},$$

which completes Step 1.

Step 2. Next, we show that the inequality $\tilde{\mathcal{S}}(R) \leq C \|f\|_{L^{2k}(S^{2k})}^{2k-2} \|f\|_{2k}^2$ holds for every $1 \leq d \leq d_0$ and every $R \in \mathcal{R}(d)$.

For this aim, fix $1 \leq d \leq d_0$ and $R \in \mathcal{R}(d)$. There is a canonical equivalence relation \mathcal{P}_R induced by the collection R on the set $\{1, 2, \dots, k\}$, defined as follows. Write $j \equiv \ell \pmod{\mathcal{P}_R}$ if and only if there exists a positive integer $t = t(j, \ell)$ and sets $\mathcal{J}_1, \dots, \mathcal{J}_t$ in the collection R such that:

- (i) $j \in \mathcal{J}_1$ and $\ell \in \mathcal{J}_t$.
- (ii) $\mathcal{J}_j \cap \mathcal{J}_{j+1} \neq \emptyset$ for $1 \leq j < t$.

Let \mathcal{P}_R also denote the corresponding partition of $\{1, 2, \dots, k\}$, and let a_R denote the number of singleton sets in \mathcal{P}_R . Below we show the following more precise inequality:

$$\tilde{\mathcal{S}}(R) = \sum_{\gamma \in \mathbb{Z}} \left\| \sum_{\substack{n \in E^k(\gamma) \\ R_n \geq R}} \tilde{x}_n \right\|_{S^2}^2 \leq C \|f\|_{L^{2k}(S^{2k})}^{2a_R} \|f\|_{2k}^{2k-2a_R}.$$

Combining (ii) in property $Z^*(k)$ with condition (ii) in our definition of the equivalence relation \mathcal{P}_R above, it is not hard to see that there are at most C vectors $v_\ell = (v_{\ell,j})_{j=1}^k \in \mathbb{Q}^k$ with the following properties:

- (i) $v_{\ell,j} = 1$ if $\{j\} \in \mathcal{P}_R$.
- (ii) For every $n = (n_1, n_2, \dots, n_k) \in E^k$, the inequality $R_n \geq R$ holds if and only if for some $1 \leq \ell \leq C$ and some $\eta = (\eta_1, \eta_2, \dots, \eta_k) \in E^k$ with $\eta_i = \eta_\ell$ whenever $i \equiv \ell \pmod{\mathcal{P}_R}$, $n_j = \eta_j v_{\ell,j}$ for all $1 \leq j \leq k$.

Consequently,

$$\begin{aligned} \tilde{\mathcal{S}}(R) &= \sum_{\gamma \in \mathbb{Z}} \left\| \sum_{\substack{n \in E^k(\gamma) \\ R_n \geq R}} \tilde{x}_n \right\|_{S^2}^2 = \sum_{\gamma \in \mathbb{Z}} \left\| \sum_{\substack{n=(n_1, \dots, n_k) \in E^k(\gamma) \\ R_n \geq R}} x_{n_1}^{\mu_1} x_{n_2}^{\mu_2} \dots x_{n_k}^{\mu_k} \right\|_{S^2}^2 \\ &= \sum_{\gamma \in \mathbb{Z}} \left\| \sum_{1 \leq \ell \leq C} \sum_{\substack{\eta=(\eta_1, \dots, \eta_k) \in E^k, \mathcal{P}_\eta \geq \mathcal{P}_R \\ \eta_1 v_{\ell,1} - \eta_2 v_{\ell,2} + \dots + (-1)^{k+1} \eta_k v_{\ell,k} = \gamma}} x_{\eta_1 v_{\ell,1}}^{\mu_1} x_{\eta_2 v_{\ell,2}}^{\mu_2} \dots x_{\eta_k v_{\ell,k}}^{\mu_k} \right\|_{S^2}^2 \\ &\leq C \sum_{1 \leq \ell \leq C} \sum_{\gamma \in \mathbb{Z}} \left\| \sum_{\substack{\eta=(\eta_1, \dots, \eta_k) \in E^k, \mathcal{P}_\eta \geq \mathcal{P}_R \\ \eta_1 v_{\ell,1} - \eta_2 v_{\ell,2} + \dots + (-1)^{k+1} \eta_k v_{\ell,k} = \gamma}} x_{\eta_1 v_{\ell,1}}^{\mu_1} x_{\eta_2 v_{\ell,2}}^{\mu_2} \dots x_{\eta_k v_{\ell,k}}^{\mu_k} \right\|_{S^2}^2 \end{aligned}$$

$$= C \sum_{1 \leq \ell \leq C} \tilde{\mathcal{S}}_\ell(R),$$

where $\tilde{\mathcal{S}}_\ell(R)$ denotes the inner summation for each ℓ .

Let $1 \leq \ell \leq C$ be fixed; then we can estimate $\tilde{\mathcal{S}}_\ell(R)$ as follows:

$$\begin{aligned} \tilde{\mathcal{S}}_\ell(R) &= \left\| \sum_{\gamma \in \mathbb{Z}} e^{i\gamma t} \sum_{\substack{\eta=(\eta_1, \dots, \eta_k) \in E^k, \mathcal{P}_\eta \geq \mathcal{P}_R \\ \eta_1 v_{\ell,1} - \eta_2 v_{\ell,2} + \dots + (-1)^{k+1} \eta_k v_{\ell,k} = \gamma}} x_{\eta_1 v_{\ell,1}}^{\mu_1} x_{\eta_2 v_{\ell,2}}^{\mu_2} \dots x_{\eta_k v_{\ell,k}}^{\mu_k} \right\|_{L^2(S^2)}^2 \\ &= \left\| \sum_{\substack{\eta=(\eta_1, \dots, \eta_k) \in E^k \\ \mathcal{P}_\eta \geq \mathcal{P}_R}} \left(x_{\eta_1 v_{\ell,1}} e^{i\eta_1 v_{\ell,1} t} \right)^{\mu_1} \dots \left(x_{\eta_k v_{\ell,k}} e^{i\eta_k v_{\ell,k} t} \right)^{\mu_k} \right\|_{L^2(S^2)}^2. \end{aligned}$$

We apply Proposition 1 with the following choices: Ω is $\{\pm 1\}^{\mathbb{N}}$ equipped with the counting probability; $\{\varepsilon_\eta\}_{\eta \in E}$ is a family of coordinate projections, where ε_η is the m_η -th projection on Ω , for some enumeration $\{m_\eta \mid \eta \in E\}$ of the set \mathbb{N} ; \mathcal{P} is the partition \mathcal{P}_R ; φ is the k -linear contractive map that is simply the k -fold product from $L^{2k}(S^{2k}) \times L^{2k}(S^{2k}) \times \dots \times L^{2k}(S^{2k})$ into $L^2(S^2)$; the functions $f_j : E \rightarrow L^{2k}(S^{2k})$ are defined by mapping $\eta \in E$ to

$$f_j(\eta) : t \mapsto \left(x_{\eta v_{\ell,j}} e^{i\eta v_{\ell,j} t} \right)^{\mu_j}$$

for each $1 \leq j \leq k$. Note that each $f_j \in L^{2k}(S^{2k})$. By the proposition, it follows that

$$\begin{aligned} &\tilde{\mathcal{S}}_\ell(R)^{1/2} \\ &\leq \prod_{\substack{1 \leq j \leq k \\ \{j\} \in \mathcal{P}_R}} \left\| \sum_{\eta \in E} f_j(\eta) \right\|_{L^{2k}(S^{2k})} \prod_{\substack{1 \leq j \leq k \\ \{j\} \notin \mathcal{P}_R}} \left(\int_{\Omega} \left\| \sum_{\eta \in E} \varepsilon_\eta f_j(\eta) \right\|_{L^{2k}(S^{2k})}^k d\mathbb{P} \right)^{1/k} \\ &= \left\| \sum_{\eta \in E} x_\eta e^{i\eta t} \right\|_{L^{2k}(S^{2k})}^{a_R} \prod_{\substack{1 \leq j \leq k \\ \{j\} \notin \mathcal{P}_R}} \left(\int_{\Omega} \left\| \sum_{\eta \in E} \varepsilon_\eta x_{\eta v_{\ell,j}} e^{i\eta v_{\ell,j} t} \right\|_{L^{2k}(S^{2k})}^{2k} d\mathbb{P} \right)^{1/2k} \\ &\leq C \left\| f \right\|_{L^{2k}(S^{2k})}^{a_R} \prod_{\substack{1 \leq j \leq k \\ \{j\} \notin \mathcal{P}_R}} \tilde{\mathcal{S}}_j, \end{aligned}$$

where for the second inequality, we have used Jensen’s inequality, and for the third one, we have used Fubini’s Theorem followed by (2.2), and where

$$\tilde{\mathcal{S}}_j = \max \left\{ \left\| \left(\sum_{\eta \in E} x_{\eta v_{\ell,j}} x_{\eta v_{\ell,j}}^* \right)^{1/2} \right\|_{S^{2k}}, \left\| \left(\sum_{\eta \in E} x_{\eta v_{\ell,j}}^* x_{\eta v_{\ell,j}} \right)^{1/2} \right\|_{S^{2k}} \right\} \leq \|f\|_{2k}$$

for every $1 \leq j \leq k$ with $\{j\} \notin \mathcal{P}_R$. Therefore, we have shown that for each $1 \leq \ell \leq C$,

$$\tilde{S}_\ell(R) \leq C \|f\|_{L^{2k}(S^{2k})}^{2a_R} \|f\|_{2k}^{2k-2a_R}.$$

It follows that

$$\tilde{S}(R) \leq C \|f\|_{L^{2k}(S^{2k})}^{2a_R} \|f\|_{2k}^{2k-2a_R} \leq C \|f\|_{L^{2k}(S^{2k})}^{2k-2} \|f\|_{2k}^2,$$

where for the second inequality, we have used (4.2). This completes Step 2.

Step 3. Combining our estimates from Steps 1 and 2, we have by (8.5):

$$\|f\|_{L^{2k}(S^{2k})}^{2k} \leq C \left(\|f\|_{2k}^{2k} + \|f\|_{L^{2k}(S^{2k})}^{2k-2} \|f\|_{2k}^2 \right),$$

which clearly implies that

$$\|f\|_{L^{2k}(S^{2k})} \leq C \|f\|_{2k}.$$

This completes the proof. □

9. *S*-unit equations

In this section, we use some known number theoretic results to show that for an arbitrary finite set Q of primes, the set E_Q is of type $\Lambda(p)_{cb}$ for $2 < p < \infty$.

Let K be an algebraic number field of degree d ; that is, K is a finite extension of the rationals \mathbb{Q} , with $d = [K : \mathbb{Q}]$. Let S be a finite collection of places of K containing all of the archimedean places, and let \mathcal{U}_S be the group of S -units inside the integral closure \mathcal{O}_K of \mathbb{Z} in K . Given nonzero elements $a_1, \dots, a_k \in K$, one is interested in counting the number of nondegenerate solutions to the S -unit equation

$$(9.1) \quad a_1x_1 + a_2x_2 + \dots + a_kx_k = 1, \quad x_1, x_2, \dots, x_k \in \mathcal{U}_S,$$

i.e., those where no proper subsum $a_{j_1}x_{j_1} + \dots + a_{j_\ell}x_{j_\ell}$ vanishes.

Mahler [9] proved that for $k = 2$ and $K = \mathbb{Q}$, (9.1) has only finitely many solutions. Van der Poorten and Schlickewei [12] and Evertse [3] independently proved that for all $k \geq 2$ and every number field K , (9.1) has only finitely many solutions. This result was later extended by Evertse and Györy [4], who showed that the number of solutions is bounded by a constant which is independent of the coefficients a_1, \dots, a_k . Later, Schlickewei showed that the constant depends only on k , on the cardinality $\#S$ of the set S , and on the degree d (see [17] for the case $K = \mathbb{Q}$, and [18] for the general case).

In particular, when $K = \mathbb{Q}$, for any finite set Q of primes, one can apply the results of [17] mentioned above with $S = Q \cup \{\infty\}$ to deduce that E_Q satisfies both properties $Z^+(k)$ and $Z^*(k)$ for all $k \geq 2$, where the constant $C > 0$ depends only on k and on the cardinality $\#Q$ of the set Q . In fact, our definition of property $Z^*(k)$ was chosen with precisely these sets in mind. Applying now Theorem 4 together with our remarks from Section 4, we obtain the following:

THEOREM 5. *Let Q be a nonempty finite set of prime numbers. Then the set E_Q is of type $\Lambda(p)_{cb}$ for every real number $2 < p < \infty$.*

We conclude this section by observing that E_Q is not a Sidon set whenever $\#Q \geq 2$. Indeed, let $s = \#Q$, and let $q_1 < q_2 < \dots < q_s$ be the primes in Q . Then for all nonnegative integers $\alpha_1, \alpha_2, \dots, \alpha_s \leq (\log N)/(s \log q_s)$, the integer $n = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_s^{\alpha_s}$ lies in E_Q and in $[1, N]$. Thus, if N is sufficiently large,

$$\#(E_Q \cap [1, N]) \geq C (\log N)^s,$$

where the constant $C > 0$ depends only on Q . This contradicts (5.1) (with $a = 0$ and $b = 1$) whenever $s = \#Q \geq 2$.

10. Remarks

The notions of $\Lambda(p)$ and $\Lambda(p)_{cb}$ sets and the properties $Z^+(k)$ and $Z(k)$ can be naturally defined for an arbitrary discrete group G . In this more general context, it has been shown that any subset of G with the $Z^+(k)$ property is necessarily of type $\Lambda(2k)$. The argument is identical to that given by Rudin in the special case $G = \mathbb{Z}$; see [16]. It is also known that any subset of G with the $Z(k)$ property is necessarily of type $\Lambda(2k)_{cb}$ by the results of [5]. It would be interesting to find a suitable generalization of the property $Z^*(k)$ for an arbitrary discrete group G and to show that any subset of G with the $Z^*(k)$ property is necessarily of type $\Lambda(2k)_{cb}$. It would also be of interest to obtain explicit examples of $\Lambda(2k)_{cb}$ sets in G that are similar to the sets E_Q considered here.

Let G be any discrete group and $k \geq 2$ a fixed integer. If a set $E \subset G$ has the $Z(k)$ property, then it is of type $\Lambda(2k)_{cb}$ as we have just mentioned. Consequently, the union of any finite number of sets with the $Z(k)$ property is also of type $\Lambda(2k)_{cb}$. It is natural to ask whether the converse statement is also true; this question was originally raised by Pisier when $G = \mathbb{Z}$ and is still open.

QUESTION 1. Let G be a discrete group, and let $E \subset G$ be a set of type $\Lambda(2k)_{cb}$, where $k > 2$ is a fixed integer. Does there exist a finite collection E_1, E_2, \dots, E_c of subsets of G such that each E_j has the $Z(k)$ property and such that E is the union of the E_j ?

Using Mihăilescu's recent proof of the Catalan conjecture (see [10], and also [1]), one can show that every set E_Q with $\#Q = 2$ can be decomposed into (at most) four sets, each with the $Z(3)$ property. In particular, this shows that E_Q is of type $\Lambda(6)_{cb}$ without using our Theorem 4. However, we do not see how to generalize this to an arbitrary set E_Q and an arbitrary integer $k \geq 2$, since the appropriate analogue to Mihăilescu's result is missing.

Finally, it has been shown in [5] that any noncommutative $\Lambda(p)$ set cannot contain the sum $A + A$ for any infinite set A . Neuwirth [11] later noticed that the arguments in [5] can be slightly modified to show that a noncommutative $\Lambda(p)$ set cannot contain the sum $A + B$ for any infinite sets A and B . By Theorem 4, this can therefore be applied to any set E with the property $Z^*(k)$. For the special sets E_Q , stronger results are known: E_Q cannot contain the sum $A + B$ for any infinite set A and any set B with at least two elements. This follows, for example, from a fairly deep result due to Mahler: for any finite set of primes Q , the gaps between consecutive integers free of primes outside of Q tend to infinity. The authors wish to thank Carl Pomerance for bringing this to our attention.

Acknowledgements. The authors wish to thank Igor Shparlinski for many useful suggestions and helpful comments. We also thank Macquarie University for its kind hospitality during the preparation of this paper. The first author was supported in part by NSF grant DMS-0070628.

REFERENCES

- [1] Y. Bilu, *Catalan's conjecture (after Mihăilescu)*, preprint.
- [2] J. Bourgain, *Bounded orthogonal systems and the $\Lambda(p)$ -set problem*, Acta Math. **162** (1989), 227–245. MR **90h**:43008
- [3] J.-H. Evertse, *On sums of S -units and linear recurrences*, Compositio Math. **53** (1984), 225–244. MR **86c**:11045
- [4] J.-H. Evertse and K. Győry, *On the numbers of solutions of weighted unit equations*, Compositio Math. **66** (1988), 329–354. MR **89i**:11082
- [5] A. Harcharras, *Fourier analysis, Schur multipliers on S^p and non-commutative $\Lambda(p)$ -sets*, Studia Math. **137** (1999), 203–260. MR **2001f**:43004
- [6] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. I*, Springer-Verlag, Berlin-New York, 1977. MR 58#17766
- [7] F. Lust-Piquard, *Inégalités de Khintchine dans C_p ($1 < p < \infty$)*, C. R. Acad. Sci. Paris Sér. I Math. **303** (1986), 289–292. MR **87j**:47032
- [8] F. Lust-Piquard and G. Pisier, *Noncommutative Khintchine and Paley inequalities*, Ark. Mat. **29** (1991), 241–260. MR **94b**:46011
- [9] K. Mahler, *Zur Approximation algebraischer Zahlen. I. (Über den grössten Primteiler binärer Formen)*, Math. Ann. **107** (1933), 691–730.
- [10] P. Mihăilescu, *Primary cyclotomic units and a proof of Catalan's conjecture. Draft*, preprint.
- [11] S. Neuwirth, *Multiplicateurs et analyse fonctionnelle*, Ph.D. Thesis, Université Paris 6, 1999.
- [12] A. van der Poorten and H. Schlickewei, *The growth condition for recurrence sequences*, Rep. no. 82-0041, Dept. Math., Macquarie Univ., North Ryde, 1982.
- [13] G. Pisier, *Non-commutative vector valued L_p -spaces and completely p -summing maps*, Astérisque (1998). MR **2000a**:46108
- [14] ———, *An inequality for p -orthogonal sums in non-commutative L_p* , Illinois J. Math. **44** (2000), 901–923. MR **2001k**:46101
- [15] G. Pisier and Q. Xu, *Non-commutative martingale inequalities*, Comm. Math. Phys. **189** (1997), 667–698. MR **98m**:46079

- [16] W. Rudin, *Trigonometric series with gaps*, J. Math. Mech. **9** (1960), 203–227. MR [22#6972](#)
- [17] H. P. Schlickewei, *An explicit upper bound for the number of solutions of the S -unit equation*, J. Reine Angew. Math. **406** (1990), 109–120. MR [91h:11022](#)
- [18] ———, *S -unit equations over number fields*, Invent. Math. **102** (1990), 95–107. MR [91h:11022](#)
- [19] M. Talagrand, *Sections of smooth convex bodies via majorizing measures*, Acta. Math. **175** (1995), 273–300. MR [96m:46016](#)
- [20] A. Zygmund, *Trigonometric series*, 2nd ed., Vols. I, II, Cambridge Univ. Press, New York, 1959. MR [21#6498](#)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MO 65211, USA
E-mail address, William D. Banks: bbanks@math.missouri.edu
E-mail address, Asma Harcharras: harchars@math.missouri.edu