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QUASI-CENTRAL BOUNDED APPROXIMATE IDENTITIES IN GROUP ALGEBRAS OF LOCALLY COMPACT GROUPS

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ABSTRACT. A net in the group algebra of a locally compact group which commutes asymptotically with elements from the measure algebra is called quasi-central. In this paper we provide new characterizations of locally compact groups whose group algebras possess quasi-central bounded approximate identities. Reiter-type and structural conditions for such groups are obtained which indicate that these groups behave much like the tractable [SIN]-groups. A general notion of an amenable action on the predual of a von Neumann algebra is developed to prove these theorems. Applications to the cohomology of group and Fourier algebras are discussed.

Introduction

Let G be a locally compact group with identity e, left Haar measure dx, and modular function Δ. If A is a Borel measurable subset of G, then $|A|$ will denote its Haar measure; if $0 < |A| < \infty$, ϕ_A is its normalized characteristic function $\frac{1}{|A|}1_A$. Let $\mathcal{K}(e) = \{U : U \text{ is a compact neighbourhood of } e\}$, and for $1 \leq p < \infty$ let $L^p(G)_1^+ = \{f \in L^p(G) : f \geq 0 \text{ and } \|f\|_p = 1\}$. A net (u_α) in $L^1(G)$ is called *weakly asymptotically central* if $\delta_x * u_\alpha - u_\alpha * \delta_x \to 0\ \ (x \in G)$, where convergence is with respect to the weak topology in $L^1(G)$. A net (u_α) in $L^1(G)$ is called *quasi-central* if $\|\mu * u_\alpha - u_\alpha * \mu\|_1 \to 0\ \ (\mu \in M(G))$.

Locally compact groups G whose group algebras $L^1(G)$ possess quasi-central bounded approximate identities (bai's) have been studied by several authors; see, for example, [12], [20], [23], [24], [25]. In particular, A. Sinclair [20, Problem A3.4] first asked the question: When does $L^1(G)$ have a quasi-central bounded approximate identity? V. Losert and H. Rindler addressed this problem in [12] and, among other things, showed that the existence of a weak asymptotically central bai in $L^1(G)$ is equivalent to the existence of

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a quasi-central bai, and that group algebras of amenable groups always possess quasi-central bai's [12, Theorem 3]. We note that important use of [12, Theorem 3] was made in the papers [17], [18], and [21].

In Section 1 we develop an amenability theory in the very general context of a group action on the predual of a von Neumann algebra. The machinery developed in Section 1 is used in Section 2 to prove Theorem 2.6, which is an analogue of Reiter's condition [7, 3.2.1] for groups whose group algebras possess quasi-central bai's. This result includes the converse direction of [12, Theorem 2]. We also provide a new proof of [12, Theorem 3].

A locally compact group G is called a [SIN]-*group* (*small invariant neighbourhood group*) if there is a base for the neighbourhood system at the identity comprised of compact sets which are invariant under inner automorphisms. A well-known theorem due to R. Mosak [13] states that $G \in$[SIN] if and only if $L^1(G)$ possesses a central bai. Moreover, every [SIN]-group is unimodular.

In Section 3 we define almost-[SIN]-groups to be those locally compact groups for which there is a base for the neighbourhood system at the identity which is asymptotically invariant under inner automorphisms. We prove our main result, Theorem 3.3, which states that G is an almost-[SIN]-group if and only if G is unimodular and $L^1(G)$ possesses a quasi-central bounded approximate identity.

Applications of this work are discussed in Section 4. First we briefly discuss implications to the cohomology of the group algebra; details are found in [21]. We then characterize locally compact groups G with group algebras admitting quasi-central bounded approximate identities in terms of the Fourier and Fourier-Stieltjes algebras of G. Finally we will discuss applications to the cohomology of the Fourier algebra.

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1. Amenable action on the predual of a W^*-algebra

In this section we briefly outline a unified approach under which the standard techniques used to develop the basic theory of amenable groups, up to and including Reiter's condition, may be used to develop the theory of several types of amenability. We omit most proofs as they may all be adapted from their classical counterparts (see, for example, [7], [15], [16]). In the special case of amenable representations (Example 1.2, part (3) below) this theory was developed by M. Bekka [2, Sections 2, 3, and 4], and the details found in this paper may be helpful. For Sections 2, 3, and 4 of this paper we only need up to Lemma 1.9 (3) in the special case of Example 1.2 part (2). We have chosen to set our presentation in this more general context because it is no more difficult to do so and because this approach does not seem to exist elsewhere in the literature.

Let $\mathcal{M}$ be a W^*-algebra with predual $\mathcal{M}_*$. Let $S(\mathcal{M})$ denote the state space of $\mathcal{M}$, $(\mathcal{M}_*)_1^+$ the normal states of $\mathcal{M}$. References for Banach G, $L^1(G)$, and $M(G)$-modules are [9, Chapter 2] and [16, Section 11].

Definition 1.1. A locally compact group G will be said to have *positive action* on $\mathcal{M}_*$, if $\mathcal{M}_*$ is a left Banach G-module such that

(i) $\|s \cdot \phi\| \leq \|\phi\| \quad (\phi \in \mathcal{M}_*,\ s \in G)$, and
(ii) $s \cdot \phi \in (\mathcal{M}_*)_1^+$ whenever $s \in G,\ \phi \in (\mathcal{M}_*)_1^+$.

Example 1.2.
(1) Let $\mathcal{M} = L^\infty(G), \quad \mathcal{M}_* = L^1(G)$, with

$$s \cdot f = \delta_s * f \quad (f \in L^1(G), s \in G).$$

(2) Let $\mathcal{M} = L^\infty(G), \quad \mathcal{M}_* = L^1(G)$, with

$$s \cdot f = \delta_s * f * \delta_{s^{-1}} \quad (f \in L^1(G), s \in G).$$

(3) Let $\{\pi, \mathcal{H}\}$ be a continuous unitary representation of G, $\mathcal{M} = B(\mathcal{H})$ the bounded linear operators on $\mathcal{H}$, $\mathcal{M}_* = T(\mathcal{H})$ the trace class operators on $\mathcal{H}$, and define

$$s \cdot T = \pi(s) T \pi(s^{-1}) \quad (T \in T(\mathcal{H}), s \in G).$$

(4) Let G be a locally compact group, H a closed subgroup of G, $X = G/H$ the left coset space of G modulo H. Let $\mathcal{M} = L^\infty(X, \nu), \mathcal{M}_* = L^1(X, \nu)$, where ν is a strongly continuous quasi-invariant positive Borel measure on X (see, for example, [5]). Define

$$s \cdot f = \delta_s * f \quad (f \in L^1(X, \nu), \quad s \in G).$$

(5) Let $(\mathcal{M}, G, \alpha)$ be a W^*-dynamical system. That is, $\mathcal{M}$ is a W^*-algebra, G is a locally compact group, and $\alpha : G \to \mathrm{Aut}(\mathcal{M})$ is a homomorphism of G into the group of *-automorphisms of $\mathcal{M}$, such that for each $x \in \mathcal{M}$, $s \to \alpha_s(x) : G \to (\mathcal{M}, \sigma(\mathcal{M}, \mathcal{M}_*))$ is continuous. Define

$$s \cdot \phi = (\alpha_{s^{-1}})^*(\phi) \quad (s \in G,\ \phi \in \mathcal{M}_*),$$

where $(\alpha_s)^* : \mathcal{M}^* \to \mathcal{M}^*$ is the adjoint map of $\alpha_s : \mathcal{M} \to \mathcal{M}$. In fact, each of our first four examples is a special case of this last example.

For the remainder of this section, G is a locally compact group, and $\mathcal{M}$ is a W^*-algebra such that G has positive action on $\mathcal{M}_*$. Note that $\mathcal{M}_*$ is a left Banach $M(G)$-module (and essential Banach $L^1(G)$-module) through the action defined by the weak integral

$$\mu \cdot \phi = \int_G s \cdot \phi \, d\mu(s) \quad (\phi \in \mathcal{M}_*, \ \mu \in M(G)).$$

Dual module operations on $\mathcal{M}$ and $\mathcal{M}^*$ are defined in canonical fashion. The next lemma is often required in the proofs of the statements which follow.

LEMMA 1.3. *Let $e_{\mathcal{M}}$ be the identity of $\mathcal{M}$, and let $M(G)_1^+$ denote the set of probability measures in $M(G)$. The following statements hold:*

(1) *$(\mathcal{M}_*)_1^+$ is w^*-dense in $S(\mathcal{M})$.*
(2) *For each $\mu \in M(G)_1^+$, $e_{\mathcal{M}} \cdot \mu = e_{\mathcal{M}}$.*
(3) *$(\mathcal{M}_*)_1^+ = G \cdot (\mathcal{M}_*)_1^+ = M(G)_1^+ \cdot (\mathcal{M}_*)_1^+$.*
(4) *$S(\mathcal{M}) = G \cdot S(\mathcal{M}) = M(G)_1^+ \cdot S(\mathcal{M})$.*

Proof. (1) This is standard and may be found, for example, in [22].

(2) Let $\mu \in M(G)_1^+$. Then for any $\phi \in (\mathcal{M}_*)_1^+$,

$$\langle \phi, e_{\mathcal{M}} \cdot \mu \rangle = \langle \mu \cdot \phi, e_{\mathcal{M}} \rangle = \int_G \langle s \cdot \phi, e_{\mathcal{M}} \rangle \, d\mu(s) = 1 = \langle \phi, e_{\mathcal{M}} \rangle,$$

because the action of G on $\mathcal{M}_*$ is positive. But $(\mathcal{M}_*)_1^+$ separates points of $\mathcal{M}$, so $e_{\mathcal{M}} \cdot \mu = e_{\mathcal{M}}$.

(3) The first equality is obvious. For the second one, note that if $\mu \in M(G)_1^+$ and $\phi \in (\mathcal{M}_*)_1^+$, then $\|\mu \cdot \phi\| \leq 1$, and from part (2), $\mu \cdot \phi(e_{\mathcal{M}}) = \phi(e_{\mathcal{M}} \cdot \mu) = \phi(e_{\mathcal{M}}) = 1$. Hence $\mu \cdot \phi$ is a normal state on $\mathcal{M}$.

(4) The dual module actions on $\mathcal{M}^*$ are $w^* - w^*$ continuous, so this follows from parts (1) and (3). □

DEFINITION 1.4. We will say that G *acts amenably on* $\mathcal{M}_*$ if there exists a state m on $\mathcal{M}$ such that

$$m(x \cdot s) = m(x) \quad (s \in G, \ x \in \mathcal{M}).$$

The state m will be called a *G-invariant mean* (G-IM) for the action.

The interpretation of this definition in Example 1.2, parts (1)–(4), is as follows:

(1) G acts amenably on $\mathcal{M}_* \iff G$ is amenable.
(2) G acts amenably on $\mathcal{M}_* \iff G$ is inner amenable.
(3) G acts amenably on $\mathcal{M}_* \iff \{\pi, \mathcal{H}\}$ is amenable [2].
(4) G acts amenably on $\mathcal{M}_* \iff G$ acts amenably on X [5].

DEFINITION 1.5. An element $x \in \mathcal{M}$ will be called *uniformly continuous* if $s \mapsto x \cdot s : G \to (\mathcal{M}, \| \cdot \|)$ is continuous. Let $UC(\mathcal{M}) = \{x \in \mathcal{M} : x \text{ is uniformly continuous}\}$.

Remarks 1.6.

(1) For Examples 1.2 (1), (3), and (4), we respectively have $UC(\mathcal{M}) = C_{ru}(G)$ as defined in [8], $UC(\mathcal{M}) = X(\mathcal{H})$ as defined in [2], and $UC(\mathcal{M}) = UCB(X)$ as defined in [5].

(2) In the case of Example 1.2 (2), $UC(\mathcal{M})$ may contain functions which are not continuous on G. For example, if there exists $U \in \mathcal{K}(e)$ which is invariant under inner automorphisms (that is, if G is an [IN]-group), then it is clear that $1_U \in UC(\mathcal{M})$.

(3) $UC(\mathcal{M})$ is always a ($\|\cdot\|$-closed) right Banach G-submodule of $\mathcal{M}$ containing $e_{\mathcal{M}}$. In the case of Example 1.2 (5) (and hence in all of our examples), it is easy to see that $UC(\mathcal{M})$ is a C^*-subalgebra of $\mathcal{M}$ (and $(UC(\mathcal{M}), G, \alpha|_{UC(\mathcal{M})})$ is a 'C^*-system').

Lemma 1.7. *We always have* $UC(\mathcal{M}) = \mathcal{M} \cdot L^1(G)$.

Definition 1.8. A state m on $\mathcal{M}$ is called a *topological invariant mean* (TIM) if

$$m(x \cdot u) = m(x) \quad (x \in \mathcal{M},\ u \in L^1(G)_1^+).$$

An element $m \in UC(\mathcal{M})^*$ such that $\|m\| = m(e_{\mathcal{M}}) = 1$ will be called a *mean.* A mean m is a TIM on $UC(\mathcal{M})$ if

$$m(x \cdot u) = m(x) \quad (x \in UC(\mathcal{M}),\ u \in L^1(G)_1^+).$$

Lemma 1.9. *The following statements hold:*

(1) *If m is a TIM on $\mathcal{M}$ (respectively $UC(\mathcal{M})$), then m is a G-IM on $\mathcal{M}$ (respectively $UC(\mathcal{M})$).*
(2) *If m is a G-IM on $UC(\mathcal{M})$, then m is a TIM on $UC(\mathcal{M})$.*
(3) *If m is a G-IM on $\mathcal{M}$ (or $UC(\mathcal{M})$) and $u \in L^1(G)_1^+$, then m_u is a TIM on $\mathcal{M}$, where*

$$m_u(x) := m(x \cdot u) \quad (x \in \mathcal{M}).$$

Proposition 1.10. *The following statements are equivalent:*

(1) *G acts amenably on $\mathcal{M}_*$.*
(2) *There is a TIM on $\mathcal{M}$.*
(3) *There is a G-IM on $UC(\mathcal{M})$.*
(4) *There is a TIM on $UC(\mathcal{M})$.*

Corollary 1.11. *The following are equivalent for a locally compact group G:*

(1) *G is amenable.*
(2) *Every positive action of G on the predual of a W^*-algebra $\mathcal{M}$ is amenable.*

Proof. (2) $\Longrightarrow$ (1) is obvious. For the implication (1) $\Longrightarrow$ (2) apply Day's fixed point theorem [7, 3.3.5] to the natural action of G on the set $\mathcal{S}$ of means on $UC(\mathcal{M})$. $\square$

COROLLARY 1.12. *The following statements are equivalent:*

(1) *G acts amenably on $\mathcal{M}_*$.*
(2) *There is a net $(\phi_\alpha) \subset (\mathcal{M}_*)_1^+$ such that $\|s \cdot \phi_\alpha - \phi_\alpha\| \to 0 \quad (s \in G)$.*
(3) *There is a net $(\phi_\alpha) \subset (\mathcal{M}_*)_1^+$ such that $\|u \cdot \phi_\alpha - \phi_\alpha\| \to 0 \quad (u \in L^1(G)_1^+)$.*

PROPOSITION 1.13 (Reiter's condition). *The following statements are equivalent:*

(1) *G acts amenably on $\mathcal{M}_*$.*
(2) *For any $\epsilon > 0$ and any compact subset K of G there exists $\phi \in \mathcal{M}_*$ such that*
$$\|s \cdot \phi - \phi\| < \epsilon \quad (s \in K).$$
(3) *There is a net $(\phi_\alpha) \subset (\mathcal{M}_*)_1^+$ such that $\|\mu \cdot \phi_\alpha - \phi_\alpha\| \to 0 \quad (\mu \in M(G)_1^+)$.*

This is precisely Reiter's condition in each of our Examples 1.2 (1)–(4).

2. A Reiter condition

Let G be a locally compact group. Throughout the sequel we will restrict our attention to the positive action
$$x \cdot f := \delta_x * f * \delta_{x^{-1}} \quad (x \in G,\ f \in L^1(G))$$
of G on $L^1(G)$ (Example 1.2 (2)). All references to TIM, G-IM, $UC(L^\infty(G))$, etc. are with respect to this action. It is easy to see that
$$\mu \cdot f(y) = \int_G \Delta(x) f(x^{-1}yx)\, d\mu(x) \ \text{ a.e. } y \quad (\mu \in M(G),\ f \in L^1(G))$$
and
$$\phi \cdot \mu(y) = \int_G \phi(xyx^{-1})\, d\mu(x) \text{ locally a.e. } y \quad (\mu \in M(G),\ \phi \in L^\infty(G))$$
describe the induced $M(G)$-module and dual $M(G)$-module operations on $L^1(G)$ and $L^\infty(G)$, respectively. In particular, we have
$$\phi \cdot x(y) = \phi \cdot \delta_x(y) = \phi(xyx^{-1}) \quad (x \in G,\ \phi \in L^\infty(G)).$$
A mean m on $L^\infty(G)$ is called *inner invariant* if
$$m(\phi \cdot x) = m(\phi) \quad (\phi \in L^\infty(G),\ x \in G),$$
and is called *an extension of the Dirac measure δ_e (from $CB(G)$ to $L^\infty(G)$)* if
$$m(\phi) = \phi(e) \qquad (\phi \in CB(G)).$$

In [12, Lemma 3] it is shown that m extends the Dirac measure at e if and only if $m(\phi) = m(\phi 1_V)$ for any $\phi \in L^\infty(G)$, $V \in \mathcal{K}(e)$, which in turn holds if and only if $m(\phi) = 0$ for any $\phi \in L^\infty(G)$ which vanishes locally a.e. on a neighbourhood of e. The following is contained in [12, Theorem 5].

LEMMA 2.1. *For $L^1(G)$ to have a quasi-central bai it is necessary and sufficient that $L^\infty(G)$ has an inner invariant mean which extends the Dirac measure at e.*

LEMMA 2.2. *If $L^1(G)$ has a quasi-central bai, then there is a TIM on $L^\infty(G)$ which extends the Dirac measure at e.*

Proof. Direct $\mathcal{K}(e)$ by reverse inclusion and consider the bai $\{\phi_U : U \in \mathcal{K}(e)\}$ for $L^1(G)$, where $\phi_U := \frac{1}{|U|}1_U$. Let m be an inner invariant mean for $L^\infty(G)$ extending δ_e. By Lemma 1.9 (2) and (3), m_U is a TIM for $L^\infty(G)$, where $m_U(\psi) = m(\psi \cdot \phi_U) \quad (\psi \in L^\infty(G))$. Let m_0 be a w^*-limit point of (m_U) in $L^\infty(G)^*$; without loss of generality assume that $m_U \to m_0 \quad w^*$. Clearly m_0 is a TIM on $L^\infty(G)$. Suppose that $\phi \in L^\infty(G)$ and $\phi(x) = 0$ locally a.e. on a neighbourhood V of e. By [12, Lemma 3] we only need to show that $m_0(\phi) = 0$. To this end take $U_0 \in \mathcal{K}(e)$, which is symmetric and satisfies $U_0^3 \subset V$. Then for any $U \subset U_0$ and almost every $x \in U_0$

$$\phi \cdot \phi_U(x) = \frac{1}{|U|} \int_U \phi(yxy^{-1})\, dy = 0.$$

That is, $(\phi \cdot \phi_U)|_{U_0} = 0$ a.e. for $U \subset U_0$. But m extends δ_e, so

$$m_0(\phi) = \lim_U m_U(\phi) = \lim_U m(\phi \cdot \phi_U) = \lim_{U \subset U_0} m(\phi \cdot \phi_U) = 0. \qquad \square$$

REMARKS 2.3. A standard compactness argument shows that $C_{00}(G) \subset UC(L^\infty(G))$. Consequently, the proof of [12, Lemma 3] shows that if m is a mean on $UC(L^\infty(G))$, then $m(\phi) = \phi(e)$ for every $\phi \in C_{00}(G)$ precisely when $m(\phi) = 0$ for every $\phi \in UC(L^\infty(G))$ which vanishes a.e. on a neighbourhood of e. Suppose now that in the proof of Lemma 2.2 we assume instead that m is a mean on $UC(L^\infty(G))$ satisfying

$$(\dagger) \qquad x \cdot m = m \text{ and } m(\phi) = \phi(e) \qquad (x \in G,\ \phi \in C_{00}(G)).$$

It is then clear that the constructed mean m_0 is still a TIM on $L^\infty(G)$ which extends the Dirac measure at e. In particular, the existence of a mean m on $UC(L^\infty(G))$ satisfying (†) ensures the existence of a quasi-central bai for $L^1(G)$. With this observation, Day's fixed point theorem yields another proof of [12, Theorem 3].

THEOREM 2.4 (Losert and Rindler). *If G is amenable, then $L^1(G)$ has a quasi-central bai.*

Proof. Let $\mathcal{M}_D$ denote the set of means m on $UC(L^\infty(G))$ such that $m(\phi) = \phi(e)$ for every $\phi \in C_{00}(G)$. Note that if m is any w^*-limit point in $L^\infty(G)^*$ of a bai in $L^1(G)_1^+$ for $L^1(G)$, then the restriction of m to $UC(L^\infty(G))$ belongs to $\mathcal{M}_D$. Thus $\mathcal{M}_D \neq \emptyset$, and $\mathcal{M}_D$ is plainly w^*-compact and convex. Now for $\phi \in C_{00}(G)$ and $x \in G$, $\phi \cdot x(e) = \phi(e)$, so

$$(x, m) \mapsto x \cdot m : G \times \mathcal{M}_D \to \mathcal{M}_D$$

defines an affine action of G on $\mathcal{M}_D$. Moreover, the action is separately continuous with respect to the relative w^*-topology on $\mathcal{M}_D$, so by Day's fixed point theorem [7, 3.3.5], there is some $m \in \mathcal{M}_D$ such that $x \cdot m = m$ $(x \in G)$. □

NOTATION. For any $U \in \mathcal{K}(e)$ let

$$\Psi(U) := \{v \in L^1(G)_1^+ : \text{support}(v) \subset U\} \cap L^\infty(G).$$

LEMMA 2.5. *Let m be a mean on $L^\infty(G)$ extending δ_e. Then for any $U \in \mathcal{K}(e)$, $m \in \overline{\Psi(U)}^{w^*}$.*

Proof. If not, then by the Hahn-Banach separation theorem we may find $f \in L^\infty(G)$ and $\epsilon > 0$ such that

$$\mathrm{Re}\langle f, m\rangle > \epsilon + \mathrm{Re}\langle f, v\rangle, \quad (v \in \Psi(U)).$$

Letting $g = (\mathrm{Re}\, f)1_U$ we have

$$(*) \qquad \langle v, g\rangle + \epsilon < m(g) \quad (v \in \Psi(U)),$$

where we have used [12, Lemma 3]. Let $\alpha = \operatorname{ess\,sup}\{g(x) : x \in U\}$ and $A = \{x \in U : g(x) > \alpha - \epsilon/2\}$. Then $|A| > 0$ and $\phi_A \in \Psi(U)$. Observe that if $g' = g + \alpha 1_{G\setminus U}$ then (again by use of [12, Lemma 3])

$$m(g) = m(g1_U) = m(g'1_U) = m(g') \leq \operatorname{ess\,sup}(g') = \alpha.$$

Hence by $(*)$

$$\alpha - \frac{\epsilon}{2} < \langle \phi_A, g\rangle < m(g) - \epsilon \leq \alpha - \epsilon,$$

a contradiction. □

We may now prove the following version of Reiter's condition for groups whose group algebras possess quasi-central bounded approximate identities. This may be seen as an improvement on the converse direction of [12, Theorem 2].

THEOREM 2.6. *Let G be a locally compact group such that $L^1(G)$ has a quasi-central bai. Then for any $\epsilon > 0$, any compact subset K of G, and any compact neighbourhood U of e there is some $u \in \Psi(U)$ such that*

$$\|\delta_x * u * \delta_{x^{-1}} - u\|_1 < \epsilon \quad (x \in K).$$

In particular, if $L^1(G)$ has a quasi-central bai (u_β), then (u_β) may be chosen so that

$$\|\delta_x * u_\beta - u_\beta * \delta_x\|_1 \to 0$$

uniformly on compact subsets of G, and for any neighbourhood U of e, there exists β_0 such that $u_\beta \in \Psi(U)$ whenever $\beta \succeq \beta_0$.

Proof. Choose a symmetric set $V \in \mathcal{K}(e)$ such that $V^3 \subset U$. Choose $E \in \mathcal{K}(e)$ such that

$$\|\phi_E * \phi_V - \phi_V\|_1 < \epsilon \quad \text{and} \quad \|\delta_x * \phi_V - \phi_V\|_1 < \epsilon \quad (x \in E).$$

Take $x_1, \ldots, x_k \in K$ such that $K \subset \bigcup_{k=1}^n x_k E$. For $k = 1, \ldots, n$ let $\psi_k = \delta_{x_k} * \phi_E$. Using Lemmas 2.2, 2.5, and an idea due to Namioka [14, 2.2] one can obtain a net $(\phi_\alpha) \subset \Psi(V)$ such that $\|\phi \cdot \phi_\alpha - \phi_\alpha\|_1 \to 0 \quad (\phi \in L^1(G)_1^+)$. In particular, for some α

$$\|\phi_V \cdot \phi_\alpha - \phi_\alpha\|_1 < \epsilon \quad \text{and} \quad \|\psi_k \cdot \phi_\alpha - \phi_\alpha\|_1 < \epsilon \quad (k = 1, \ldots, n).$$

Let $\phi = \phi_V \cdot \phi_\alpha$, that is

$$\phi(y) = \frac{1}{|V|} \int_V \Delta(x) \phi_\alpha(x^{-1} y x)\, dx, \quad \text{a.e. } y.$$

Then by Lemma 1.3 (3), $\phi \in L^1(G)_1^+$, support$(\phi) \subset V^3 \subset U$, and it is clear that $\phi \in L^\infty(G)$. Thus $\phi \in \Psi(U)$.

As in the proof of the classical version of Reiter's condition [7, 3.2.1] one can now show that

$$\|\delta_x * \phi * \delta_{x^{-1}} - \phi\|_1 = \|x \cdot \phi - \phi\|_1 < 5\epsilon \quad (x \in K). \qquad \square$$

Remarks 2.7.

(1) By Lemmas 1.9 (1), 2.1, and 2.2, $L^1(G)$ has a quasi-central bai if and only if there is a TIM on $L^\infty(G)$ extending the Dirac measure at e.

(2) A net (u_α) satisfying the convergence property of Theorem 2.6 is necessarily a quasi-central bai. This can be seen by arguing as in [15, 4.3].

(3) In [1], a [QSIN]-group (standing for quasi-[SIN]-group) is defined to be any locally compact group whose group algebra has a quasi-central bai.

3. The Main Theorem

We begin with a definition.

Definition 3.1. A net (U_α) of measurable subsets of G with $0 < |U_\alpha| < \infty$ will be called *asymptotically invariant* (under inner automorphisms) if

$$\frac{|x U_\alpha x^{-1} \bigtriangleup U_\alpha|}{|U_\alpha|} \to 0 \qquad (x \in G).$$

We will call G an *almost-*[SIN]*-group* if it possesses an asymptotically invariant net $(U_\alpha) \subset \mathcal{K}(e)$ which comprises a base for the neighbourhood system at e.

Lemma 3.2. *If G possesses an asymptotically invariant net of subsets, then G is unimodular.*

Proof. For each $x \in G$ and each α,

$$\Delta(x) = \frac{|x^{-1}U_\alpha x|}{|U_\alpha|} = \frac{1}{|U_\alpha|}[|x^{-1}U_\alpha x \setminus U_\alpha| + |U_\alpha| - |U_\alpha \setminus x^{-1}U_\alpha x|].$$

Taking the limit of the right hand side of this equation, we obtain $\Delta(x) = 1$ for each $x \in G$. □

It follows that asymptotic invariance of the net (U_α) is equivalent to the condition

$$\frac{|xU_\alpha \triangle U_\alpha x|}{|U_\alpha|} \to 0 \qquad (x \in G).$$

Theorem 3.3. *The following are equivalent for a locally compact group G.*

(1) *G is unimodular and $L^1(G)$ has a quasi-central bai.*
(2) *There exists a net $(U_\alpha) \subset \mathcal{K}(e)$ comprising a base for the neighbourhood system at e such that*
$$\frac{|xU_\alpha x^{-1} \triangle U_\alpha|}{|U_\alpha|} \to 0$$
uniformly on compact subsets of G. The sets U_α may be chosen to be symmetric.
(3) *G is an almost-*[SIN]*-group.*
(4) *$L^1(G)$ has a quasi-central bai comprised of normalized characteristic functions (of compact symmetric neighbourhoods of the identity).*

Note that unimodularity does not follow from the existence of a quasi-central bai alone. Indeed, the group algebra of any amenable group has a quasi-central bai [12, Theorem 3].

Proof. (1) $\Longrightarrow$ (2) We begin by proving some lemmas, in which we assume that condition (1) is satisfied and $U \in \mathcal{K}(e)$ is *fixed*. If v is a function on G, we set $\widetilde{v}(x) := v(x^{-1})$.

Lemma 3.4. *There is a net $(\phi_\alpha) \subset \Psi(U) \cap C_{00}(G)$ such that for each α, $\|\phi_\alpha\|_\infty = \phi_\alpha(e)$, $\widetilde{\phi_\alpha} = \phi_\alpha$, and $\|\delta_x * \phi_\alpha - \phi_\alpha * \delta_x\|_1 \to 0$ uniformly on compact subsets of G.*

Proof. Let $V \in \mathcal{K}(e)$ be symmetric and such that $V^2 \subset U$. Using Theorem 2.6 choose a net $(v_\alpha) \subset \Psi(V)$ such that $\|\delta_x * v_\alpha - v_\alpha * \delta_x\|_1 \to 0$ uniformly on compacta. Let $\phi_\alpha = v_\alpha * \widetilde{v_\alpha}$. It is then easy to see that for each α, $\phi_\alpha \in \Psi(U)$,

$\|\phi_\alpha\|_\infty = \phi_\alpha(e)$ (for example, ϕ_α is positive definite), $\widetilde{\phi_\alpha} = \phi_\alpha$, and because $\Psi(V) \subset L^2(G)$, $\phi_\alpha \in C_{00}(G)$. Finally,

$$\begin{aligned}\|\delta_x * \phi_\alpha - \phi_\alpha * \delta_x\|_1 &\leq \|\delta_x * v_\alpha * \widetilde{v_\alpha} - v_\alpha * \delta_x * \widetilde{v_\alpha}\|_1 \\ &\quad + \ \|v_\alpha * \delta_x * \widetilde{v_\alpha} - v_\alpha * \widetilde{v_\alpha} * \delta_x\|_1 \\ &\leq \|\delta_x * v_\alpha - v_\alpha * \delta_x\|_1 \\ &\quad + \|(v_\alpha * \delta_{x^{-1}} - \delta_{x^{-1}} * v_\alpha)^{\sim}\|_1 \\ &= \|\delta_x * v_\alpha - v_\alpha * \delta_x\|_1 \\ &\quad + \|v_\alpha * \delta_{x^{-1}} - \delta_{x^{-1}} * v_\alpha\|_1,\end{aligned}$$

from which the uniform convergence on compacta follows. We note that unimodularity was used in this proof. □

Notation. We denote the convex hull of a subset S of a linear space by $\mathrm{co}(S)$. Let

$$\Phi(U) = \mathrm{co}\{\phi_K : K \subset U, K \text{ a compact symmetric neighbourhood of } e\}.$$

Lemma 3.5. *There is a net $(\phi_\beta) \subset \Phi(U)$ such that $\|\delta_x * \phi_\beta - \phi_\beta * \delta_x\|_1 \to 0$ uniformly on compact subsets of G.*

Proof. Let (ϕ_α) be a net as in Lemma 3.4 and *fix* α. As $\phi_\alpha \in \Psi(U) \cap C_{00}(G)$, $\|\phi_\alpha\|_\infty = \phi_\alpha(e)$, and $\widetilde{\phi_\alpha} = \phi_\alpha$, it follows that for each positive integer n, and each $k = 0, 1, \ldots, n-1$,

$$A_k^\alpha = \left\{x \in U : \phi_\alpha(x) \geq \frac{k}{n}\phi_\alpha(e)\right\}$$

is a compact symmetric neighbourhood of e. Note that $A_{n-1}^\alpha \subset \cdots \subset A_0^\alpha = U$. Let

$$\phi'_{\alpha,n} := \sum_{k=0}^{n-1} \frac{\phi_\alpha(e)}{n} 1_{A_k^\alpha}.$$

Then

$$\phi_{\alpha,n} := \frac{1}{\|\phi'_{\alpha,n}\|_1}\phi'_{\alpha,n} = \sum_{k=0}^{n-1} \lambda_k \phi_{A_k^\alpha}, \quad \text{where} \quad \lambda_k = \frac{\phi_\alpha(e)}{n\|\phi'_{\alpha,n}\|_1}|A_k^\alpha|.$$

Observe that

$$\sum_{k=0}^{n-1} \lambda_k = \sum_{k=0}^{n-1} \lambda_k \int_G \phi_{A_k^\alpha} = \|\phi_{\alpha,n}\|_1 = 1,$$

so $\phi_{\alpha,n} \in \Phi(U)$. Now it is not difficult to see that $\|\phi_\alpha - \phi'_{\alpha,n}\|_\infty \leq \phi_\alpha(e)/n$, so

$$\|\phi_\alpha - \phi'_{\alpha,n}\|_1 \leq \frac{\phi_\alpha(e)}{n}|U| \to 0 \text{ as } n \to \infty.$$

Therefore $\lim_{n\to\infty} \|\phi'_{\alpha,n}\|_1 = \|\phi_\alpha\|_1 = 1$ and it follows that $\lim_{n\to\infty} \|\phi_\alpha - \phi_{\alpha,n}\|_1 = 0$. Let $\mathcal{F} = \{(\epsilon, K) : \epsilon > 0,\ K \subset G \text{ is compact}\}$. For each $\beta = (\epsilon, K) \in \mathcal{F}$ take ϕ_α such that $\|\delta_x * \phi_\alpha - \phi_\alpha * \delta_x\|_1 < \epsilon/3$, and take n such that $\|\phi_\alpha - \phi_{\alpha,n}\|_1 < \epsilon/3$. Then letting $\phi_\beta = \phi_{\alpha,n}$, we have $\|\delta_x * \phi_\beta - \phi_\beta * \delta_x\|_1 < \epsilon \ \ (x \in K)$. Thus $(\phi_\beta)_{\beta\in\mathcal{F}}$ is the net we want. □

Observe that in establishing Lemma 3.5, we showed that each ϕ_β may be written in the form

$$(*) \qquad \phi_\beta = \sum_{k=1}^{n} \lambda_k \phi_{A_k},$$

where each $\lambda_k > 0$, $\sum_{k=1}^n \lambda_k = 1$, and $U \supset A_1 \supset A_2 \supset \cdots \supset A_n$, with each set A_k a compact symmetric neighbourhood of e.

Lemma 3.6. *Let $\phi \in \Phi(U)$ be written in the form $(*)$. Then*

$$\|\delta_x * \phi - \phi * \delta_x\|_1 = \sum_{k=1}^{n} \lambda_k \frac{|xA_k \triangle A_k x|}{|A_k|} \quad (x \in G).$$

Proof. This is similar to the proof of [14, 3.3]. For any Borel measurable set A, $x \in G$,

$$(\delta_x * \phi_A - \phi_A * \delta_x)(y) = \begin{cases} \frac{1}{|A|} & \text{if } y \in xA \setminus Ax, \\ \frac{-1}{|A|} & \text{if } y \in Ax \setminus xA, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, noting that the sets $\bigcup_{k=1}^n xA_k \setminus A_k x$, $\bigcup_{k=1}^n A_k x \setminus xA_k$ are disjoint, it is clear that

$$P = \{y : (\delta_x * \phi - \phi * \delta_x)(y) > 0\} = \bigcup_{k=1}^{n} (xA_k \setminus A_k x),$$

and

$$N = \{y : (\delta_x * \phi - \phi * \delta_x)(y) < 0\} = \bigcup_{k=1}^{n} (A_k x \setminus xA_k).$$

Hence

$$\begin{aligned}\|\delta_x * \phi - \phi * \delta_x\|_1 &= \sum_{k=1}^{n} \lambda_k \left[\int_P (\delta_x * \phi_{A_k} - \phi_{A_k} * \delta_x)(y)\ dy \right. \\ &\qquad \left. - \int_N (\delta_x * \phi_{A_k} - \phi_{A_k} * \delta_x)(y)\ dy \right] \\ &= \sum_{k=1}^{n} \lambda_k \left[\int_{xA_k \setminus A_k x} \frac{1}{|A_k|} - \int_{A_k x \setminus xA_k} \frac{-1}{|A_k|} \right] \\ &= \sum_{k=1}^{n} \lambda_k \frac{|xA_k \triangle A_k x|}{|A_k|}.\end{aligned}$$

We note that unimodularity was used in this proof. □

We can now prove the implication (1) $\implies$ (2) of the theorem. Let $\mathcal{T} = \{(\epsilon, K, U) : \epsilon > 0,\ K \subset G \text{ is compact},\ U \in \mathcal{K}(e)\}$. It suffices to prove that the following statement holds:

(†) For every $(\epsilon, K, U) \in \mathcal{T}$ there is a compact symmetric neighbourhood A of e such that $A \subset U$ and
$$\frac{|xA \triangle Ax|}{|A|} < \epsilon \quad (x \in K).$$

This is established from Lemmas 3.5 and 3.6 by use of an argument similar to the usual proof of the classical Følner condition, as found, for example, in [7, 3.6.2, 3.6.4]. We first show that the following statement (†*) holds:

(†*) For every $(\epsilon, K, U) \in \mathcal{T}$, and every $\delta > 0$, there is a compact symmetric neighbourhood A of e with $A \subset U$ and a measurable set $N \subset K$ with $|N| < \delta$ such that
$$\frac{|xA \triangle Ax|}{|A|} < \epsilon \quad (x \in K \setminus N).$$

To see this, let $(\epsilon, K, U) \in \mathcal{T}$, $\delta > 0$, and choose $\phi \in \Phi(U)$ such that for every $x \in K$, $\|\delta_x * \phi - \phi * \delta_x\|_1 < \epsilon\delta/|K|$. If we write ϕ in the form $(*)$, and then integrate the continuous function $x \mapsto \|\delta_x * \phi - \phi * \delta_x\|_1$ over K, we obtain
$$\sum_{k=1}^{n} \lambda_k \int_K \frac{|xA_k \triangle A_k x|}{|A_k|}\ dx < \epsilon\delta.$$
As $\sum_{k=1}^{n} \lambda_k = 1$ and each $\lambda_k > 0$, we must have
$$\int_K \frac{|xA \triangle Ax|}{|A|}\ dx < \epsilon\delta$$
for some $A = A_k$. Letting $N = \{x \in K : |xA \triangle Ax|/|A| \geq \epsilon\}$, the sets A and N satisfy (†*) for $(\epsilon, K, U) \in \mathcal{T}$ and δ. We will now show that (†*) $\implies$ (†).

Given $(\epsilon, K, U) \in \mathcal{T}$, apply $(\dagger^*)$ to the triple $(\epsilon/2, L = K \cup K^2, U) \in \mathcal{T}$ and $\delta = (1/2)|K|$ to obtain sets A and N. Let $M = L \setminus N$. Observe that for any $k \in K$, $kL \cap L \subset (kM \cap M) \cup (L \setminus M) \cup (kL \setminus kM)$; also $kK \subset kL \cap L$, so $|kL \cap L| \geq |K|$. Therefore

$$2\delta = |K| \leq |kM \cap M| + 2|N| < |kM \cap M| + 2\delta,$$

whence $kM \cap M \neq \emptyset$ $(k \in K)$. Thus $K \subset MM^{-1}$. But for any $x, y \in M = L \setminus N$,

$$\begin{aligned}\frac{|xy^{-1}A \triangle Axy^{-1}|}{|A|} &= \|\delta_x * \delta_{y^{-1}} * \phi_A - \phi_A * \delta_x * \delta_{y^{-1}}\|_1 \\ &\leq \|\delta_x * (\delta_{y^{-1}} * \phi_A - \phi_A * \delta_{y^{-1}})\|_1 \\ &\quad + \|(\delta_x * \phi_A - \phi_A * \delta_x) * \delta_{y^{-1}}\|_1 \\ &= \frac{|Ay \triangle yA|}{|A|} + \frac{|xA \triangle Ax|}{|A|} < \epsilon.\end{aligned}$$

(2) $\Longrightarrow$ (3) is obvious.

(3) $\Longrightarrow$ (1) Let (U_α) be an asymptotically invariant base for the neighbourhood system at e, and consider the net of normalized characteristic functions $\phi_\alpha = \phi_{U_\alpha}$. By Lemma 3.2, G is unimodular, so $\|\delta_x * \phi_\alpha - \phi_\alpha * \delta_x\|_1 = |xU_\alpha \triangle U_\alpha x|/|U_\alpha|$, which converges to zero. Hence (ϕ_α) is an asymptotically central bai and so, by [12, Theorem 2], $L^1(G)$ has a quasi-central bounded approximate identity.

(2) $\Longrightarrow$ (4) This follows from Remark 2.7(2) and the argument used in the proof of (3) $\Longrightarrow$ (1).

(4) $\Longrightarrow$ (1) Let (ϕ_{U_α}) be such a bai. By Lemma 3.2 we only need to show that the net (U_α) is asymptotically invariant. Observe that

$$\begin{aligned}\|\delta_x * \phi_{U_\alpha} - \phi_{U_\alpha} * \delta_x\|_1 &= \frac{1}{|U_\alpha|}\int \left|1_{xU_\alpha}(y) - \frac{1}{\Delta(x)} 1_{U_\alpha x}(y)\right| dy \\ &= \frac{|xU_\alpha \setminus U_\alpha x|}{|U_\alpha|} + \left|1 - \frac{1}{\Delta(x)}\right| \frac{|xU_\alpha \cap U_\alpha x|}{|U_\alpha|} \\ &\quad + \frac{1}{\Delta(x)} \frac{|U_\alpha x \setminus xU_\alpha|}{|U_\alpha|}.\end{aligned}$$

As $\|\delta_x * \phi_{U_\alpha} - \phi_{U_\alpha} * \delta_x\|_1 \to 0$ and the modular function Δ is always positive (nonzero), it follows that for each $x \in G$

$$\frac{|xU_\alpha \setminus U_\alpha x|}{|U_\alpha|} \to 0 \quad \text{and} \quad \frac{|U_\alpha x \setminus xU_\alpha|}{|U_\alpha|} \to 0.$$

Hence (U_α) is asymptotically invariant. □

Remarks 3.7.

(1) In the proof of (4) $\Longrightarrow$ (1) we only needed $\|\delta_x * \phi_{U_\alpha} - \phi_{U_\alpha} * \delta_x\|_1 \to 0$ $(x \in G)$ and [12, Theorem 2].

(2) By Remark 2.7(2) the nets of Lemmas 3.4 and 3.5 are necessarily quasi-central bai's. Hence the existence of such nets in the group algebra of a unimodular group also characterize almost-[SIN]-groups.

(3) If G is σ-compact and first countable (i.e., metrizable), then the net in the second part of Theorem 3.3 may be taken to be a sequence.

(4) By [12, Theorem 3], we know that every unimodular amenable group is an almost-[SIN]-group. The Heisenberg group is an example of an almost-[SIN]-group which is not even an [IN]-group.

(5) The following statement can be proved by use of Proposition 1.13, [14, 3.1], Lemma 3.2, and the arguments used in the proofs of Lemma 3.6 and the implications (1) $\Longrightarrow$ (2) and (4) $\Longrightarrow$ (1) of Theorem 3.3:

Proposition 3.8. *The following are equivalent for a locally compact group G.*

(1) *G is inner amenable (defined in Section 1) and unimodular.*
(2) *The following Følner-type condition is satisfied:*
For every $\epsilon > 0$ and every compact subset K of G, there is a compact subset A of G such that

$$\frac{|xA \triangle Ax|}{|A|} < \epsilon \quad (x \in K).$$

(3) *G has an asymptotically invariant net of subsets.*
(4) *There is a net of normalized characteristic functions (ϕ_{A_β}) in $L^1(G)$ such that $\|\delta_x * \phi_{A_\beta} - \phi_{A_\beta} * \delta_x\|_1 \to 0$, $(x \in G)$ (or uniformly on compacta of G).*

4. Applications

In this section we will discuss applications of our work to the cohomology of the group and Fourier algebras.

(1) The details of the following remarks may be found in the preprint [21]. Let G be a locally compact group. Then Johnson's theorem states that the group algebra $L^1(G)$ is amenable precisely when G is amenable [9]. In particular, when G is amenable, $L^1(G)$ has an approximate diagonal (m^δ) in $L^1(G)\hat{\otimes}L^1(G)$; see [10]. In [21] it is shown how this follows from Reiter's condition characterizing amenable groups (Proposition 1.13 applied to Example 1.2 (1)), Theorem 2.4, and Theorem 2.6. Indeed, suppose that $(f_\alpha) \in L^1(G)_1^+$ is a net such that $\|\delta_x * f_\alpha - f_\alpha\|_1 \to 0$ uniformly on compact subsets of G and $(u_\beta) \in L^1(G)_1^+$ is a quasi-central bai for $L^1(G)$ as described in Theorem 2.6. Identify $L^1(G)\hat{\otimes}L^1(G)$ with $L^1(G \times G)$ in canonical fashion and

let $\pi : L^1(G \times G) \to L^1(G)$ denote the multiplication operator. For each $\delta = (\alpha, \beta)$, define

$$m^\delta(s,t) = f_\alpha(s)u_\beta(st), \qquad (s,t) \in G \times G.$$

Then (m^δ) is a (strong form of) approximate diagonal for $L^1(G)$ contained in $L^1(G \times G)_1^+$ such that the net $(\pi(m^\delta))$ is a quasi-central bai satisfying the properties described in Theorem 2.6. We note that in [21] it is also shown how one can directly prove from the existence of an approximate diagonal for $L^1(G)$ that G is amenable, thus giving a new proof of Johnson's theorem in terms of approximate diagonals. For discrete groups such a proof may be found in [3].

Moreover, in [21] it is shown that when G is a unimodular amenable group, one can obtain an approximate diagonal (m^δ) for $L^1(G)$ which is comprised of normalized characteristic functions of compact subsets of $G \times G$. This is done by use of the Følner condition [7, 3.6.2] and our Theorem 3.3. Indeed, let (K_α) be a net of compact subsets of G such that

$$\frac{|xK_\alpha \triangle K_\alpha|}{|K_\alpha|} \to 0$$

uniformly on compact subsets of G. By Theorem 2.4, G is an almost-[SIN]-group, so there is a net (U_β) satisfying statement (2) of Theorem 3.3. Let

$$A_{\alpha,\beta} = \{(s,t) : s \in K_\alpha \text{ and } st \in U_\beta\}.$$

Then the net of normalized characteristic functions $(\phi_{A_{\alpha,\beta}})$ is a (strong form of) approximate diagonal for $L^1(G)$. This result may be interpreted as a Følner condition characterizing amenable group algebras.

(2) The Fourier and Fourier-Stieltjes algebras of G are denoted by $A(G)$ and $B(G)$, respectively [4]. Let $\{\lambda, L^2(G)\}$ and $\{\rho, L^2(G)\}$, respectively, denote the left and right regular representations of G. Then the conjugation representation $\{\beta, L^2(G)\}$ of G is defined by $\beta(s) = \lambda(s)\rho(s) \quad (s \in G)$. For $\xi \in L^2(G)$ we define the coefficient of ξ with respect to β by

$$e_\xi(s) = \langle \beta(s)\xi, \xi \rangle \qquad (s \in G).$$

Note that, by definition, any $e_\xi \in B(G)$. Let (f_α) be a net of complex-valued functions on G. We will write support$(f_\alpha) \to \{e\}$ if for each neighbourhood U of e there is some α_0 such that support$(f_\alpha) \subset U$ whenever $\alpha \succeq \alpha_0$. The following result describes when $L^1(G)$ has a quasi-central bai in terms of $A(G)$ and $B(G)$.

PROPOSITION 4.1. *The following are equivalent for a locally compact group G.*

(1) *$L^1(G)$ has a quasi-central bounded approximate identity.*

(2) *There exists a net* (ξ_α) *in* $L^2(G)_1^+$ *such that support*$(\xi_\alpha) \to \{e\}$, *and*

$$\|ve_{\xi_\alpha} - v\|_{A(G)} \to 0 \qquad (v \in A(G)).$$

Proof. (1) $\Longrightarrow$ (2) Let (u_α) be a quasi-central bai for $L^1(G)$ as described in Theorem 2.6. Let $\xi_\alpha := u_\alpha^{1/2}$. Then $(\xi_\alpha) \subset L^2(G)_1^+$, support$(\xi_\alpha) \to \{e\}$, and by a standard inequality (see, for example, [19, Exercise 4.4.5]), $\|\beta(s)\xi_\alpha - \xi_\alpha\|_2^2 \le \|\delta_x * u_\alpha - u_\alpha * \delta_x\|_1$, which converges to 0 uniformly on compact subsets of G. It follows that $e_{\xi_\alpha} \to 1$ uniformly on compact subsets of G. The conclusion is now a consequence of [6, Theorem B2].

(2) $\Longrightarrow$ (1) Let (ξ_α) be a net as described in statement (2). Let K be any compact subset of G and choose $v \in A(G)$ so that v is identically 1 on K. Then

$$\sup\{|e_{\xi_\alpha}(s) - 1| : s \in K\} \le \|ve_{\xi_\alpha} - v\|_{A(G)} \to 0.$$

Observe that

$$\|\beta(s)\xi_\alpha - \xi_\alpha\|_2^2 = 2|1 - \mathrm{Re}\langle\beta(s)\xi_\alpha, \xi_\alpha\rangle| \le 2|1 - e_{\xi_\alpha}(s)|,$$

so $\|\beta(s)\xi_\alpha - \xi_\alpha\|_2 \to 0$ uniformly on compact subsets of G. Now let $u_\alpha := \xi_\alpha^2$. Then $(u_\alpha) \subset L^1(G)_1^+$ and support$(u_\alpha) \to \{e\}$, so (u_α) is a bounded approximate identity for $L^1(G)$. Moreover, by a standard inequality,

$$\|\delta_x * u_\alpha - u_\alpha * \delta_x\|_1 = \|(\beta(x)\xi_\alpha)^2 - (\xi_\alpha)^2\|_1 \le 4\|\beta(x)\xi_\alpha - \xi_\alpha\|_2 \to 0$$

uniformly on compact subsets of G. □

(3) In [17] Z.-J. Ruan proved that a locally compact group G is amenable precisely when its associated Fourier algebra $A(G)$ is operator amenable. We will now indicate how Theorem 2.6 allows for a simplification of Ruan's proof. References for the terminology used below are [17] and [19].

The operator projective tensor product $A(G)\hat{\otimes}A(G)$ can be identified with $A(G \times G)$ through the identity

$$(u \otimes v)(s,t) = u(s)v(t) \qquad (u, v \in A(G),\ s,t \in G).$$

Doing this, $A(G \times G)$ has canonical operator $A(G)$-bimodule operations defined by

$$(u \cdot w)(s,t) = u(s)w(s,t) \ \text{ and } \ (w \cdot u)(s,t) = w(s,t)u(t),$$

where $w \in A(G \times G)$, $u \in A(G)$, and $s,t \in G$. The multiplication operator

$$\Pi : A(G \times G) \to A(G)$$

is given by restricting functions in $A(G\times G)$ to the diagonal $\{(s,s) : s \in G\}$. In an obvious way one can extend these module operations on $A(G\times G)$ to module operations on $B(G\times G)$, and one can extend Π to a map $\Pi : B(G\times G) \to B(G)$.

It is easy to see that $\{\gamma, L^2(G)\}$ defines a continuous unitary representation of $G \times G$, where $\gamma(s,t) := \lambda(s)\rho(t)$. For $\xi \in L^2(G)$ we define the coefficient of ξ with respect to γ by

$$m_\xi(s,t) = \langle \gamma(s,t)\xi, \xi\rangle, \quad (s,t) \in G \times G.$$

Proposition 4.2. *The following are equivalent for a locally compact group G:*

(1) *$L^1(G)$ has a quasi-central bounded approximate identity.*
(2) *There is a net (ξ_α) in $L^2(G)^+_1$ with support$(\xi_\alpha) \to \{e\}$ such that for every $u \in A(G)$*

$$(*) \quad \|u \cdot m_{\xi_\alpha} - m_{\xi_\alpha} \cdot u\|_{B(G\times G)} \to 0 \quad \text{and} \quad \|u\Pi(m_{\xi_\alpha}) - u\|_{B(G)} \to 0.$$

Define $W \in \mathcal{B}(L^2(G \times G))$ by

$$W\xi(s,t) = \xi(s, st), \qquad (\xi \in L^2(G \times G),\ (s,t) \in G \times G).$$

In [17] it is shown that when G is amenable, there is a net $(\xi_\alpha) \subset L^2(G)^+_1$ such that (m_{ξ_α}) satisfies the condition $(*)$. A major part of the proof of this fact is the following nontrivial lemma, which is proved for amenable groups in [17]. As stated below, this lemma is [19, Lemma 7.4.2], where V. Runde observed that the amenability condition may be dropped.

Lemma 4.3. *Let G be a locally compact group and suppose that there is a net of unit vectors (ξ_α) in $L^2(G)$ such that*

$$\|W(\xi_\alpha \otimes \eta) - (\xi_\alpha \otimes \eta)\|_2 \to 0 \quad (\eta \in L^2(G))$$

and

$$\|\gamma(s,s)\xi_\alpha - \xi_\alpha\|_2 \to 0$$

uniformly on compact subsets of G. Then the net (m_{ξ_α}) in $B(G\times G)$ satisfies condition $(*)$ of Proposition 4.2.

We will now show how the existence of a net (ξ_α) as described in Lemma 4.3 follows easily from our Theorem 2.6.

Proof of Proposition 4.2.

(1) $\Longrightarrow$ (2) Let (u_α) be a net as described in Theorem 2.6 and let $\xi_\alpha := u_\alpha^{1/2}$. As shown in the proof of Proposition 4.1, $(\xi_\alpha) \subset L^2(G)^+_1$, support$(\xi_\alpha) \to \{e\}$ and $\|\gamma(s,s)\xi_\alpha - \xi_\alpha\|_2 = \|\beta(s)\xi_\alpha - \xi_\alpha\|_2 \to 0$ uniformly on compact subsets of G. Now let $\eta \in L^2(G)$ be arbitrary. Let U be a symmetric neighbourhood of e such that $\|\lambda(s)\eta - \eta\|_2 < \epsilon$ whenever $s \in U$, and take α_0 such that

support$(\xi_\alpha) \subset U$ whenever $\alpha \succeq \alpha_0$. Then for $\alpha \succeq \alpha_0$

$$\begin{aligned}\|W(\xi_\alpha \otimes \eta) - (\xi_\alpha \otimes \eta)\|_2^2 &= \iint |\xi_\alpha(s)(\eta(st) - \eta(t))|^2 \, dtds \\ &= \int_U \xi_\alpha^2(s)\|\lambda(s^{-1})\eta - \eta\|_2^2 \, ds \le \epsilon^2.\end{aligned}$$

(2) $\Longrightarrow$ (1) Observe that $\Pi m_{\xi_\alpha}(s) = e_{\xi_\alpha}(s)$ $(s \in G)$. Now the implication follows from Proposition 4.1. □

Remarks 4.4.

(1) Using Theorem 2.4 Ruan proved that when G is amenable one can construct a net (ξ_α) as described in Lemma 4.3. To accomplish this, Ruan required Losert and Rindler's explicit construction of a quasi-central bai for $L^1(G)$ from the Reiter condition characterizing amenable locally compact groups.

(2) By Theorem 2.4, condition (2) of Proposition 4.2 is satisfied when G is amenable. That amenability combined with condition (2) of Proposition 4.2 implies that $A(G)$ is operator amenable follows easily from Leptin's theorem [11]. The details are found in [17, Lemma 3.1].

(3) A further application of this work is found in [1]. Indeed, Theorem 2.6 is one of the tools used there to prove that if G is a locally compact group which can be continuously embedded into another group H such that $L^1(H)$ has a quasi-central bounded approximate identity, then $A(G)$ is operator biflat.

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