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COMPLEXIFICATIONS OF REAL OPERATOR SPACES

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Abstract. We study the complexifications of real operator spaces. We show that for every real operator space V there exists a unique complex operator space matrix norm $\{\|\cdot\|_n\}$ on its complexification $V_c = V \dot{+} \mathrm{i}V$ which extends the original matrix norm on V and satisfies the condition $\|x + \mathrm{i}y\|_n = \|x - \mathrm{i}y\|_n$ for all $x + \mathrm{i}y \in M_n(V_c) = M_n(V) \dot{+} \mathrm{i}M_n(V)$. As a consequence of this result, we characterize complex operator spaces which can be expressed as the complexification of some real operator space. Finally, we show that some properties of real operator spaces are closely related to the corresponding properties of their complexifications.

1. Introduction

In a recent paper [18], the author investigated a possible development in real operator spaces. Let us first recall that if H is a real Hilbert space, we let $B(H)$ denote the space of all bounded (real linear) operators on H, equipped with the operator norm on H. Then for each $n \in \mathbb{N}$ the n-fold direct sum $H^n = H \oplus \cdots \oplus H$ is again a real Hilbert space and we may identify the $n \times n$ matrix space $M_n(B(H))$ over $B(H)$ with the space $B(H^n)$ of all bounded operators on H^n. A *real operator space* on a real Hilbert space H is a norm closed subspace V of $B(H)$ together with the operator norm $\|\cdot\|_n$ on each matrix space $M_n(V)$ inherited from $M_n(B(H)) = B(H^n)$. It is easy to verify that this family of matrix norms $\{\|\cdot\|_n\}$ satisfies

(M1) $\|x \oplus y\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}$,
(M2) $\|\alpha x \beta\|_n \leq \|\alpha\|\|x\|_n\|\beta\|$,

for all $x \in M_n(V)$, $y \in M_m(V)$ and $\alpha, \beta \in M_n(\mathbb{R})$.

As in the complex case, we can define an *abstract real operator space* to be a real space V together with a Banach space norm $\|\cdot\|_n$ on each matrix space $M_n(V)$ such that the conditions (M1) and (M2) are satisfied. It was shown in [18, Theorem 5.1] that if V is an abstract real operator space, then we may

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completely isometrically identify V with a real operator space on some real Hilbert space H. This provides us with an abstract matrix norm characterization of real operator spaces and allows us to develop a corresponding theory for real operator spaces (see [18]).

Since the theory of complex operator spaces has been intensively studied in recent years and is now well established, it is desirable to obtain some kind of connection between real operator spaces and complex operator spaces. A standard way to do this is to consider the complexifications of real operator spaces. In this paper we prove the surprising result (Theorem 3.1) that for every real operator space V there exists a unique complex operator space matrix norm on its complexification $V_c = V \dot{+} \mathrm{i}V$, which extends the original matrix norm on V and satisfies the *reasonable* condition

$$\|x + \mathrm{i}y\|_n = \|x - \mathrm{i}y\|_n \tag{1.1}$$

for all $x + \mathrm{i}y \in M_n(V_c) = M_n(V) + \mathrm{i}M_n(V)$ and $n \in \mathbb{N}$. Using this result, we can show that some properties of real operator spaces are closely related to the corresponding properties of their complexifications.

In Section 2 we first show that for every real operator space V there exists a canonical complex operator space matrix norm on its complexification V_c, which extends the original matrix norm on V and satisfies the reasonable condition (1.1). This canonical complex operator space structure on V_c can be explicitly expressed in terms of the matricial structure on V, i.e., we have the complete isometry

$$V_c = \left\{ \begin{bmatrix} x & -y \\ y & x \end{bmatrix} : x, y \in V \right\} \subseteq M_2(V). \tag{1.2}$$

In Section 3 we prove (Theorem 3.1) that any complex operator space matrix norm extension satisfying the reasonable condition (1.1) on V_c must be equal to the canonical complex operator space matrix norm given in (1.2). Therefore, up to complete isometry, there is a unique reasonable complex operator space structure on V_c. In Section 4, we show that some properties of real operator spaces are closely related to the corresponding properties of their complexifications.

We assume that readers are familiar with complex operator spaces and operator algebras. The fundamental results of complex operator spaces can be found in the recent books of Effros and Ruan [5], Paulsen [11], and Pisier [15]. The fundamentals of (complex) operator algebras can be found in Dixmier [4], Kadison and Ringrose [7], Pedersen [12], and Takesaki [22]. Real C^*-algebras and real von Neumann algebras have also been studied in the literature. Readers are referred to Goodearl [6], Li [8], Schröder [19], and Størmer [20], [21] for details.

2. Complexifications of real operator spaces

Let V be a real vector space. The *complexification* V_c of V is defined as the direct sum

$$V_c = V \dot{+} \mathrm{i}V = \{x + \mathrm{i}y : \ x, y \in V\}.$$

There is a natural complex linear structure on V_c given by

$$(x_1 + \mathrm{i}y_1) + (x_2 + \mathrm{i}y_2) = (x_1 + x_2) + \mathrm{i}(y_1 + y_2)$$

and

$$(\alpha + \mathrm{i}\beta)(x + \mathrm{i}y) = (\alpha x - \beta y) + \mathrm{i}(\beta x + \alpha y).$$

We can also define a natural conjugation on V_c by letting

$$\overline{x + \mathrm{i}y} = x - \mathrm{i}y.$$

Then up to the identification $x = x + \mathrm{i}0$, we may identify V with the *real part* of V_c since an element $z = x + \mathrm{i}y \in V_c$ is contained in V if and only if $\overline{z} = z$.

In general, a (complex) *conjugation* on a complex vector space W is defined to be a conjugate linear isomorphism $\bar{\ }$ on W such that $\bar{\ }^2 = \mathrm{id}_W$. This definition is the same as the *involution* on complex vector spaces. However, the two notions have different meanings on operator spaces. If W is a complex operator space, then we may define a conjugation on the $n \times n$ matrix space $M_n(W)$ by letting $\overline{[z_{ij}]} = [\,\overline{z_{ij}}\,]$. In this case, we assume

$$\|[z_{ij}]\| = \|\overline{[z_{ij}]}\| = \|[\,\overline{z_{ij}}\,]\|.$$

On the other hand, if * is an involution on W, then the induced involution on $M_n(W)$ is usually given by $[z_{ij}]^* = [z_{ji}^*]$ and we usually assume

$$\|[z_{ij}]\| = \|[z_{ij}]^*\| = \|[z_{ji}^*]\|.$$

The two notions are also different on complex algebras. If A is a complex algebra, then the conjugation $\bar{\ }$ on A should satisfy $\overline{ab} = \overline{a}\overline{b}$ and the involution * on A should satisfy $(ab)^* = b^* a^*$ for all $a, b \in A$.

Given real spaces V and W, we let $L(V, W)$ denote the space of all real linear maps from V into W and $L(V_c, W_c)$ the space of all complex linear maps from V_c into W_c. Then every real linear map $T \in L(V, W)$ can be uniquely extended to a complex linear map $T_c \in L(V_c, W_c)$, defined by

$$T_c(x + \mathrm{i}y) = T(x) + \mathrm{i}T(y).$$

On the other hand, a map $S \in L(V_c, W_c)$ has the form $S = T_c$ for some $T \in L(V, W)$ if and only if it is a *real map*, i.e., if it satisfies $\overline{S} = S$, where $\bar{\ }$ is the conjugation defined by

$$\overline{S}(x + \mathrm{i}y) = \overline{S(x - \mathrm{i}y)}.$$

Therefore, we may identify $L(V, W)$ with a real subspace of $L(V_c, W_c)$. Furthermore, every $S \in L(V_c, W_c)$ can be written as

$$S = S^R + \mathrm{i}S^I,$$

where $S^R = \frac{1}{2}(S+\overline{S})$ and $S^I = \frac{1}{2\mathrm{i}}(S-\overline{S})$ are real maps contained in $L(V,W)$. Since $L(V,W)\bigcap \mathrm{i}L(V,W) = \{0\}$, we can decompose $L(V_c, W_c)$ into a direct sum

$$L(V_c, W_c) = L(V,W) \dot{+} \mathrm{i}L(V,W). \tag{2.1}$$

The complexifications of real Banach spaces have been studied in the literature (cf. [6], [8] and [10]). Let V be a real Banach space. A complex Banach space norm $\|\cdot\|_{V_c}$ on V_c is called a *reasonable complex extension* of the original norm $\|\cdot\|$ on V if it satisfies

$$\|x+\mathrm{i}0\|_{V_c} = \|x\| \tag{2.2}$$

and

$$\|x+\mathrm{i}y\|_{V_c} = \|x-\mathrm{i}y\|_{V_c} \tag{2.3}$$

for all $x+\mathrm{i}y \in V_c$.

Given a real Banach space $(V, \|\cdot\|)$, we can define a real Banach space norm $|\cdot|_\infty$ on V_c by letting

$$|x+\mathrm{i}y|_\infty = \max\{\|x\|, \|y\|\}.$$

Then it is easy to verify that

$$\begin{aligned}\|x+\mathrm{i}y\|_\infty &= \sup\{|e^{\mathrm{i}\theta}(x+\mathrm{i}y)|_\infty : \theta \in [0,2\pi]\}\\ &= \sup\{\|x\cos\theta - y\sin\theta\| : \theta \in [0,2\pi]\}\\ &= \sup\{|f(x)\cos\theta - f(y)\sin\theta| : \theta\in[0,2\pi], f \in V^* \text{ with } \|f\|\le 1\}\\ &= \sup\{\sqrt{f(x)^2+f(y)^2} : f\in V^* \text{ with } \|f\|\le 1\}\end{aligned}$$

is a complex Banach space norm on V_c. This complex norm, which was first considered by A.E. Taylor (see [9]), is a reasonable complex extension of the original norm on V.

For $1\le p<\infty$, we can define a real Banach space norm

$$|x+\mathrm{i}y|_p = (\|x\|^p + \|y\|^p)^{1/p}$$

on V_c. Then

$$\|x+\mathrm{i}y\|_p = \frac{1}{c_p}\sup\{|e^{\mathrm{i}\theta}(x+\mathrm{i}y)|_p : \theta\in[0,2\pi]\}$$

with $c_p = \sup\{(|\cos\theta|^p + |\sin\theta|^p)^{1/p} : \theta \in [0,2\pi]\}$ is a complex Banach space norm on V_c (see [8]). It is easy to see that these complex Banach space norms are distinct reasonable complex extensions of the original norm on V with

$$\|x+\mathrm{i}y\|_\infty \le \|x+\mathrm{i}y\|_p \le 2\|x+\mathrm{i}y\|_\infty$$

for all $x+\mathrm{i}y\in V_c$ (see [10]).

If A is a real C^*-algebra, it is known that there exists a unique C^*-algebra norm on A_c which satisfies (2.2) and (2.3). In this case the induced conjugation $\bar{}$ is a conjugate automorphism on A_c, i.e., it is a conjugate linear isomorphism on A_c such that $\overline{ab} = \overline{a}\overline{b}$ for all $a, b \in A_c$. Therefore, a complex C^*-algebra B is the complexification of some real C^*-algebra A if and only if there exists a conjugate automorphism $\_$ on B. We note that $\kappa = {}^* \circ -$, the composition of the involution and the conjugation, is a *-anti-automorphism on B. Then a complex C^*-algebra B is the complexification of some real C^*-algebra A if and only if there exists a *-anti-automorphism on B. It follows that every group C^*-algebra (respectively, every group von Neumann algebra or, more generally, every Kac algebra) is a complexification of some real C^*-algebra (respectively, a complexification of some real von Neumann algebra). However, it was shown by Connes [3] and Phillips [16][17] that there exist von Neumann algebras and C^*-algebras which do not have any *-anti-automorphism. Thus there exist von Neumann algebras or C^*-algebras which are not the complexification of any real von Neumann algebras or any real C^*-algebras.

We now consider the complexifications of real operator spaces. Let us assume that V is a real operator space. Then for each $n \in \mathbb{N}$, we may decompose $M_n(V_c)$ into the direct sum

$$M_n(V_c) = M_n(V) \dot{+} \mathrm{i} M_n(V).$$

There is a canonical conjugation on $M_n(V_c)$ given by

$$\overline{x + \mathrm{i}y} = \overline{[\, x_{ij} + \mathrm{i}\, y_{ij} \,]} = [\, x_{ij} - \mathrm{i}\, y_{ij} \,] = x - \mathrm{i}y$$

for all $x + \mathrm{i}y = [\, x_{ij} + \mathrm{i}\, y_{ij} \,] \in M_n(V_c)$. A complex operator operator space matrix norm $\{\| \cdot \|_n\}$ on V_c is called a *reasonable complex extension* of the matrix norm on V if it satisfies

$$\|x + \mathrm{i}\, 0\|_n = \|x\|_n \tag{2.4}$$

and

$$\|x + \mathrm{i}\, y\|_n = \|x - \mathrm{i}\, y\|_n. \tag{2.5}$$

for all $x + \mathrm{i}\, y \in M_n(V_c)$ and $n \in \mathbb{N}$.

If H is a real Hilbert space with an inner product $\langle \mid \rangle$, we may define a complex inner product on H_c by letting

$$\langle (\xi_1 + \mathrm{i}\eta_1) \mid (\xi_2 + \mathrm{i}\eta_2) \rangle = \langle \xi_1 \mid \xi_2 \rangle + \langle \eta_1 \mid \eta_2 \rangle + \mathrm{i}\langle \eta_1 \mid \xi_2 \rangle - \mathrm{i}\langle \xi_1 \mid \eta_2 \rangle.$$

The induced Hilbert space norm $\| \cdot \|$ is equal to the reasonable complex extension norm $\| \cdot \|_2$ (discussed above) on H_c and satisfies

$$\|\xi + \mathrm{i}\eta\|^2 = \|\xi\|^2 + \|\eta\|^2 = \|\xi - \mathrm{i}\eta\|^2.$$

For every bounded operator $x \in B(H)$, the operator norm of its complex extension x_c on H_c is the same as the operator norm of x on H, i.e., we have $\|x_c\| = \|x\|$. Therefore, we may isometrically identify $B(H)$ with the real part of $B(H_c)$. It follows from (2.1) that we have the identification

$$B(H_c) = B(H) \dot{+} \mathrm{i}B(H) = B(H)_c.$$

The induced conjugation on $B(H_c)$ is given by $\overline{x+\mathrm{i}y} = x - \mathrm{i}y$ and satisfies

$$\|x+\mathrm{i}y\| = \|x-\mathrm{i}y\|$$

for all $x+\mathrm{i}y \in B(H_c)$. Moreover, for each $n \in \mathbb{N}$ we have the isometry $(H_c)^n = (H^n)_c$, and we obtain the isometric identifications

$$M_n(B(H_c)) = B(H^n) \dotplus \mathrm{i}\, B(H^n) = M_n(B(H)) \dotplus \mathrm{i}\, M_n(B(H)).$$

The canonical conjugation on $M_n(B(H_c))$ is given by

$$\overline{x+\mathrm{i}\,y} = \overline{[\,x_{ij}+\mathrm{i}\,y_{ij}\,]} = [x_{ij}-\mathrm{i}\,y_{ij}] = x-\mathrm{i}\,y,$$

and its norm satisfies the reasonable condition

$$\|x+\mathrm{i}\,y\|_n = \|x-\mathrm{i}\,y\|_n.$$

Therefore, the canonical complex operator space matrix norm on $B(H_c)$ is a reasonable complex extension of the operator space matrix norm on $B(H)$.

If $V \subseteq B(H)$ is a real operator subspace of $B(H)$, we may obtain a complex operator space matrix norm $\{\|\cdot\|_n\}$ on V_c by identifying V_c with the complex operator subspace of $B(H)_c = B(H_c)$, i.e., we have the identification

$$V \dotplus \mathrm{i}V \subseteq B(H) \dotplus \mathrm{i}B(H) = B(H_c). \tag{2.6}$$

This complex operator space matrix norm on V_c satisfies (2.4) and (2.5) and thus is a reasonable complex extension of the original matrix norm on V.

Theorem 2.1. *Let V and W be real operator spaces on real Hilbert spaces H and K and let $T : V \to W$ be a complete contraction* (*respectively, a complete isometry from V into W*). *If V_c and W_c are equipped with the canonical complex operator space matrix norms from $B(H_c)$ and $B(K_c)$, then $T_c : V_c \to W_c$ is a complete contraction with $\|T_c\|_{cb} = \|T\|_{cb}$* (*respectively, a complete isometry from V_c into W_c*).

Proof. Let us first assume that $T : V \to W \subseteq B(K)$ is a complete contraction. Since $B(K)$ is an injective real operator space (see [18, Theorem 3.1]), T has a completely contractive extension $\Phi : B(H) \to B(K)$. It follows from [18, Theorem 4.3] that there exist a real Hilbert space L, a unital *-representation $\pi : B(H) \to B(L)$ and contractive operators $s, t \in B(K, L)$ such that

$$\Phi(x) = s^*\pi(x)t$$

for all $x \in B(H)$. Then $\pi_c : B(H_c) = B(H)_c \to B(L_c) = B(L)_c$ defined by

$$\pi_c(x+\mathrm{i}y) = \pi(x) + \mathrm{i}\pi(y)$$

is a unital *-representation from $B(H_c)$ into $B(L_c)$ and s_c and t_c are contractive operators in $B(K_c, L_c)$. Given $\xi + \mathrm{i}\eta \in K_c$, we have

$$\begin{aligned}
&s_c^*\pi_c(x+\mathrm{i}y)t_c(\xi+\mathrm{i}\eta)\\
&\qquad= s_c^*(\pi(x)+\mathrm{i}\pi(y))(t(\xi)+\mathrm{i}t(\eta))\\
&\qquad= s_c^*[(\pi(x)t(\xi)-\pi(y)t(\eta))+\mathrm{i}(\pi(y)t(\xi)+\pi(x)t(\eta))]\\
&\qquad= (s^*\pi(x)t(\xi)-s^*\pi(y)t(\eta))+\mathrm{i}(s^*\pi(y)t(\xi)+s^*\pi(x)t(\eta))\\
&\qquad= (\Phi(x)\xi-\Phi(y)\eta)+\mathrm{i}(\Phi(y)\xi+\Phi(x)\eta)\\
&\qquad= \Phi_c(x+\mathrm{i}y)(\xi+\mathrm{i}\eta).
\end{aligned}$$

This shows that $\Phi_c = s_c^*\pi_c t_c$ is a complete contraction from $B(H_c)$ into $B(K_c)$. The restriction of Φ_c to V_c is equal to $T_c : V_c \to W_c$. Therefore, T_c is a complete contraction. This shows that $\|T_c\|_{cb} \le \|T\|_{cb}$. It is obvious that $\|T\|_{cb} \le \|T_c\|_{cb}$. Therefore we have $\|T_c\|_{cb} = \|T\|_{cb}$.

If T is a complete isometry, we may apply the above argument to $T^{-1} : T(V) \to V$ and show that T_c^{-1} is also a complete contraction. Therefore, T_c must be a complete isometry. □

As a consequence of Theorem 2.1, we see that the complex operator space matrix norm on V_c determined by (2.6) is actually independent of the choice of Hilbert spaces. We call it the *canonical reasonable complex extension* of the matrix norm on V. In the following, we show that this canonical operator space structure on V_c can be explicitly expressed in terms of the matricial structure on V.

Let us recall that the complex field $\mathbb{C} = \{\alpha + \mathrm{i}\beta\}$ with the natural norm $|\alpha+\mathrm{i}\beta| = (\alpha^2+\beta^2)^{1/2}$ and the involution $(\alpha+\mathrm{i}\beta)^* = \alpha-\mathrm{i}\beta$ is a 2-dimensional commutative real C^*-algebra. It can be identified with the real C^*-subalgebra

$$\mathbb{C} = \left\{ \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \in B(\mathbb{R}^2) : \ \alpha, \beta \in \mathbb{R} \right\}$$

of $M_2(\mathbb{R})$. Since we may define a complete contraction

$$P : \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \to \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

from $\mathbb{C}$ onto the column Hilbert space $C(2) = M_{2,1}(\mathbb{R})$, and a complete contraction

$$Q : \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \to \begin{bmatrix} \alpha & -\beta \end{bmatrix}$$

from $\mathbb{C}$ onto the row Hilbert space $R(2) = M_{1,2}(\mathbb{R})$, the canonical real operator space matrix norm on $\mathbb{C}$ dominates the column and row Hilbert space matrix norms on $\ell_2(2)$, i.e., we have

$$\|x\|_{M_n(\mathbb{C})} \ge \max\left\{ \left\| \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\|, \left\| \begin{bmatrix} \alpha & \beta \end{bmatrix} \right\| \right\} = \max\{\|x\|_{M_n(C(2))}, \|x\|_{M_n(R(2))}\}$$

for all $x = \alpha + \mathrm{i}\beta$ with $\alpha, \beta \in M_n(\mathbb{R})$. The referee pointed out to the author that if we let $x = \alpha + \mathrm{i}\beta \in M_2(\mathbb{C})$ with $\alpha = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $\beta = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ in $M_2(\mathbb{R})$, then we actually have

$$2 = \|x\|_{M_n(\mathbb{C})} > \max\left\{\left\|\begin{bmatrix}\alpha \\ \beta\end{bmatrix}\right\|, \left\|\begin{bmatrix}\alpha & \beta\end{bmatrix}\right\|\right\} = \sqrt{2}.$$

Therefore, the canonical real operator space matrix norm on $\mathbb{C}$ is greater than the real operator matrix norm on $C(2) \bigcap R(2)$.

For every $n \in \mathbb{N}$, $M_n(\mathbb{C})$ is also a real C^*-algebra and can be identified with the real C^*-subalgebra

$$M_n(\mathbb{C}) = M_n(\mathbb{R}) \dot{+} \mathrm{i}M_n(\mathbb{R}) = \left\{\begin{bmatrix}\alpha & -\beta \\ \beta & \alpha\end{bmatrix} : \alpha, \beta \in M_n(\mathbb{R})\right\}$$

of $M_2(M_n(\mathbb{R}))$. In general, if H is an (infinite-dimensional) real Hilbert space (say, $H = \ell_2(I, \mathbb{R})$), then we have the identification

$$B(H_c) = B(H) \dot{+} \mathrm{i}B(H) = \left\{\begin{bmatrix}x & -y \\ y & x\end{bmatrix} : x, y \in B(H)\right\}.$$

If $x \in B(H)$, then we have the identification

$$x \cong \begin{bmatrix}x & 0 \\ 0 & x\end{bmatrix} \text{ and } ix \cong \begin{bmatrix}0 & -x \\ x & 0\end{bmatrix}.$$

The complex scalar multiplication on $B(H_c)$ is given by

$$(\alpha + \mathrm{i}\beta)(x + \mathrm{i}y) \cong \begin{bmatrix}\alpha & -\beta \\ \beta & \alpha\end{bmatrix}\begin{bmatrix}x & -y \\ y & x\end{bmatrix}.$$

With this observation, we see that the canonical complex operator space structure on V_c is determined by the identification

$$V_c = V \dot{+} \mathrm{i}V = \left\{\begin{bmatrix}x & -y \\ y & x\end{bmatrix} : x, y \in V\right\}, \tag{2.7}$$

where the latter space is a real subspace of $M_2(V)$.

3. Uniqueness of reasonable matrix norm extensions

Let V be a real operator space and let $\{\|\cdot\|_n\}$ be the canonical complex operator space matrix norm on V_c given by (2.7). The following theorem shows that, up to complete isometry, this is the only reasonable complex extension of the original operator space matrix norm on V.

Theorem 3.1. *Let V be a real operator space. If a complex operator space matrix norm $\{|||\cdot|||_n\}$ on V_c is a reasonable complex extension of the original matrix norm on V, then $\{|||\cdot|||_n\}$ must be equal to the canonical matrix norm $\{\|\cdot\|_n\}$ on V_c.*

Proof. Let us assume that there exist a complex Hilbert space K and a complex completely isometric inclusion

$$\Phi : (V_c, \{|\|\cdot\||_n\}) \hookrightarrow (B(K), \{\|\cdot\|_n\}).$$

Since we may write $K = H + \mathrm{i}H$ for some real Hilbert space H, we have a canonical conjugation $\bar{\ }$ on $B(K) = B(H)_c$. Together with the conjugation given on V_c, we can define a conjugation of Φ by letting

$$\overline{\Phi}(x + \mathrm{i}y) = \overline{\Phi(x - \mathrm{i}y)}$$

for all $x + \mathrm{i}y \in V_c$. If Φ is a real map, i.e., if it completely isometrically maps V into $B(H)$, then the complex matrix norm $\{|\|\cdot\||_n\}$ is just the canonical complex operator space matrix norm on V_c obtained by identifying V with $\Phi(V) \subseteq B(H)$.

If Φ is not a real map, we let $\Phi^R = \frac{1}{2}(\Phi + \overline{\Phi})$ and $\Phi^I = \frac{1}{2\mathrm{i}}(\Phi - \overline{\Phi})$. Then Φ^R and Φ^I are real maps from V_c into $B(K)$ and we can write

$$\Phi = \Phi^R + \mathrm{i}\Phi^I.$$

Since $\Phi(x + \mathrm{i}y) = (\Phi^R(x) - \Phi^I(y)) + \mathrm{i}(\Phi^R(y) + \Phi^I(x))$, the map Φ can be expressed in the following real matrix form:

$$\tilde{\Phi}(x + \mathrm{i}y) = \begin{bmatrix} \Phi^R(x) - \Phi^I(y) & -\Phi^R(y) - \Phi^I(x) \\ \Phi^R(y) + \Phi^I(x) & \Phi^R(x) - \Phi^I(y) \end{bmatrix} \in M_2(B(H)). \tag{3.1}$$

Then $\tilde{\Phi}$, restricted to V, is a real completely isometric inclusion from V into $M_2(B(H))$. It follows from Theorem 2.1 that its complex extension $\tilde{\Phi}_c : V_c \to M_2(B(H))_c$ is a completely isometric inclusion from V_c (with the canonical complex matrix norm $\{\|\cdot\|_n\}$) into $M_2(B(H))_c \subseteq M_4(B(H))$ and can be written as

$$\tilde{\Phi}_c(x + \mathrm{i}y) = \tilde{\Phi}(x) + \mathrm{i}\tilde{\Phi}(y) = \begin{bmatrix} \tilde{\Phi}(x) & -\tilde{\Phi}(y) \\ \tilde{\Phi}(y) & \tilde{\Phi}(x) \end{bmatrix} \in M_4(B(H)).$$

Then we have

$$\|[\tilde{\Phi}_c(x_{ij} + \mathrm{i}y_{ij})]\|_n = \|[x_{ij} + \mathrm{i}y_{ij}]\|_n$$

for all $[x_{ij} + \mathrm{i}y_{ij}] \in M_n(V_c)$ and $n \in \mathbb{N}$.

If we let U be the unitary matrix

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \in M_4(\mathbb{R}),$$

then

$$\tilde{\Phi}_c(x+\mathrm{i}y) = \begin{bmatrix} \tilde{\Phi}(x) & -\tilde{\Phi}(y) \\ \tilde{\Phi}(y) & \tilde{\Phi}(x) \end{bmatrix} = \begin{bmatrix} \Phi^R(x) & -\Phi^I(x) & -\Phi^R(y) & \Phi^I(y) \\ \Phi^I(x) & \Phi^R(x) & -\Phi^I(y) & -\Phi^R(y) \\ \Phi^R(y) & -\Phi^I(y) & \Phi^R(x) & -\Phi^I(x) \\ \Phi^I(y) & \Phi^R(y) & \Phi^I(x) & \Phi^R(x) \end{bmatrix}$$

$$= U^* \begin{bmatrix} \Phi^R(x)-\Phi^I(y) & -\Phi^R(y)-\Phi^I(x) & 0 & 0 \\ \Phi^R(y)+\Phi^I(x) & \Phi^R(x)-\Phi^I(y) & 0 & 0 \\ 0 & 0 & \Phi^R(x)+\Phi^I(y) & \Phi^R(y)-\Phi^I(x) \\ 0 & 0 & -\Phi^R(y)+\Phi^I(x) & \Phi^R(x)+\Phi^I(y) \end{bmatrix} U.$$

We note that the upper left 2×2 corner in the last 4×4 matrix can be expressed as

$$\tilde{\Phi}(x+\mathrm{i}y) = \begin{bmatrix} \Phi^R(x)-\Phi^I(y) & -\Phi^R(y)-\Phi^I(x)) \\ \Phi^R(y)+\Phi^I(x) & \Phi^R(x)-\Phi^I(y) \end{bmatrix} \in M_2(B(H))$$

(see (3.1)), and the lower right 2×2 corner as

$$\tilde{\Phi}(x-\mathrm{i}y) = \begin{bmatrix} \Phi^R(x)+\Phi^I(y) & \Phi^R(y)-\Phi^I(x)) \\ -\Phi^R(y)+\Phi^I(x) & \Phi^R(x)+\Phi^I(y) \end{bmatrix} \in M_2(B(H)).$$

Since $|||x+\mathrm{i}y||| = |||x-\mathrm{i}y|||$ for every $x+\mathrm{i}y \in V_c$, we conclude that

$$|||x+\mathrm{i}y||| = \max\{|||x+\mathrm{i}y|||, |||x-\mathrm{i}y|||\} = \left\| \begin{bmatrix} \tilde{\Phi}(x+\mathrm{i}y) & 0 \\ 0 & \tilde{\Phi}(x-\mathrm{i}y) \end{bmatrix} \right\|$$
$$= \|\tilde{\Phi}_c(x+\mathrm{i}y)\| = \|x+\mathrm{i}y\|.$$

This shows that $|||\cdot|||$ is equal to the canonical norm $\|\cdot\|$ on V_c. Applying the same argument to the matrix spaces $M_n(V_c)$, we conclude that the reasonable complex extension $\{|||\cdot|||_n\}$ is equal to the canonical operator space matrix norm $\{\|\cdot\|_n\}$ on V_c. □

If W is a complex vector space with a conjugation $\bar{\ }$, we let

$$\operatorname{Re} W = \{z \in W \text{ such that } \overline{z} = z\}$$

and

$$\operatorname{Im} W = \{z \in W \text{ such that } \overline{z} = -z\}.$$

It is easy to see that $\operatorname{Re} W \cap \operatorname{Im} W = \{0\}$ and $\mathrm{i}\operatorname{Re} W = \operatorname{Im} W$. If we let $V = \operatorname{Re} W$, then W can be decomposed into the (vector space) direct sum

$$W = V \dotplus \mathrm{i}V, \tag{3.2}$$

and the conjugation on W is given by $\overline{x+\mathrm{i}y} = x - \mathrm{i}y$ for all $z = x+\mathrm{i}y \in W$.

THEOREM 3.2. *Let $(W, \{\|\cdot\|_n\})$ be a complex operator space. Then W has a conjugation $\bar{\ }$ such that*

$$|||[\, z_{ij}\,]|||_n = |||[\, \overline{z_{ij}}\,]|||_n \tag{3.3}$$

for all $[z_{ij}] \in M_n(W)$ *and* $n \in \mathbb{N}$ *if and only if there exists a real operator space* V *such that we have the complete isometry* $W = V_c$ *and the conjugation on* W *is given by*

$$\overline{z} = \overline{x + \mathrm{i}y} = x - \mathrm{i}y$$

for all $z = x + \mathrm{i}y \in W$.

Proof. Let W be a complex operator space with a conjugation $\bar{\ }$. Then $V = \operatorname{Re} W$ is a real subspace of W. Since the matrix norm on W restricted to V satisfies the conditions (M1) and (M2), V is an abstract real operator space, and thus there exists a (uniquely defined) canonical complex operator space matrix norm on its complexification V_c. It follows from (3.2) that as vector spaces we have $W = V_c$ and every element z in W can be uniquely written as $z = x + \mathrm{i}y$ for some $x, y \in V$. In this case, the conjugation on W is given by

$$\overline{z} = \overline{x + \mathrm{i}y} = x - \mathrm{i}y.$$

The induced conjugation on the matrix space $M_n(W)$ is given by

$$\overline{[\, z_{ij} \,]} = \overline{[\, x_{ij} + \mathrm{i}\, y_{ij} \,]} = [\, x_{ij} - \mathrm{i}\, y_{ij} \,].$$

Then condition (3.3) implies that the complex operator space matrix norm $\{\| \cdot \|_n\}$ on W is a reasonable complex extension of the matrix norm on V. Thus we have the complete isometry $W = V_c$ by Theorem 3.1.

The other direction is obvious. □

Remark 3.3. Let us recall that the real quaternion ring $\mathbb{H} = \{\alpha + \beta \mathrm{i} + \gamma \mathrm{j} + \delta \mathrm{k}\}$ with the Hilbert space norm

$$|\alpha + \beta \mathrm{i} + \gamma \mathrm{j} + \delta \mathrm{k}| = (\alpha^2 + \beta^2 + \gamma^2 + \delta^2)^{1/2}$$

and the involution

$$(\alpha + \beta \mathrm{i} + \gamma \mathrm{j} + \delta \mathrm{k})^* = \alpha - \beta \mathrm{i} - \gamma \mathrm{j} - \delta \mathrm{k}$$

is a 4-dimensional non-commutative real C^*-algebra (see [6], [8] and [20]). Its complexification $\mathbb{H}_c$ is a non-commutative 4-dimensional complex C^*-algebra, and thus we must have $\mathbb{H}_c = M_2(\mathbb{C})$. On the other hand, $M_2(\mathbb{C})$ is also the complexification of the non-commutative real C^*-algebra $M_2(\mathbb{R})$. It is clear that $\mathbb{H}$ is not isometric to $M_2(\mathbb{R})$ since $M_2(\mathbb{R})$ is not a Hilbert space. This shows that there exists two distinct real C^*-algebras (or real operator spaces) $\mathbb{H}$ and $M_2(\mathbb{R})$ such that

$$\mathbb{H}_c = M_2(\mathbb{C}) = M_2(\mathbb{R})_c.$$

Therefore, we may have two distinct conjugations on a complex C^*-algebra (or a complex operator space).

4. Properties of real operator spaces associated with their complexifications

If V is a real operator space, we let V_c denote the complexification of V equipped with the canonical complex operator space matrix norm. Then we may identify V_c with

$$V_c = \left\{ \begin{bmatrix} x & -y \\ y & x \end{bmatrix} : x, y \in V \right\} \subseteq M_2(V).$$

For any completely bounded map $T : V \to W$, its complex extension T_c can be expressed as

$$T_c \left(\begin{bmatrix} x & -y \\ y & x \end{bmatrix} \right) = \left(\begin{bmatrix} T(x) & -T(y) \\ T(y) & T(x) \end{bmatrix} \right). \tag{4.1}$$

In the following, we discuss some connections between V and V_c.

Proposition 4.1. *Let V be a real operator space. Then V is an injective real operator space if and only if V_c is an injective complex operator space.*

Proof. Let us first assume that V is an injective real operator space contained in some $B(H)$, and let $P : B(H) \to V$ be a completely contractive projection from $B(H)$ onto V. Then V_c is a complex operator subspace of $B(H_c)$ and it follows from Theorem 2.1 that P_c is a completely contractive projection from $B(H_c)$ onto V_c. This shows that V_c is an injective complex operator space.

On the other hand, let us assume that V_c is an injective complex operator space, and let $Q : B(H_c) \to V_c$ be a complex completely contractive projection from $B(H_c)$ onto V_c. Then $Q^R = \frac{1}{2}(Q + \overline{Q})$ is a completely contractive projection from $B(H)$ onto V. This shows that V is an injective real operator space. □

A complex (respectively, real) operator space V is said to be λ-*nuclear* (for some $\lambda \geq 1$) if there exist two nets of completely bounded maps $S_\alpha : V \to M_n(\mathbb{F})$ and $T_\alpha : M_n(\mathbb{F}) \to V$ with $\mathbb{F} = \mathbb{C}$ (respectively, $\mathbb{F} = \mathbb{R}$) such that $\|S_\alpha\|_{cb}\|T_\alpha\|_{cb} \leq \lambda$ and $\|T_\alpha \circ S_\alpha(x) - x\| \to 0$ for all $x \in V$.

Proposition 4.2. *A real operator space V is λ-nuclear if and only if its complexification V_c is λ-nuclear.*

Proof. Let us first assume that V is λ-nuclear. Then there exist maps $S_\alpha : V \to M_n(\mathbb{R})$ and $T_\alpha : M_n(\mathbb{R}) \to V$ such that $\|S_\alpha\|_{cb}\|T_\alpha\|_{cb} \leq \lambda$ and $\|T_\alpha \circ S_\alpha(x) - x\| \to 0$ for all $x \in V$. It follows from Theorem 2.1 that $\tilde{S}_\alpha = (S_\alpha)_c : V_c \to M_n(\mathbb{C})$ and $\tilde{T}_\alpha = (T_\alpha)_c : M_n(\mathbb{C}) \to V_c$ are completely bounded complex linear maps such that

$$\|\tilde{S}_\alpha\|_{cb}\|\tilde{T}_\alpha\|_{cb} = \|S_\alpha\|_{cb}\|T_\alpha\|_{cb} \leq \lambda$$

and

$$\|\tilde{T}_\alpha \circ \tilde{S}_\alpha(x + \mathrm{i}y) - (x + \mathrm{i}y)\| \leq \|T_\alpha \circ S_\alpha(x) - x\| + \|T_\alpha \circ S_\alpha(y) - y\| \to 0$$

for all $x+\mathrm{i}y \in V_c$. This shows that the complex operator space V_c is λ-nuclear.

On the other hand, if V_c is a λ-nuclear complex operator space, then there exist completely bounded complex linear maps $\tilde{S}_\alpha : V_c \to M_n(\mathbb{C})$ and $\tilde{T}_\alpha : M_n(\mathbb{C}) \to V_c$ such that $\|\tilde{S}_\alpha\|_{cb}\|\tilde{T}_\alpha\|_{cb} \leq \lambda$ and $\|\tilde{T}_\alpha \circ \tilde{S}_\alpha(x+\mathrm{i}y)-(x+\mathrm{i}y)\| \to 0$. Let $S_\alpha : V \to M_n(\mathbb{C}) \subseteq M_{2n}(\mathbb{R})$ denote the restriction of $\tilde{S}_\alpha$ to V, and let $T_\alpha = Q \circ \tilde{T}_\alpha \circ P_\mathbb{C} : M_{2n}(\mathbb{R}) \to V$, where $P_\mathbb{C}$ is the completely contractive projection from $M_n(M_2(\mathbb{R}))$ onto $M_n(\mathbb{C})$ given by

$$P_\mathbb{C}\left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2}(\alpha + \delta) & \frac{1}{2}(\beta - \gamma) \\ \frac{1}{2}(\gamma - \beta) & \frac{1}{2}(\alpha + \delta) \end{bmatrix}$$

and Q is the completely contractive map from V_c onto V given by

$$Q\left(\begin{bmatrix} x & -y \\ y & x \end{bmatrix}\right) = x.$$

Then it is rountine to verify that $\|S_\alpha\|_{cb}\|T_\alpha\|_{cb} \leq \lambda$ and $\|T_\alpha \circ S_\alpha(x)-(x)\| \to 0$ for all $x \in V$. Therefore, V is a λ-nuclear real operator space. ☐

Similarly, we can study CCAP, CBAP, OAP, WEP for real operator spaces. We can also study the corresponding local properties such as exactness and local reflexivity for real operator spaces (for definitions see [5]). For example, a real operator space V is said to have the *OAP* if for every $[v_{ij}] \in K_\infty(V) = K_\infty(\mathbb{R})\check{\otimes}V$ and $\varepsilon > 0$ there exists a finite-rank map $T : V \to V$ such that

$$\|[T(v_{ij})] - [v_{ij}]\|_{K_\infty(V)} < \varepsilon.$$

A real operator space V is said to be *exact* if for every finite-dimensional subspace E of V and $\varepsilon > 0$ there exists a real linear isomorphism $T : E \to F$ from E onto an operator subspace F of some real matrix space $M_n(\mathbb{R})$ such that

$$\|T\|_{cb}\|T^{-1}\|_{cb} < 1 + \varepsilon.$$

We can easily obtain the following results, which are left as exercises for the reader.

Proposition 4.3. *Let V be a real operator space and let V_c be its complexification.*

1. *V has CCAP (respectively, CBAP, OAP, WEP) in the category of real operator spaces if and only if V_c has CCAP (respectively, CBAP, OAP, WEP) in the category of complex operator spaces.*
2. *V is real exact (respectively, real locally reflexive) if and only if V_c is complex exact (respectively, complex locally reflexive).*

Let A and B be two unital real C^*-algebras. If A is completely isometric to B, then it follows from Theorem 2.1 that A_c is completely isometric to B_c and thus A_c is *-isomorphic to B_c. It is natural to expect that the same result holds for real C^*-algebras, i.e., we expect that A is *-isomorphic to B. In the following, we show (directly) that this is indeed the case.

THEOREM 4.4. *Let A and B be unital real C^*-algebras. If $T : A \to B$ is a completely isometric isomorphism from A onto B, then there exists a unital *-isomorphism $\pi : A \to B$ and a unitary operator $u \in B$ such that*

$$T(a) = u\pi(a)$$

for all $a \in A$.

Proof. Let

$$\mathcal{L}_A = \left\{ \begin{bmatrix} \alpha 1 & a \\ b & \beta 1 \end{bmatrix} : \alpha, \beta \in \mathbb{R}, a, b \in A \right\}$$

be the real operator system induced by A and let

$$\mathcal{L}_B = \left\{ \begin{bmatrix} \alpha 1 & a \\ b & \beta 1 \end{bmatrix} : \alpha, \beta \in \mathbb{R}, a, b \in B \right\}$$

be the real operator system induced by B. Then $\Phi : \mathcal{L}_A \to \mathcal{L}_B$ defined by

$$\Phi\left(\begin{bmatrix} \alpha 1 & a \\ b & \beta 1 \end{bmatrix} \right) = \begin{bmatrix} \alpha I_H & T(a) \\ T(b^*)^* & \beta I_H \end{bmatrix}.$$

is a self-adjoint unital complete order isomorphism from $\mathcal{L}_A$ onto $\mathcal{L}_B$.

Let $\mathcal{I}_A$ and $\mathcal{I}_B$ denote the real injective envelopes of $\mathcal{L}_A$ and $\mathcal{L}_B$, respectively. Then $\mathcal{I}_A$ and $\mathcal{I}_B$ are injective real C^*-algebras. There exist a self-adjoint unital completely positive extension $\tilde{\Phi} : \mathcal{I}_A \to \mathcal{I}_B$ of Φ and a self-adjoint unital completely positive extension $\tilde{\Psi} : \mathcal{I}_B \to \mathcal{I}_A$ of Φ^{-1} (see [18, §4]). Since $\tilde{\Psi} \circ \tilde{\Phi}_{|\mathcal{L}_A} = \mathrm{id}_{\mathcal{L}_A}$, we must have $\tilde{\Psi} \circ \tilde{\Phi} = \mathrm{id}_{\mathcal{I}_A}$. Similarly, we must have $\tilde{\Phi} \circ \tilde{\Psi} = \mathrm{id}_{\mathcal{I}_B}$. Therefore, $\tilde{\Phi}$ is a unital complete order isomorphism, and thus a unital *-isomorphism from $\mathcal{I}_A$ onto $\mathcal{I}_B$. In this case, the map T can be expressed as

$$T(a) = \begin{bmatrix} 1 & 0 \end{bmatrix} \tilde{\Phi}\left(\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

If we let $u = T(1)$, then u is an element in B such that

$$uu^*T(a) = T(a) = T(a)u^*u$$

for all $a \in A$. It follows that $u^*u = 1 = uu^*$ and

$$\pi(a) = u^*T(a)$$

is a unital *-isomorphism from A onto B. Therefore, we have $T(a) = u\pi(a)$ for all $a \in A$. □

To end this section, we make the following remark on the duality of complexifications. Let V be a real operator space. Then every bounded complex linear functional $f \in (V_c)^* = B(V_c, \mathbb{C})$ can be written as $f = f^R + \mathrm{i} f^I$. For any $x + \mathrm{i}y \in V_c$, we have

$$f(x+\mathrm{i}y) = f(x) + \mathrm{i}f(y) = (f^R(x) - f^I(y)) + \mathrm{i}(f^I(x) + f^R(y)). \tag{4.2}$$

This complex duality (4.2) can also be expressed in real matrix form as follows:

$$\langle \begin{bmatrix} f^R & -f^I \\ f^I & f^R \end{bmatrix}, \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \rangle = \begin{bmatrix} f^R(x) - f^I(y) & -f^I(x) - f^R(y) \\ f^I(x) + f^R(y) & f^R(x) - f^I(y) \end{bmatrix}. \tag{4.3}$$

This shows that the complex dual $(V_c)^*$ of V_c can be identified with the complexification $(V^*)_c$ of V^*. In particular, bounded linear functionals in V^* correspond to the diagonal elements in $(V_c)^*$. Using this observation, we can turn some real duality problems to complex duality problems. We can also use the complex interpolation introduced by Pisier [13][14] to induce a canonical real operator space matrix norm on real (non-commutative) L_p spaces. We will discuss this elsewhere.

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