HOMOLOGICAL PROPERTIES OF BIGRADED ALGEBRAS

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Abstract. We investigate the $x$- and $y$-regularity of a bigraded $K$-algebra $R$ as introduced in [2]. These notions are used to study asymptotic properties of certain finitely generated bigraded modules. As an application we get for any equigenerated graded ideal $I$ upper bounds for the number $j_0$ for which $\text{reg}(I^j)$ is a linear function for $j \geq j_0$. Finally, we give upper bounds for the $x$- and $y$-regularity of generalized Veronese algebras.

Introduction

Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ be a standard bigraded polynomial ring with $\text{deg}(x_i) = (1,0)$ and $\text{deg}(y_j) = (0,1)$, and let $J \subset S$ be a bigraded ideal. In this paper we study homological properties of the bigraded algebra $R = S/J$.

First we consider the $x$- and the $y$-regularity of $R$. According to [2] they are defined as follows:

$$\text{reg}_x^S(R) = \max \{ a \in \mathbb{Z} : \beta_{i(a+i,b)}^S(R) \neq 0 \text{ for some } i, b \in \mathbb{Z} \},$$

$$\text{reg}_y^S(R) = \max \{ b \in \mathbb{Z} : \beta_{i(a+b+i)}^S(R) \neq 0 \text{ for some } i, a \in \mathbb{Z} \},$$

where $\beta_{i(a,b)}^S(R) = \dim_K \text{Tor}_i^S(K,R)_{(a,b)}$ is the $i$th bigraded Betti number of $R$ in bidegree $(a,b)$. We give a homological characterization of these regularities similarly as in the graded case (see [3]). As an application we generalize a result of Trung [13] concerning $d$-sequences. Furthermore we prove that

$$\text{reg}_x^S(S/J) = \text{reg}_x^S(S/\text{bigin}(J))$$

where $\text{bigin}(J)$ is the bigeneric initial ideal of $J$ with respect to the bigraded reverse lexicographic order induced by $y_1 > \cdots > y_m > x_1 > \cdots > x_n$.

It was shown in [7] (or [12]) that for $j \gg 0$, $\text{reg}(I^j)$ is a linear function $cj + d$ in $j$, for a graded ideal $I$ in the polynomial ring. In [12] the constant $c$ is described in terms of invariants of $I$. In this paper we give, in case $I$ is equigenerated, bounds $j_0$ such that for $j \geq j_0$ the function is linear, and we

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also give a bound for \(d\). Our methods can also be applied to \(\text{reg}(S^j(I))\), where \(S^j(I)\) is the \(j\)th symmetric power of \(I\).

In the last section we introduce a generalized Veronese algebra in the bigraded setting. For a bigraded algebra \(R\) and \(\Delta = (s, t) \in \mathbb{N}^2\) with \((s, t) \neq (0, 0)\) we set

\[
R_{\Delta} = \bigoplus_{(a, b) \in \mathbb{N}^2} R_{(as, bt)}.
\]

In the same manner as it was done in [6] for diagonal subalgebras, we prove that for these algebras

\[
\text{reg}_{x}(R_{\Delta}) = 0 \quad \text{and} \quad \text{reg}_{y}(R_{\Delta}) = 0 \quad \text{if} \quad s \gg 0 \quad \text{and} \quad t \gg 0.
\]

1. Preliminaries

Throughout this paper, let \(K\) be an infinite field and let \(S = K[x_1, \ldots, x_n, y_1, \ldots, y_m]\) be a standard bigraded polynomial ring with \(\deg(x_i) = (1, 0)\) and \(\deg(y_j) = (0, 1)\). Let \(M\) be a finitely generated bigraded \(S\)-module. For some bihomogeneous \(w \in M\) with \(\deg(w) = (a, b)\) we set \(\deg_x(w) = a\) and \(\deg_y(w) = b\). Sometimes we will consider the \(\mathbb{Z}\)-graded modules \(M_{(a, *)} = \bigoplus_{b \in \mathbb{Z}} M_{(a, b)}\) or \(M_{(*, b)} = \bigoplus_{a \in \mathbb{Z}} M_{(a, b)}\). If in addition \(M\) is \(\mathbb{N}^n \times \mathbb{N}^m\)-graded, we write \(M_{u, v}\) for the homogeneous component in bidegree \(u, v\), where \(u \in \mathbb{N}^n\) and \(v \in \mathbb{N}^m\). For \(u \in \mathbb{N}^n\) we set \(\text{supp}(u) = \{i : u_i > 0\}\).

Define \(m_x = (x_1, \ldots, x_n) = (x)\), \(m_y = (y_1, \ldots, y_m) = (y)\), and \(m = m_x + m_y\). Let \(S_x = K[x_1, \ldots, x_n]\) and \(S_y = K[y_1, \ldots, y_m]\) be the polynomial rings with respect to the \(x\)-variables and the \(y\)-variables.

For any \(u = (u_1, \ldots, u_n) \in \mathbb{N}^n\) and \(v = (v_1, \ldots, v_m) \in \mathbb{N}^m\) we write \(x^u y^v\) for the monomial \(x_1^{u_1} \cdots x_n^{u_n} y_1^{v_1} \cdots y_m^{v_m}\). For \(u, u' \in \mathbb{N}^n\) we let \(u \preceq u'\) if \(u_i \leq u'_i\) for all \(i\). Furthermore we set \(|u| = u_1 + \cdots + u_n\). Let \(\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}^n\), where the entry 1 is at the \(i\)th position. For \(t \in \mathbb{N}\) define \([t] = \{1, \ldots, t\}\).

We consider bigraded algebras \(R = S/J\), which are quotients of \(S\) by some bigraded ideal \(J\). For a finitely generated bigraded \(R\)-module \(M\) and \(a, b \in \mathbb{N}\) let \(\beta_{i, (a, b)}^R(M) = \dim_K \text{Tor}^R_i(M, K)_{(a, b)}\) be the \(i\)th bigraded Betti number in bidegree \((a, b)\). We recall from [2] that

\[
\text{reg}_x^R(M) = \sup \{a \in \mathbb{Z} : \beta_{i, (a+i, b)}^R(M) \neq 0 \text{ for some } i, b \in \mathbb{Z}\},
\]

\[
\text{reg}_y^R(M) = \sup \{b \in \mathbb{Z} : \beta_{i, (a, b+i)}^R(M) \neq 0 \text{ for some } i, a \in \mathbb{Z}\}
\]

is the \(x\)- and \(y\)-regularity of \(M\). In the case \(R = S\) we set \(\text{reg}_x(M) = \text{reg}_x^S(M)\) and \(\text{reg}_y(M) = \text{reg}_y^S(M)\).

Let \(K_s(k, l; M)\) denote the Koszul complex of \(M\) and \(H_s(k, l; M)\) the Koszul homology of \(M\) with respect to \(x_1, \ldots, x_k\) and \(y_1, \ldots, y_l\) (see [5] for details). If it is clear from the context, we write \(K_s(k, l)\) and \(H_s(k, l)\) instead of \(K_s(k, l; M)\) and \(H_s(k, l; M)\). Note that \(K_s(k, l; M) = K_s(k, l; S) \otimes_M M\) where \(K_s(k, l; S)\) is the exterior algebra on \(e_1, \ldots, e_k\) and \(f_1, \ldots, f_l\) with
we can ensure that a
Furthermore
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deg(e_i) = (1, 0) and deg(f_j) = (0, 1) together with a differential \( \partial \) induced by \( \partial(e_i) = x_i \) and \( \partial(f_j) = y_j \). For a cycle \( z \in K_i(k, l; M) \) we denote by \([z] \in H_i(k, l; M)\) the corresponding homology class. There are two long exact sequences relating the homology groups:

\[
\ldots \rightarrow H_i(k, l; M)(-1, 0) \xrightarrow{\partial_{k+1}} H_i(k, l; M) \\
\rightarrow H_i(k + 1, l; M) \rightarrow H_{i-1}(k, l; M)(-1, 0) \xrightarrow{\partial_{k+1}} \ldots \\
\rightarrow H_0(k, l; M)(-1, 0) \xrightarrow{\partial_{k+1}} H_0(k, l; M) \rightarrow H_0(k + 1, l; M) \rightarrow 0
\]

and

\[
\ldots \rightarrow H_i(k, l; M)(0, -1) \xrightarrow{\partial_{l+1}} H_i(k, l; M) \\
\rightarrow H_i(k, l + 1; M) \rightarrow H_{i-1}(k, l; M)(0, -1) \xrightarrow{\partial_{l+1}} \ldots \\
\rightarrow H_0(k, l; M)(0, -1) \xrightarrow{\partial_{l+1}} H_0(k, l; M) \rightarrow H_0(k, l + 1; M) \rightarrow 0.
\]

The map \( H_i(k, l; M) \rightarrow H_i(k + 1, l; M) \) is induced by the inclusion of the corresponding Koszul complexes. Every homogeneous element \( z \in K_i(k + 1, l; M) \) can be uniquely written as \( e_{k+1} \wedge z' + z'' \) with \( z', z'' \in K_i(k, l; M) \). Then \( H_i(k + 1, l; M) \rightarrow H_{i-1}(k, l; M)(-1, 0) \) is given by sending \([z]\) to \([z']\).

Furthermore \( H_i(k, l; M)(-1, 0) \xrightarrow{\partial_{k+1}} H_i(k, l; M) \) is just the multiplication with \( x_{k+1} \). The maps in the other exact sequence are defined analogously.

2. Regularity

Let \( R \) be a bigraded algebra. To simplify the notation we do not distinguish between the polynomial ring variables \( x_i \) or \( y_j \) and the corresponding residue classes in \( R \). Following [3] (or [13] under the name filter regular element) we call an element \( x \in R_{(1,0)} \) an almost regular element for \( R \) (with respect to the \( x \)-degree) if

\[
(0 :_R x)_{(a, *)} = 0 \text{ for } a \gg 0.
\]

A sequence \( x_1, \ldots, x_t \in R_{(1,0)} \) is an almost regular sequence (with respect to the \( x \)-degree) if for all \( i \in [t] \) the element \( x_i \) is almost regular for \( R/(x_1, \ldots, x_{i-1})R \).

Analogously we call an element \( y \in R_{(0,1)} \) an almost regular element for \( R \) (with respect to the \( y \)-degree) if

\[
(0 :_R y)_{(*, b)} = 0 \text{ for } b \gg 0.
\]

A sequence \( y_1, \ldots, y_t \in R_{(0,1)} \) is an almost regular sequence (with respect to the \( y \)-degree) if for all \( i \in [t] \) the element \( y_i \) is almost regular for \( R/(y_1, \ldots, y_{i-1})R \).

It is well-known that, provided \(|K| = \infty\), by a generic choice of coordinates we can ensure that a \( K \)-basis of \( R_{(1,0)} \) is almost regular for \( R \) with respect to the \( x \)-degree and a \( K \)-basis of \( R_{(0,1)} \) is almost regular for \( R \) with respect to the \( y \)-degree. For the convenience of the reader we give a proof of this fact, which follows from the following lemma (see also [13]).
Lemma 2.1. Let R be a bigraded algebra. If \( \dim_K R_{(1,0)} > 0 \) (resp.\( \dim_K R_{(0,1)} > 0 \)), then there exists an element \( x \in R_{(1,0)} \) (resp. \( y \in R_{(0,1)} \)) which is almost regular for \( R \).

Proof. By symmetry it is enough to prove the existence of \( x \). We claim that it is possible to choose \( 0 \neq x \in R_{(1,0)} \) such that for all \( Q \in \text{Ass}_S(0 :_R x) \) one has \( Q \supseteq \mathfrak{m}_x \). It follows that \( \text{Rad}_S(\text{Ann}_S(0 :_R x)) \supseteq \mathfrak{m}_x \). Hence there exists an integer \( i \) such that \( \mathfrak{m}_x^i(0 :_R x) = 0 \) and this proves the lemma.

It remains to show the claim. If \( P \supseteq \mathfrak{m}_x \) for all \( P \in \text{Ass}_S(R) \), then we may choose \( 0 \neq x \in R_{(1,0)} \) arbitrarily because \( \text{Ass}_S(0 :_R x) \subseteq \text{Ass}_S(R) \). Otherwise there exists an ideal \( P \in \text{Ass}_S(R) \) with \( P \not\supseteq \mathfrak{m}_x \). In this case we may choose \( x \in R_{(1,0)} \) such that

\[
x \notin \bigcup_{P \in \text{Ass}_S(R), P \not\supseteq \mathfrak{m}_x} P
\]

since \( |K| = \infty \). Let \( Q \in \text{Ass}_S(0 :_R x) \) be arbitrary. Then \( x \in Q \) because \( x \in \text{Ann}_S(0 :_R x) \). We also have that \( Q \in \text{Ass}_S(R) \), and this implies \( Q \supseteq \mathfrak{m}_x \) by the choice of \( x \), as claimed. \( \square \)

Let \( x \) and \( y \) be almost regular for \( R \) with respect to the \( x \)- and \( y \)-degree. Define

\[
\begin{align*}
s^x = & \max\{ a : (0 :_R (x_1, \ldots, x_{i-1})_R x_i)_{(a,s)} \neq 0 \} \cup \{ 0 \}, \\
\end{align*}
\]

and

\[
\begin{align*}
s^y = & \max\{ b : (0 :_R (y_1, \ldots, y_{i-1})_R y_i)_{(s,b)} \neq 0 \} \cup \{ 0 \}, \\
\end{align*}
\]

\[
\begin{align*}
s^x = & \max\{ a : (0 :_R (x_1, \ldots, x_{i-1})_R x_i)_{(a,s)} \neq 0 \} \cup \{ 0 \}, \\
\end{align*}
\]

The following theorem characterizes the \( x \)- and \( y \)-regularity. It is the analogue of the corresponding graded version proved in [3].

For its proof we consider \( \bar{H}_0(n, k-1) = (0 :_R (x_1, \ldots, x_{k-1})_R x_k) \) for \( k \in [n] \) and \( \bar{H}_0(n, k-1) = (0 :_R (\mathfrak{m}_x + y_1, \ldots, y_{k-1})_R y_k) \) for \( k \in [m] \). Then the beginning of the long exact Koszul sequence of the Koszul homology groups of \( R \) for \( k \in [n] \) is modified to

\[
\begin{align*}
\ldots \rightarrow & \ H_1(k-1, 0)(-1,0) \xrightarrow{x_k} H_1(k-1,0) \\
& \rightarrow H_1(k,0) \rightarrow \bar{H}_0(k-1,0)(-1,0) \rightarrow 0,
\end{align*}
\]

and for \( k \in [m] \) to

\[
\begin{align*}
\ldots \rightarrow & \ H_1(n, k-1)(0,-1) \xrightarrow{y_k} H_1(n,k-1) \\
& \rightarrow H_1(n,k) \rightarrow \bar{H}_0(n,k-1)(0,-1) \rightarrow 0.
\end{align*}
\]

Note that for \( k \in [n] \) and \( i \geq 1 \) one has \( \bar{H}_i(k,0)_{(a,s)} = 0 \) for \( a \gg 0 \). Similarly for \( k \in [m] \) and \( i \geq 1 \) one has \( \bar{H}_i(n,k)_{(s,b)} = 0 \) for \( b \gg 0 \).
Theorem 2.2. Let $R$ be a bigraded algebra, $x$ almost regular for $R$ with respect to the $x$-degree and $y$ almost regular for $R$ with respect to the $y$-degree. Then

$$\text{reg}_x(R) = s^x \text{ and } \text{reg}_y(R) = s^y.$$ 

Proof. By symmetry it is enough to show this theorem only for $x$. Let

$$r_{(k,0)} = \max\{a : H_i(k,0)_{(a+i,s)} \neq 0 \text{ for } i \in [k] \cup \{0\}\}$$

for $k \in [n]$ and

$$r_{(n,k)} = \max\{a : H_i(n,k)_{(a+i,s)} \neq 0 \text{ for } i \in [n + k] \cup \{0\}\}$$

for $k \in [m]$. Then $r_{(n,m)}$ is $\text{reg}_x(R)$ because $H_0(n,m) = K$. We will prove:

(i) For $k \in [n]$ one has $r_{(k,0)} = \max\{s_1^x, \ldots, s_n^x\}$.

(ii) For $k \in [m]$ one has $r_{(n,k)} = \max\{s_1^x, \ldots, s_m^x\}$.

These two assertions yield the theorem.

We show (i) by induction on $k \in [n]$. For $k = 1$ we have the following exact sequence

$$0 \longrightarrow H_1(1,0) \longrightarrow H_0(0,0)(-1,0) \longrightarrow 0,$$

which proves this case. Let $k > 1$. Since

$$\ldots \longrightarrow H_1(k,0) \longrightarrow H_0(k - 1,0)(-1,0) \longrightarrow 0,$$

we get $r_{(k,0)} \geq s_k^x$. If $r_{(k-1,0)} = 0$, then $r_{(k,0)} \geq r_{(k-1,0)}$. Assume that $r_{(k-1,0)} > 0$. There exists an integer $i$ such that $H_i(k-1,0)_{(r_{(k-1,0)}+i,s)} \neq 0$. Since $H_i(k-1,0)_{(r_{(k-1,0)}+i+1,s)} = 0$ and since we have the exact sequence

$$\ldots \longrightarrow H_i+1(k,0)_{(r_{(k-1,0)}+i+1,s)} \longrightarrow H_i(k-1,0)_{(r_{(k-1,0)}+i,s)} \longrightarrow H_i(k-1,0)_{(r_{(k-1,0)}+i+1,s)} \longrightarrow \ldots,$$

it follows that $H_{i+1}(k,0)_{(r_{(k-1,0)}+i+1,s)} \neq 0$. This gives again $r_{(k,0)} \geq r_{(k-1,0)}$. On the other hand, let $a > \max\{r_{(k-1,0)}, s_k^x\}$. If $i \geq 2$, then by the exact sequence

$$H_i(k-1,0)_{(a+i,s)} \longrightarrow H_i(k,0)_{(a+i,s)} \longrightarrow H_{i-1}(k-1,0)_{(a+i-1,s)} \longrightarrow$$

we get $H_i(k,0)_{(a+i,s)} = 0$ because $H_i(k-1,0)_{(a+i,s)} = H_{i-1}(k-1,0)_{(a+i-1,s)} = 0$. Similarly $H_i(k,0)_{(a+1,s)} = 0$. Therefore, by the induction hypothesis, we obtain $r_{(k,0)} = \max\{r_{(k-1,0)}, s_k^x\} = \max\{s_1^x, \ldots, s_k^x\}$.

We prove (ii) also by induction on $k \in \{0, \ldots, m\}$. The case $k = 0$ was shown in (i), so let $k > 0$. Assume that $a > s^y$. For $i \geq 2$ one has

$$H_i(k,0)_{(a+i,s)} = 0$$

Then $H_i(n,k)_{(a+i,s)} = 0$ because $H_i(n,k-1)_{(a+i,s)} = H_{i-1}(n,k-1)_{(a+i,s)} = 0$. Similarly $H_i(n,k)_{(a+1,s)} = 0$ and therefore $r_{(n,k)} \leq s^y$. If $s^y = 0$, then
Let $n$ of $s$, the ideal generated by the $(0 = x f$ of $I S$ has this property.

This concludes the proof. \hfill $\Box$

3. $d$-sequences and $s$-sequences

The concept of a $d$-sequence was introduced by Huneke [11]. Recall that a sequence of elements $f_1, \ldots, f_r$ in a ring is called a $d$-sequence, if

(i) $f_1, \ldots, f_r$ is a minimal system of generators of the ideal $I = (f_1, \ldots, f_r)$.

(ii) $(f_1, \ldots, f_{i-1}) : f_i \cap I = (f_1, \ldots, f_{i-1})$.

A result in [13] motivated the following theorem. For a bigraded algebra $R$ let $n_x$ denote the ideal generated by the $(1, 0)$-forms of $R$ and let $n_y$ denote the ideal generated by the $(0, 1)$-forms of $R$.

**Proposition 3.1.** Let $R$ be a bigraded algebra. Then:

(i) $\reg_R(R) = 0$ if and only if a generic minimal system of generators of $(1, 0)$-forms for $n_x$ is a $d$-sequence.

(ii) $\reg_{S_y}(R) = 0$ if and only if a generic minimal system of generators of $(0, 1)$-forms for $n_y$ is a $d$-sequence.

**Proof.** By symmetry we only have to prove (i). Without loss of generality $x = x_1, \ldots, x_n$ is an almost regular sequence for $R$ with respect to the $x$-degree because a generic minimal system of generators of $(1, 0)$-forms for $n_x$ has this property.

By Theorem 2.2 one has $\reg_R(R) = 0$ if and only if $s^x = 0$. By the definition of $s^x$ this is equivalent to the fact that, for all $i \in [n]$ and all $a > 0$, we have

$$\left( (x_1, \ldots, x_{i-1}) :_R x_i \right)_{(a, \ast)} = 0.$$

Equivalently, for all $i \in [n]$ we obtain $(x_1, \ldots, x_{i-1}) :_R x_i \cap n_x = (x_1, \ldots, x_{i-1})$. This concludes the proof. \hfill $\Box$

If $n_x$ (resp. $n_y$) can be generated by a $d$-sequence (not necessarily generic), then the proof of Proposition 3.1 shows that $\reg_R(R) = 0$ (resp. $\reg_{S_y}(R) = 0$).

For an application we recall some further definitions. Let $I = (f_1, \ldots, f_m) \subset S_x$ be a graded ideal generated in degree $d$. Let $R(I)$ denote the Rees algebra of $I$ and let $S(I)$ denote the symmetric algebra of $I$. It is well known that
both algebras are bigraded and have a presentation $S/J$ for a bigraded ideal $J \subset S$. For example, let $R(I) = S_x[I]^t \subset S_x$. Define
\[
\varphi : S \longrightarrow R(I), \quad x_i \mapsto x_i, \quad y_j \mapsto f_j t,
\]
and let $J = \text{Ker}(\varphi)$. Under the assumption that $I$ is generated in one degree we have that $J$ is a bigraded ideal. We will always assume that $R(I) = S/J$. Note that then $I^j \cong (S/J)_{(s,j)}(-jd)$ for all $j \in \mathbb{N}$. Similarly we may assume that $S(I) = S/J$ for a bigraded ideal $J \subset S$. We also consider the finitely generated $S_x$-module $S^d(I) = (S/J)_{(s,j)}(-jd)$, which we call the $j^{th}$ symmetric power of $I$.

For the notion of an $s$-sequence see [10]. The following results were shown in [10] and [13].

**Corollary 3.2.** Let $I = (f_1, \ldots, f_m) \subset S_x$ be a graded ideal generated in degree $d$. Then:

(i) $I$ can be generated by an $s$-sequence (with respect to the reverse lexicographic order) if and only if $\text{reg}_y(S(I)) = 0$.

(ii) $I$ can be generated by a $d$-sequence if and only if $\text{reg}_y(R(I)) = 0$.

**Proof.** In [10] and [13] it was shown that

(i) $I$ can be generated by an $s$-sequence (with respect to the reverse lexicographic order) if and only if $n_y \subseteq S(I)$ can be generated by a $d$-sequence;

(ii) $I$ can be generated by a $d$-sequence if and only if $n_y \subseteq R(I)$ can be generated by a $d$-sequence.

Combining these results with Proposition 3.1 concludes the proof. \qed

4. Bigeneric initial ideals

We recall the following definitions from [2]. For a monomial $x^u y^v \in S$ we set
\[
m_x(x^u y^v) = m(u) = \max\{0, i \text{ with } u_i > 0\},
\]
\[
m_y(x^u y^v) = m(v) = \max\{0, i \text{ with } v_i > 0\}.
\]

Similarly we define for $L \subseteq [n]$
\[
m(L) = \max\{0, i \text{ with } i \in L\}.
\]

Let $J \subset S$ be a monomial ideal. Let $G(J)$ denote the unique minimal system of generators of $J$. If $G(J) = \{z_1, \ldots, z_t\}$ with $\deg(z_i) = (u^i, v^i) \in \mathbb{N}^n \times \mathbb{N}^m$, then we set $m_x(J) = \max\{|u^i|\}$ and $m_y(J) = \max\{|v^i|\}$.

$J$ is called bistable if for all monomials $z \in J$, all $i \leq m_x(z)$, all $j \leq m_y(z)$ one has $x_i z / x_{m_x(z)} \in J$ and $y_j z / y_{m_y(z)} \in J$. $J$ is called strongly bistable if for all monomials $z \in J$, all $i \leq s$ with $x_s$ divides $z$, all $j \leq t$ with $y_t$ divides $z$ one has $x_i z / x_s \in J$ and $y_j z / y_t \in J$. 
Lemma 4.1. Let $J \subset S$ be a bistable ideal and $R = S/J$. Then:

(i) $x_n, \ldots, x_1$ is an almost regular sequence for $R$ with respect to the $x$-degree.

(ii) $y_m, \ldots, y_1$ is an almost regular sequence for $R$ with respect to the $y$-degree.

Proof. This follows easily from the fact that $J$ is bistable.

We fix a term order $>$ on $S$ by defining $x^u y^v > x^{u'} y^{v'}$ if either $(|u| + |v|, |v|, |u|) > (|u'| + |v'|, |v'|, |u'|)$ lexicographically or $(|u| + |v|, |v|, |u|) = (|u'| + |v'|, |v'|, |u'|)$ and $x^u y^v > x^{u'} y^{v'}$ reverse lexicographically with respect to the order induced by $y_1 > \cdots > y_m > x_1 > \cdots > x_n$. (See [8] for details on monomial orders.) For a bigraded ideal $J$ let $\text{in}(J)$ denote the monomial ideal generated by $\text{in}(f)$ for all $f \in J$. In [2] the bigeneric initial ideal $\text{bigin}(J)$ was constructed in the following way: For $t \in \mathbb{N}$ let $\text{GL}(t, K)$ be the general linear group of the $t \times t$-matrices with entries in $K$. Let $G = \text{GL}(n, K) \times \text{GL}(m, K)$ and $g = (d_{ij}, e_{kl}) \in G$. Then $g$ defines an $S$-automorphism by extending $g(x_i) = \sum_j d_{ij} x_j$ and $g(y_i) = \sum_k e_{kl} y_k$. There exists a non-empty Zariski open set $U \subset G$ such that for all $g \in U$ we have $\text{bigin}(J) = \text{in}(gJ)$. We call these $g \in U$ generic for $J$. If $\text{char}(K) = 0$, then $\text{bigin}(J)$ is strongly bistable for every bigraded ideal $J$. See, for example, [3] for similar results in the graded case.

Proposition 4.2. Let $J \subset S$ be a bigraded ideal. If $\text{char}(K) = 0$, then

$$\text{reg}_x(S/J) = \text{reg}_x(S/\text{bigin}(J)).$$

Proof. Set $x = x_n, \ldots, x_1$, choose $g \in G$ generic for $J$ and let $\tilde{x} = \tilde{x}_n, \ldots, \tilde{x}_1$ such that $x_i = g(\tilde{x}_i)$. We may assume that the sequence $\tilde{x}$ is almost regular for $S/J$ with respect to the $x$-degree. Furthermore, by Lemma 4.1 the sequence $x$ is almost regular for $S/\text{bigin}(J)$ with respect to the $x$-degree. We have

$$(0 : S/((x_n, \ldots, x_{i+1}) + J) \tilde{x}_i) \cong (0 : S/((x_n, \ldots, x_{i+1}) + g(J)) x_i).$$

It follows from [8, 15.12] that

$$(0 : S/((x_n, \ldots, x_{i+1}) + g(J)) x_i) \cong (0 : S/((x_n, \ldots, x_{i+1}) + \text{bigin}(J)) x_i).$$

By Theorem 2.2 we get the desired result.

Remark 4.3.

(i) In general it is not true that

$$\text{reg}_y(S/J) = \text{reg}_y(S/\text{bigin}(J)).$$

For example, let $S = K[x_1, \ldots, x_3, y_1, \ldots, y_3]$ and $J = (y_2 x_2 - y_1 x_3, y_3 x_1 - y_1 x_3)$. Then the minimal bigraded free resolution of $S/J$ is given by

$$0 \longrightarrow S(-2, -2) \longrightarrow S(-1, -1) \oplus S(-1, -1) \longrightarrow S \longrightarrow 0.$$
Then generated in because |x| if x by Lemma 4.1 the sequence x is again a monomial ideal, which is stable in the usual sense, that is, if x is in S, then x/x is a bistable ideal, which is stable in the usual sense, that is, for all N, we have

\[ m_j = \max\{a \in \mathbb{N} : (0 : R/(x_{a+n}, \ldots, x_{i+1})R x_i)_{(a,j)} \neq 0\} \cup \{0\} \]

Furthermore, for a bistable ideal J and v \in \mathbb{N} we set J(J, v) = I_v y^v where I_v \subset S is again a monomial ideal, which is stable in the usual sense, that is, if x^v \in I_v, then x/x^v = x_m(u) \in I_v for i \leq m(u).

**Proposition 5.1.** Let J \subset S be a bistable ideal and R = S/J. Then:

(i) For every i \in [n] and for j \geq 0 we have m_j^i \leq \max\{m_x(J) - 1, 0\}.

(ii) For every i \in [n] and for j \geq m_y(J) we have m_j^i = m_{m_y(J)}^i.

**Proof.** If G(J) = \{x^{uk} y^v : k = 1, \ldots, r\}, then \( I_v = (x^{uk} : v^k \leq v) \) for v \in \mathbb{N}. This means that for all x^v \in G(I_v) one has \|u\| \leq m_x(J). For fixed v with \|v\| = j we have

\[ (0 : R/(x_{n+1}, \ldots, x_{i+1})R x_i)_{(v, v)} = \frac{\left((x_n, \ldots, x_{i+1}) + I_v : x_{i+1} \right)}{\left(x_n, \ldots, x_{i+1} + I_v \right)} y^v \]

As a K-vector space

\[ \frac{\left((x_n, \ldots, x_{i+1}) + I_v : x_{i+1} \right)}{\left(x_n, \ldots, x_{i+1} + I_v \right)} y^v = \bigoplus_{x^v \in G(I_v), m(u) = 1} K(x^v/x_{m(u)}) y^v \]

because I_v is stable. Thus m_j^i \leq \max\{m_x(J) - 1, 0\}, which is (i).

To prove (ii) we replace J by J(J, m_y(J)) and may assume that J is generated in y-degree t = m_y(J). Then G(J) = \{x^{uk} y^v : k = 1, \ldots, r\} where \|u^k\| = t for all k \in [r]. Let \|u^k\| be maximal with m(u^k) = i and define \( c^i = \max\{|u^k| - 1, 0\} \). We will show that m_j^i = c^i for j \geq t. This gives (ii).

By a similar argument as in (i) we have m_j^i + t \leq c^i for all s \geq 0. If c^i = 0, then m_j^i + t = 0. Assume that c^i \neq 0. We claim that

\[ 0 \neq (x^{uk}/x_i)y^v y_n \in (0 : R/(x_{n+1}, \ldots, x_{i+1})R x_i)_{(v, s+t)} \]

for s \geq 0.
Assume this is not the case, then either
\[ (x^{u_k}/x_i)y^{v_k}y_n^s = x_l x^{u'} y^{v'} \]
for some $u', v'$ and $l \geq i + 1$, which contradicts to $m(u^k) = i$, or
\[ (x^{u_k}/x_i)y^{v_k}y_n^s = x^{u_k'} y^{v_k'} x^{u'} y^{v'} \]
for $x^{u_k'} y^{v_k'} \in G(J)$. It follows that $|v'| = s$. Let $k_1$ be the largest integer $l$
such that $y_{u_1}^s | y^{v_k'}$. Then
\[ (x^{u_k}/x_i)y^{v_k} = ((x^{u_k'} y^{v_k'} x^{u'})/y_n^{k_1}) y^{v'} / y_{n-k_1} \in J \]
because $J$ is bistable, and this is again a contradiction. Therefore $(*)$ is true
and we get $m_{j+t}^{(s)} \geq c'$ for $s \geq 0$. This concludes the proof. \hfill \Box

**Remark 5.2.** Proposition 5.1 can also be formulated with the roles of $x$ and $y$ interchanged.

Let $A$ be a standard graded $K$-algebra. For a finitely generated graded
$A$-module $M$ the usual Castelnuovo-Mumford regularity is defined as
\[ \text{reg}^A(M) = \sup \{ r \in \mathbb{Z} : \beta_i^A(M) \neq 0 \text{ for some integer } i \} \].
In [7] and [12] it was shown that for a graded ideal $I \subset S_x$ the function
$\text{reg}^S_x(I)$ is a linear function $pj + c$ for $j \gg 0$. In the case that $I$ is generated
in one degree we give an upper bound for $c$ and find an integer $j_0$ such that
$\text{reg}^S_x(I)$ is a linear function for all $j \geq j_0$.

**Theorem 5.3.** Let $I = (f_1, \ldots, f_m) \subset S_x$ be a graded ideal generated in
degree $d \in \mathbb{N}$. Let $R(I) = S/J$ for a bigraded ideal $J$. Then:
(i) $\text{reg}^S_x(I) \leq jd + \text{reg}^S_x(R(I))$.
(ii) $\text{reg}^S_x(I) = jd + c$ for $j \geq m_y(\text{bigin}(J))$ and some constant $0 \leq c \leq \text{reg}^S_x(R(I))$.

**Proof.** We choose an almost regular sequence $\tilde{x} = \tilde{x}_n, \ldots, \tilde{x}_1$ for $R(I)$ over $S$ with respect to the $x$-degree. For all $j \in \mathbb{N}$ the sequence $\tilde{x}$ is almost regular for $I^j$ over $S_x$ in the sense of [3] because $R(I)_{(*,j)}(-dj) \cong I^j$ as graded $S_x$-
modules and
\[ (0 : R(I)/(\tilde{x}_n, \ldots, \tilde{x}_{i+1})R(I) \tilde{x}_i)_{(*,j)}(-dj) = (0 : R(I)/(\tilde{x}_n, \ldots, \tilde{x}_{i+1})R(I) \tilde{x}_i). \]
Define $m_j^i$ for $\text{bigin}(J)$ as in Proposition 5.1. Since
\[ (0 : R(I)/(\tilde{x}_n, \ldots, \tilde{x}_{i+1})R(I) \tilde{x}_i) \cong (0 : S/(\text{bigin}(J) + \tilde{x}_i)) \tilde{x}_i), \]
it follows that
\[ jd + m_j^i = r_j^i = \max \{ l : (0 : R(I)/(\tilde{x}_n, \ldots, \tilde{x}_{i+1})R(I) \tilde{x}_i)_l \neq 0 \} \cup \{ 0 \}. \]
By a characterization of the regularity of graded modules in [3] we have
\[ \text{reg}^S_x(I^j) = \max\{jd, r^1_j, \ldots, r^n_j\}. \]
Hence the assertion follows from Proposition 4.2, Remark 4.3(ii) and Proposition 5.1. \(\square\)

Analogously to Theorem 5.3 one has:

**Theorem 5.4.** Let \( I = (f_1, \ldots, f_m) \subset S_x \) be a graded ideal generated in degree \( d \in \mathbb{N} \). Let \( S(I) = S/J \) for a bigraded ideal \( J \). Then:

(i) \( \text{reg}^S_x(S^j(I)) \leq jd + \text{reg}^S_x(S(I)) \).

(ii) \( \text{reg}^S_x(S^j(I)) = jd + c \) for \( j \geq m_y(\text{bigin}(J)) \) and some constant \( 0 \leq c \leq \text{reg}^S_x(S(I)) \).

Blum [4] proved the following result with different methods.

**Corollary 5.5.** Let \( I = (f_1, \ldots, f_m) \subset S_x \) be a graded ideal generated in degree \( d \in \mathbb{N} \).

(i) If \( \text{reg}_{x_y}(R(I)) = 0 \), then \( \text{reg}^S_x(I^j) = jd \) for \( j \geq 1 \).

(ii) If \( \text{reg}_{x_y}(S(I)) = 0 \), then \( \text{reg}^S_x(S^j(I)) = jd \) for \( j \geq 1 \).

**Proof.** This follows from Theorems 5.3 and 5.4. \(\square\)

Next we give a more theoretical bound for the regularity function becoming linear. Consider a bigraded algebra \( R \). Let \( y \) be almost regular for all \( \text{Tor}_i^S(S/m_x, R) \) with respect to the \( y \)-degree. Define
\[ w(R) = \max\{b: (0 : \text{Tor}_i^S(S/m_x, R) y)_i(b, b) \neq 0 \text{ for some } i \in [n]\}. \]

**Lemma 5.6.** Let \( I = (f_1, \ldots, f_m) \subset S_x \) be a graded ideal generated in degree \( d \in \mathbb{N} \).

(i) For \( j > w(R(I)) \) we have \( \text{reg}^S_x(I^{j+1}) \geq \text{reg}^S_x(I^j) + d \).

(ii) For \( j > w(S(I)) \) we have \( \text{reg}^S_x(S^{j+1}(I)) \geq \text{reg}^S_x(S^j(I)) + d \).

**Proof.** We prove the case \( R = R(I) \). For \( j > w(R) \) one has the exact sequence
\[ 0 \rightarrow \text{Tor}_i^S(S/m_x, R)_{(s,j)} \rightarrow \text{Tor}_i^S(S/m_x, R)_{(s,j+1)}. \]
In [7, 3.3] it was shown that
\[ \text{Tor}_i^S(S/m_x, R)_{(a,j)} \cong \text{Tor}_i^S(K, I^j)_{a+jd}, \]
and this concludes the proof. \(\square\)

**Lemma 5.7.** Let \( R \) be a bigraded algebra. Then
\[ H_*(0, m)_{(s,j)} = 0 \text{ for } j > \text{reg}_{y}(R) + m. \]
Proof. We know that
\[ H_\ast(0,m) \cong \text{Tor}_\ast^S(S/m_y, R) \cong H_\ast(S/m_y \otimes_S F). \]
where \( F \) is the minimal bigraded free resolution of \( R \) over \( S \). Let
\[ F_i = \bigoplus S(-a,-b) \beta_{i,(a,b)}^S(R). \]
Then, by the definition of \( y \)-regularity, we have \( b \leq \text{reg}_y(R) + m \) for all \( \beta_{i,(a,b)}^S(R) \neq 0 \). Thus \( (S/(y) \otimes_S F_i)_{(s,j)} = 0 \) for \( j > \text{reg}_y(R) + m \). The assertion now follows.
\[ \square \]

We obtain the following exact sequences.

**Corollary 5.8.** Let \( I = (f_1, \ldots, f_m) \subset S_x \) be a graded ideal generated in degree \( d \in \mathbb{N} \).

(i) For \( j \geq \text{reg}_y(R(I)) + m \) we have the exact sequence
\[
0 \rightarrow P^j - m(-md) \rightarrow \bigoplus_m P^{j-m+1}(-(m-1)d) \rightarrow \ldots \rightarrow \bigoplus_m P^{j-1}(-d) \rightarrow P^j \rightarrow 0.
\]

(ii) For \( j \geq \text{reg}_y(S(I)) + m \) we have the exact sequence
\[
0 \rightarrow S^j - m(I) - md \rightarrow \bigoplus_m S^{j-m+1}(I)(-(m-1)d) \rightarrow \ldots \rightarrow \bigoplus_m S^{j-1}(I)(-d) \rightarrow S^j(I) \rightarrow 0.
\]

**Proof.** This statement follows from Lemma 5.7 since \( R(I)(s,j)(-jd) \cong P^j \) or \( S(I)(s,j)(-jd) \cong S^j(I) \), respectively.
\[ \square \]

**Corollary 5.9.** Let \( I = (f_1, \ldots, f_m) \subset S_x \) be a graded ideal generated in degree \( d \in \mathbb{N} \). Then:

(i) For \( j \geq \max\{\text{reg}_y(R(I)) + m, w(R(I)) + m\} \) we have
\[ \text{reg}_{S_z}(P^{j+1}) = d + \text{reg}_{S_z}(P^j). \]

(ii) For \( j \geq \max\{\text{reg}_y(S(I)) + m, w(S(I)) + m\} \) we have
\[ \text{reg}_{S_z}(S^{j+1}(I)) = d + \text{reg}_{S_z}(S^j(I)). \]

**Proof.** We prove the corollary for \( R(I) \). By Corollary 5.8 and standard arguments (see Lemma 6.1 for the bigraded case) we obtain for \( j \geq \text{reg}_y(R(I)) + m \)
\[ \text{reg}_{S_z}(P^{j+1}) \leq \max\{\text{reg}_{S_z}(P^{j+1-i}) + id - i + 1: i \in [m]\}. \]
Since \( j + 1 - i > w(R(I)) \), it follows from Lemma 5.6 that

\[
\reg^S(I^{j+1-i}) \leq \reg^S(I^{j+1-i+1}) - d \leq \cdots \leq \reg^S(I^{j+1}) - id.
\]

Hence \( \reg^S(I^{j+1}) = \reg^S(I) + d \). \( \square \)

We now consider a special case where \( \reg^S(I^j) \) can be computed precisely.

**Proposition 5.10.** Let \( R = S/J \) be a bigraded algebra which is a complete intersection. Let \( \{z_1, \ldots, z_t\} \) be a homogeneous minimal system of generators of \( J \) which is a regular sequence. Assume that \( \deg_x(z_i) \geq \cdots \geq \deg_x(z_1) > 0 \) and \( \deg_y(z_k) = 1 \) for all \( k \in [t] \). Then for all \( j \geq t \)

\[
\reg^S(R_{(s,j+1)}) = \reg^S(R_{(s,j)}).
\]

If in addition \( \deg_x(z_k) = 1 \) for all \( k \in [t] \), then for \( j \geq 1 \)

\[
\reg^S(R_{(s,j)}) = 0.
\]

**Proof.** The Koszul \( K_s(z) \) complex with respect to \( \{z_1, \ldots, z_t\} \) provides a minimal bigraded free resolution of \( R \) because these elements form a regular sequence. Observe that \( (s,j) \) is an exact functor on complexes of bigraded modules. Note that \( K_s(z)_{(s,j)} \) is a complex of free \( S_x \)-modules because

\[
K_s(z)_{(s,j)} \cong \bigoplus_{\{j_1, \ldots, j_t\} \leq [t]} \bigoplus_{x} S(-\deg(z_{j_1}) - \cdots - \deg(z_{j_t})),
\]

and

\[
S(-a, -b)_{(s,j)} \cong \bigoplus_{|v|=j-b} S_x(-a)y^v \text{ as graded } S_x\text{-modules}.
\]

Furthermore \( K_s(z)_{(s,j)} \) is minimal by the additional assumption \( \deg_x(z_k) > 0 \). We have for \( j \geq t \)

\[
\reg^S(R_{(s,j)}) = \max\{\deg_x(z_i) + \cdots + \deg_x(z_{i+j-1}) - i : i \in [t]\},
\]

and this is independent of \( j \). If in addition \( \deg_x(z_k) = 1 \) for all \( k \), then we obtain

\[
\reg^S(R_{(s,j)}) = 0 \text{ for } j \geq 1.
\] \( \square \)

Recall that a graded ideal \( I \) is said to be of linear type, if \( R(I) = S(I) \). For example, ideals generated by \( d \)-sequences are of linear type. Let \( I = (f_1, \ldots, f_m) \subset S_x \) be a graded ideal, which is Cohen-Macaulay of codim 2. By the Hilbert-Burch theorem \( S_x/I \) has a minimal graded free resolution

\[
0 \rightarrow \bigoplus_{i=1}^{m-1} S_x(-b_i) \xrightarrow{B} \bigoplus_{i=1}^{m} S_x(-a_i) \rightarrow S_x \rightarrow S_x/I \rightarrow 0,
\]

where \( B = (b_{ij}) \) is a \( m \times m - 1 \)-matrix with \( b_{ij} \in m \) and we may assume that the ideal \( I \) is generated by the maximal minors of \( B \). The matrix \( B \) is
said to be the Hilbert-Burch matrix of $I$. If $I$ is generated in degree $d$, then $S(I) = S/J$ where $J$ is the bigraded ideal $(\sum_{i=1}^{m} b_{ij}y_i: j = 1, \ldots, m - 1)$.

**Corollary 5.11.** Let $I = (f_1, \ldots, f_m) \subset S_x$ be a graded ideal generated in degree $d \in \mathbb{N}$, which is Cohen-Macaulay of codim 2 with $m \times m - 1$ Hilbert-Burch matrix $B = (b_{ij})$ and of linear type. Then for $j \geq m - 1$

$$\text{reg}^{S_x}(I^{j+1}) = \text{reg}^{S_x}(I^j) + d.$$  

If additionally $\deg_x(b_{ij}) = 1$ for $b_{ij} \neq 0$, then the equality holds for $j \geq 1$.

**Proof.** Since $I$ is of linear type, we have $R(I) = S(I) = S/J$ with the ideal $J = (\sum_{i=1}^{m} b_{ij}y_i: j = 1, \ldots, m - 1)$. One knows that $(\text{Krull-) dim}(R(I)) = n + 1$. Since $J$ is defined by $m - 1$ equations, we conclude that $R(I)$ is a complete intersection. Now apply Proposition 5.10. □

6. Bigraded Veronese algebras

Let $R$ be a bigraded algebra and fix $\Delta = (s,t) \in \mathbb{N}^2$ with $(s,t) \neq (0,0)$. We call $R_\Delta = \bigoplus_{(a,b) \in \mathbb{N}^2} R_{(as, bt)}$ the bigraded Veronese algebra of $R$ with respect to $\Delta$ (see, for example, [9] for the graded case and [6] for similar constructions in the bigraded case). Note that $R_\Delta$ is again a bigraded algebra. We want to relate $\text{reg}^{S_{(s,t)}}(R_\Delta)$ and $\text{reg}^{S_y}(R_\Delta)$ to $\text{reg}^{S_x}(R)$ and $\text{reg}^{S_y}(R)$. We follow the presentation in [6] for the case of diagonals.

**Lemma 6.1.** Let $R$ be a bigraded algebra and let

$$0 \rightarrow M_r \rightarrow \ldots \rightarrow M_0 \rightarrow N \rightarrow 0$$

be an exact complex of finitely generated bigraded $R$-modules. Then

$$\text{reg}^{R}(N) \leq \sup\{\text{reg}^{R}(M_k) - k: 0 \leq k \leq r\}$$

and

$$\text{reg}^{R}(N) \leq \sup\{\text{reg}^{R}(M_k) - k: 0 \leq k \leq r\}.$$  

**Proof.** We prove by induction on $r \in \mathbb{N}$ the above inequality for $\text{reg}^{R}(N)$. The case $r = 0$ is trivial. Now let $r > 0$, and consider

$$0 \rightarrow N' \rightarrow M_0 \rightarrow N \rightarrow 0,$$

where $N'$ is the kernel of $M_0 \rightarrow N$. Then for every integer $a$ we have the exact sequence

$$\ldots \rightarrow \text{Tor}^R_i(M_0, K)_{(a+i, s)} \rightarrow \text{Tor}^R_i(N, K)_{(a+i, s)} \rightarrow \text{Tor}^{R}_{i-1}(N', K)_{(a+i-1, s)} \rightarrow \ldots$$
We get
\[
\text{reg}_x^R(N) \leq \sup\{\text{reg}_x^R(M_0), \; \text{reg}_x^R(N') - 1\}
\]
\[
\leq \sup\{\text{reg}_x^R(M_k) - k: 0 \leq k \leq r\},
\]
where the last inequality follows from the induction hypothesis. Analogously we obtain the inequality for \(\text{reg}_y^R(N)\).

**Lemma 6.2.** Let \(A\) and \(B\) be graded \(K\)-algebras, let \(M\) be a finitely generated graded \(A\)-module and let \(N\) be a finitely generated graded \(B\)-module. Then \(M \otimes_K N\) is a finitely generated bigraded \(A \otimes_K B\)-module with
\[
\text{reg}_{A \otimes_K B}^R(M \otimes_K N) = \text{reg}^A(M) \text{ and } \text{reg}_{A \otimes_K B}^R(M \otimes_K N) = \text{reg}^B(N).
\]

**Proof.** Let \(F\), be the minimal graded free resolution of \(M\) over \(A\) and \(G\), be the minimal graded free resolution of \(N\) over \(B\). Then \(H_i = F_i \otimes_B G\) is the minimal bigraded free resolution of \(M \otimes_K N\) over \(A \otimes_K B\) with \(H_i = \bigoplus_{k+l=i} F_k \otimes G_l\). Since \(A(-a) \otimes_K B(-b) = (A \otimes_K B)(-a, -b)\), the assertion follows.

**Theorem 6.3.** Let \(R\) be a bigraded algebra, \(\bar{\Delta} = (s, t) \in \mathbb{N}^2\) with \((s, t) \neq (0, 0)\). Then
\[
\text{reg}_{S_{\bar{\Delta}}}^S(R_{\bar{\Delta}}) \leq \max\{c: c = [a/s] - i, \beta_{i,(a,b)}^S(R) \neq 0 \text{ for } i,b \in \mathbb{N}\}
\]
and
\[
\text{reg}_{y}^S(R_{\bar{\Delta}}) \leq \max\{c: c = [b/t] - i, \beta_{i,(a,b)}^S(R) \neq 0 \text{ for } i,a \in \mathbb{N}\}.
\]

**Proof.** It suffices to show the inequality for \(\text{reg}_{S_{\bar{\Delta}}}^S(R_{\bar{\Delta}})\). Let
\[
0 \longrightarrow F_r \longrightarrow \ldots \longrightarrow F_0 \longrightarrow R \longrightarrow 0
\]
be the minimal bigraded free resolution of \(R\) over \(S\). Since \((\cdot)_{\bar{\Delta}}\) is an exact functor, we obtain the exact complex of finitely generated \(S_{\bar{\Delta}}\)-modules
\[
0 \longrightarrow (F_r)_{\bar{\Delta}} \longrightarrow \ldots \longrightarrow (F_0)_{\bar{\Delta}} \longrightarrow R_{\bar{\Delta}} \longrightarrow 0.
\]
By Lemma 6.1 we have
\[
\text{reg}_{S_{\bar{\Delta}}}^S(R_{\bar{\Delta}}) \leq \max\{\text{reg}_{S_{\bar{\Delta}}}^S((F_i)_{\bar{\Delta}}) - i\}.
\]
Since
\[
F_i = \bigoplus_{(a,b) \in \mathbb{N}^2} S(-a, -b)^{\beta_{i,(a,b)}^S(R)},
\]
one has
\[
\text{reg}_{S_{\bar{\Delta}}}^S((F_i)_{\bar{\Delta}}) = \max\{\text{reg}_{\bar{\Delta}}^S(S(-a, -b)_{\bar{\Delta}}): \beta_{i,(a,b)}^S(R) \neq 0\}.
\]
We have to compute \(\text{reg}_{y}^S(S(-a, -b)_{\bar{\Delta}})\). Let \(M_0, \ldots, M_{s-1}\) the relative Veronese modules of \(S_x\) and \(N_0, \ldots, N_{t-1}\) be the relative Veronese modules of
$S_y$. That is, $M_j = \bigoplus_{k \in \mathbb{N}} (S_x)_{ks+j}$ for $j = 0, \ldots, s-1$ and $N_j = \bigoplus_{k \in \mathbb{N}} (S_y)_{kt+j}$ for $j = 0, \ldots, t-1$. Then

$$S(-a, -b)_\Delta = \bigoplus_{(k,t) \in \mathbb{N}^2} (S_x)_{ks-a} \otimes_K (S_y)_{lt-b} = M \cdot \left(-\left\lfloor \frac{a}{s} \right\rfloor \right) \otimes_K \left(N \cdot \left(-\left\lfloor \frac{b}{t} \right\rfloor \right)\right),$$

where $i \equiv -a \mod s$ for $0 \leq i \leq s-1$ and $j \equiv -b \mod t$ for $0 \leq j \leq t-1$.

By [1] the relative Veronese modules over a polynomial ring have a linear resolution over the Veronese algebra. Hence Lemma 6.2 implies that $\text{reg}_{x}^{S_{\Delta}}(S(-a, -b)_\Delta) = \left\lfloor \frac{a}{s} \right\rfloor$. This concludes the proof.

**Corollary 6.4.** Let $R$ be a bigraded algebra.

(i) For $s \gg 0$, $t \in \mathbb{N}$ and $\Delta = (s, t)$ one has $\text{reg}_{x}^{S_{\Delta}}(R_{\Delta}) = 0$.

(ii) For $t \gg 0$, $s \in \mathbb{N}$ and $\Delta = (s, t)$ one has $\text{reg}_{y}^{S_{\Delta}}(R_{\Delta}) = 0$.

**References**


