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BEST WEAK–TYPE (p,p) CONSTANTS, $1 \leq p \leq 2$, FOR ORTHOGONAL HARMONIC FUNCTIONS AND MARTINGALES

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Abstract. We prove that the best weak–type (p,p) constant, $1 \leq p \leq 2$, for orthogonal harmonic functions u and v with v differentially subordinate to u is

$$K_p = \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\left| \frac{2}{\pi} \log |t| \right|^p}{t^2+1} dt \right)^{-1}.$$

1. Introduction

A celebrated theorem of Kolmogorov [10] states that if $\tilde{f}$ is the conjugate function of an integrable real-valued function f on the unit circle, then $m\{|\tilde{f}| > \lambda\} \leq \frac{K}{\lambda}\|f\|_1$, where m is the Lebesgue measure on the circle and K is a positive constant independent of f. This is the weak–type (1,1) inequality for the conjugate function operator on the unit circle. In general, given a measure space (X,ν) and an operator T acting on $L^p(X,\nu)$, $1 \leq p < \infty$, T is said to be a strong–type (p,p) operator if there exists a constant $A_p > 0$ such that

$$\|Tf\|_{L^p(X,\nu)} \leq A_p \|f\|_{L^p(X,\nu)} \tag{1.1}$$

for all $f \in L^p(X,\nu)$. T is said to be a weak–type (p,p) operator if there exists constant $K_p > 0$ such that

$$\nu\{\omega \in X : |Tf(\omega)| > 1\} \leq K_p \|f\|^p_{L^p(X,\nu)} \tag{1.2}$$

for all $f \in L^p(X,\nu)$. The strong–type property implies the weak–type property but not vice versa.

An area of research for well known operators satisfying these properties is to find the best constants A_p and K_p in (1.1) and (1.2), respectively. The best

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constant in Kolmogorov's weak–type (1,1) inequality was shown by Davis [7] to be

$$K = \frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots}{1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \cdots}. \tag{1.3}$$

Davis used Brownian motion techniques to prove this result and also showed that the same inequality holds for the conjugate function operator (commonly called the Hilbert transform) on the real line. The same was reproved by Baernstein [1] using complex analysis. Tomaszewski [16] was the first to extend these results to the case $1 \leq p \leq 2$. Let D be the unit disk in $\mathbb{R}^2$, f analytic in D and continuous to the boundary with $\operatorname{Im} f(0) = 0$. Let σ be the Lebesgue measure on ∂D. Then Tomaszewski proved that

$$\sigma\{z \in \partial D : |f(z)| \geq 1\} \leq C_p \| \operatorname{Re}(f)\|_p^p, \tag{1.4}$$

where

$$C_p = \frac{\sqrt{\pi}}{2} \frac{p\Gamma(p/2)}{\Gamma((p+1)/2)}$$

is the best constant possible. The objective of the present paper is to prove the $1 \leq p \leq 2$ result in a setting of orthogonal harmonic functions introduced by Bañuelos and Wang [2]. Let D be a domain in $\mathbb{R}^n$, where n is a positive integer. Let D_0 be a bounded subdomain of D with $\partial D_0 \subset D$ and $\xi \in D_0$. Let μ be the harmonic measure on ∂D_0 with respect to ξ. Then the following theorem holds.

THEOREM 1.1. *Let $1 \leq p \leq 2$. If u and v are harmonic functions on D such that*

(i) $|v(\xi)| \leq |u(\xi)|$,
(ii) $|\nabla v| \leq |\nabla u|$ *on D,*
(iii) $\nabla v \cdot \nabla u = 0$ *on D,*

then

$$\mu(|v| \geq 1) \leq K_p \int_{\partial D_0} |u|^p d\mu. \tag{1.5}$$

Here K_p is the best constant given by

$$K_p^{-1} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\left|\frac{2}{\pi} \log |t|\right|^p}{t^2 + 1} dt. \tag{1.6}$$

This agrees with (1.3) when $p = 1$. To show that K_p is the best constant possible, let $D = \mathbb{R}^2_+$ and $\xi = (0, 1)$. Let u and v be the harmonic extensions of $\frac{2}{\pi} \log |t|$ and $\operatorname{sgn} t$, respectively. Then u and v satisfy the conditions of the theorem, and (1.5) holds only if the constant is at least as large as K_p defined in (1.6).

Notice that in this theorem the condition that u and v be conjugate harmonic functions is dropped, and u and v are only assumed to be harmonic functions in a domain in $\mathbb{R}^n$, continuous to the boundary. The standard requirements are that the gradients are orthogonal and satisfy a subordination condition as in the theorem. The weak–type (1,1) version of Theorem 1.1 was proved by Choi [5]. The present paper follows the basic outline of Choi. However different methods of proof are introduced to prove some of the needed lemmas. This includes an application of the characterization of maximal subharmonic functions via minimal functions (Lemma 2.4) to prove Lemma 2.5. Lemma 2.4 is a corollary to a result of Kinnunen and Martio [9]; the theory of such characterizations is explored in depth in their paper.

In Section 5, a probabilistic analogue is proved for orthogonal martingales. Let X and Y be two real-valued continuous time parameter martingales with a common filtration $F = \{F_t\}_{t\geq 0}$ (a family of increasing, right-continuous sub-σ-fields in a probability space $\{\Omega, A, P\}$). Assume F_0 contains all null sets. Denote the quadratic covariation process $\{[X,Y]_t\}_{t\geq 0}$ between X and Y by $[X,Y]$, and denote $[X,X]$ by $[X]$. The processes X and Y are *orthogonal* if $[X,Y] = 0$. The process Y is *differentially subordinate* to X if $[X]_t - [Y]_t$ is non-decreasing and non-negative as a function of t. Note that the differential subordination and orthogonality conditions correspond to (ii) and (iii) in Theorem 1.1. The next theorem will be proved in Section 5.

THEOREM 1.2. *Let $1 \leq p \leq 2$. If X and Y are two $\mathbb{R}$-valued continuous time parameter orthogonal martingales such that Y is differentially subordinate to X, then for each $t > 0$,*

$$P(|Y_t| \geq 1) \leq K_p \|X_t\|_p^p, \tag{1.7}$$

where K_p is the constant given by (1.6). *The inequality is sharp.*

Bañuelos and Wang [3] prove this for $p = 1$. In this paper, their proof is made simpler by the use of Itô's formula. Without the orthogonality assumption in Theorem 1.2, the best constant is $2/\Gamma(p+1)$. This follows from the work of Burkholder; see Sections 1, 12 and 13 of [4]. In Theorem 1.1, without the orthogonality assumption, the constant $2/\Gamma(p+1)$ is an upper bound for the best constant. For one of the methods that can be used to establish this, see the proof of Theorem 7.1 in Suh [15].

The methods used in the present paper break down when $p > 2$. The reasons are given in Section 4, and it is shown that the best constant for $p > 2$ behaves like $(cp)^p$. Suh (Theorem 1.2, [15]) proved the following precise result when $p > 2$: Let M and N be right-continuous martingales with limits from the left, adapted to the filtration $\{F_t\}_{t\geq 0}$, and suppose $[M]_t - [N]_t$ is nonnegative and nondecreasing in t. Then

$$\lambda^p P(\sup_{t\geq 0} N_t \geq \lambda) \leq \frac{p^{p-1}}{2} \|M\|_p^p, \tag{1.8}$$

and the constant $p^{p-1}/2$ is the best possible. Hence the best constant for martingales not necessarily orthogonal is known for $p > 2$. Translated to the harmonic function setting, Suh showed (Theorem 7.1, [15]) when u and v satisfy (i) and (ii) but not (iii) in Theorem 1.1, $p^{p-1}/2$ is an upper bound for the weak–type (p, p) constant.

2. Proof of Theorem 1.1

Consider the function on $\mathbb{R}^2$ given by

$$V(x,y) = \begin{cases} -K_p|x|^p, & |y| < 1, \\ 1 - K_p|x|^p, & |y| \geq 1. \end{cases}$$

Then

$$\mu(|v| \geq 1) - K_p \int_{\partial D_0} |u|^p d\mu = \int_{\partial D_0} V(u,v) d\mu.$$

Hence the goal is to show that the right hand side is ≤ 0. This is done by finding a continuous function $U \geq V$ that satisfies the required condition.

Lemma 2.1. *There is a continuous function U on $\mathbb{R}^2$ such that*
 (a) $V \leq U$ *on* $\mathbb{R}^2$,
 (b) $U(u,v)$ *is superharmonic on* D,
 (c) $U(x,y) \leq 0$ *when* $|y| \leq |x|$.

From (a) and (b) it follows that

$$\int_{\partial D_0} V(u,v) d\mu \leq \int_{\partial D_0} U(u,v) d\mu \leq U(u(\xi), v(\xi)),$$

because μ is the harmonic measure on ∂D_0 with respect to ξ. By (c) and assumption (i) of the theorem, $U(u(\xi), v(\xi)) \leq 0$, which proves the theorem.

The proof of the lemma will be given after some preliminaries. Let $H = \{(\alpha, \beta) : \beta > 0\}$ be the upper half-space in $\mathbb{R}^2$, $S = \{(x, y) : |y| \leq 1\}$ and $S^+ = \{(x,y) \in S : x > 0\}$. Define

$$Q(\alpha,\beta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\beta \left|\frac{2}{\pi} \log|t|\right|^p}{(\alpha - t)^2 + \beta^2} dt. \tag{2.1}$$

Then Q is the harmonic function on H that vanishes as $\beta \to \infty$ and satisfies

$$\lim_{(\alpha,\beta)\to(t,0)} Q(\alpha,\beta) = \left|\frac{2}{\pi} \log|t|\right|^p \quad \text{if } t \neq 0.$$

Next let φ on S be the conformal map $\varphi(z) = ie^{\pi z/2}$ mapping S onto H. Define $W : \mathbb{R}^2 \to \mathbb{R}$ by

$$W(x,y) = \begin{cases} |x|^p, & |y| \geq 1, \\ Q(\varphi(x,y)), & |y| < 1. \end{cases} \tag{2.2}$$

Note that W on S is the harmonic lifting of the subharmonic function $|x|^p$ with same boundary values. Hence W is continuous as is the function U defined by

$$U(x,y) = 1 - K_p W(x,y) \quad \text{for } (x,y) \in \mathbb{R}^2.$$

The main lemma (Lemma 2.1) will be proved after first proving various properties of W.

Lemma 2.2. *If $(x,y) \in S$, then $W(x,y) = W(-x,y) = W(x,-y)$ and*

$$W_x(0,y) = W_y(x,0) = W_{xy}(x,0) = W_{xy}(0,y) = 0.$$

The proof of this lemma is omitted. It is an easy consequence of the characterization of superharmonic functions via maximal functions and is similar to that of Lemma 2.5.

Lemma 2.3. *If $-1 < y < 0$, then both $W_{xx}(x,y)$ and $W(x,y) - x^p \to 0$ as $x \to \infty$.*

Proof. Under a change of variables,

$$W(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos\left(\frac{\pi}{2}y\right)\left|\frac{2}{\pi}\log|u| + x\right|^p}{(u+\sin\left(\frac{\pi}{2}y\right))^2 + \cos^2\left(\frac{\pi}{2}y\right)}du.$$

By taking the x derivatives inside the integral, we get

$$W_x(x,y) = \frac{p}{\pi}\int_{-\infty}^{\infty} \frac{\cos\left(\frac{\pi}{2}y\right)\left|\frac{2}{\pi}\log|u| + x\right|^{p-1}\operatorname{sgn}(\frac{2}{\pi}\log|u|+x)}{(u+\sin\left(\frac{\pi}{2}y\right))^2 + \cos^2\left(\frac{\pi}{2}y\right)}du,$$

and

$$W_{xx}(x,y) = \frac{p(p-1)}{\pi}\int_{-\infty}^{\infty} \frac{\cos\left(\frac{\pi}{2}y\right)\left|\frac{2}{\pi}\log|u| + x\right|^{p-2}}{(u+\sin\left(\frac{\pi}{2}y\right))^2 + \cos^2\left(\frac{\pi}{2}y\right)}du.$$

Since $p-2<0$, some basic analysis arguments show that

$$\lim_{x\to\infty} W_{xx}(x,y) = 0.$$

The convergence is uniform with respect to y for $|y|<1$; this can be shown by applying the maximum principle to the harmonic function W_{xx}. Since W is harmonic in S, this implies that $W_{yy} \to 0$ as $x \to \infty$ uniformly in y. The same is true for the function $I(x,y) = W(x,y) - |x|^p$, that is, $I_{yy} \to 0$ as $x \to \infty$ uniformly in y. This implies that $I(x,y_1) - I(x,y_0) \to 0$ as $x \to \infty$ for each y_0, $y_1 \in (-1,1)$. Since $I(x,1) = I(x,-1) = 0$ for all x, $I(x,y) \to 0$ as $x \to \infty$. □

The justification for taking derivatives inside integrals to compute W_x and W_{xx} follows from Fubini's theorem. For example,

$$\begin{aligned}
W_x(b,y) - W_x(a,y) &= \int_a^b W_{xx}(x,y)dx \\
&= \int_a^b \left(\frac{p(p-1)}{\pi} \int_{-\infty}^{\infty} \frac{\cos\left(\frac{\pi}{2}y\right) \left|\frac{2}{\pi}\log|u| + x\right|^{p-2}}{(u+\sin\left(\frac{\pi}{2}y\right))^2 + \cos^2\left(\frac{\pi}{2}y\right)} du \right) dx \\
&= \frac{p(p-1)}{\pi} \int_{-\infty}^{\infty} \frac{\cos\left(\frac{\pi}{2}y\right)}{(u+\sin\left(\frac{\pi}{2}y\right))^2 + \cos^2\left(\frac{\pi}{2}y\right)} \int_a^b \left|\frac{2}{\pi}\log|u| + x\right|^{p-2} dxdu \\
&= \frac{p}{\pi} \int_{-\infty}^{\infty} \frac{\cos\left(\frac{\pi}{2}y\right) \left[\left|\frac{2}{\pi}\log|u| + x\right|^{p-1} \operatorname{sgn}\left(\frac{2}{\pi}\log|u| + x\right)\right]_a^b}{(u+\sin\left(\frac{\pi}{2}y\right))^2 + \cos^2\left(\frac{\pi}{2}y\right)} du.
\end{aligned}$$

This verifies the formula for W_{xx} given the formula for W_x. A similar argument would verify the formula for W_x.

Notice that the proof also gives that $W_{xx} \geq 0$ in S. This will be used in the proof of the main lemma.

Next, we show that $W_{xy} \leq 0$ on $\Omega = \{(x,y) : x > 0 \text{ and } -1 < y < 0\}$. The proof will use the fact that $W_x(x,y)$ is the harmonic drop of $p|x|^{p-1}$ in S^+ via maximal functions. The required lemma is the following; it is a corollary of the results in [9].

LEMMA 2.4. *Let D be a region in $\mathbb{R}^n$, $z \in D$ and $R_z = \operatorname{dist}(z, \partial D)$. If g is a superharmonic function continuous to the boundary and G is the harmonic function with same boundary values as g, then*

$$G(z) = \lim_{k\to\infty} m_D^k g(z).$$

Here $m_D^0 g(z) = g(z)$ on D, and for positive integers k, $m_D^k g$ is defined inductively by

$$m_D^{k+1} g(z) = \inf_{0<r<R_z} \frac{1}{|B(z,r)|} \int_{B(z,r)} m_D^k g(\zeta) d\zeta.$$

LEMMA 2.5. *On the set $\{(x,y) : x > 0 \text{ and } -1 < y < 0\}$, denoted by Ω, the partial derivative W_{xy} exists and satisfies $W_{xy} \leq 0$.*

Proof. Because $1 \leq p \leq 2$, the function g defined by $g(z) = px^{p-1}$ is superharmonic on S^+. Furthermore, g and the harmonic function W_x, the partial derivative with respect to x of the harmonic function W defined above, satisfy the same boundary conditions on the closure of S^+. So, by Lemma 2.4,

$$W_x(z) = \lim_{k\to\infty} m_{S^+}^k g(x) \text{ for all } z \text{ in } S^+. \tag{2.3}$$

If $z = (x, y)$, let z^* denote $(x, -y)$. If $k = 0$, then

$$m^k_{S+}g(z) = m^k_{S+}g(z^*) \text{ for all } z \text{ in } S^+. \tag{2.4}$$

If k is any nonnegative integer, then (2.4) holds also as can be shown by induction. The easy proof is omitted. However, the proof of the following result will be given:

$$m^k_{S+}g(z_2) \leq m^k_{S+}g(z_1) \tag{2.5}$$

for all z_1 and z_2 such that $z_1 = (x, y_1)$ and $z_2 = (x, y_2)$ with $x > 0$ and $-1 < y_1 < y_2 \leq 0$. It will follow from this after an additional argument that $W_x(z_1) \leq W_x(z_2)$.

The inequality (2.5) holds for $k = 0$. Suppose that k is any nonnegative integer such that (2.5) holds. Let $R_{z_1} = \min\{x, y_1 + 1\}$ and $R_{z_2} = \min\{x, y_2 + 1\}$, so that $R_{z_1} \leq R_{z_2}$. Let $0 < r < R_{z_1}$ and $z = (x, y)$, where x is as before and y is chosen so that $z \in B(z_1, r)$. Then $\tilde{z} \in B(z_2, r)$, where $\tilde{z}$ denotes the reflection of z across the horizontal line containing $(x, (y_1 + y_2)/2)$. If ζ is any point in Ω below this horizontal line, then $\tilde{\zeta}$ is above it. Letting $\tilde{\zeta} = (\tilde{\alpha}, \tilde{\beta})$, there are two possibilities: $\tilde{\beta} \leq 0$ or $\tilde{\beta} > 0$. If $\tilde{\beta} \leq 0$, then by the induction hypothesis,

$$m^k_{S+}g(\tilde{\zeta}) \leq m^k_{S+}g(\zeta). \tag{2.6}$$

If $\tilde{\beta} > 0$, then both ζ and $(\tilde{\zeta})^*$ are in Ω with $(\tilde{\zeta})^*$ above ζ. Consequently, by (2.4) and the induction hypothesis,

$$m^k_{S+}g(\tilde{\zeta}) = m^k_{S+}g((\tilde{\zeta})^*) \leq m^k_{S+}g(\zeta). \tag{2.7}$$

(2.6) and (2.7) will both be important in the proof of the following inequality in which z_1 and z_2 are as above:

$$\frac{1}{|B(z_2, r)|}\int_{B(z_2,r)} m^k_{S+}g(\zeta)d\zeta \leq \frac{1}{|B(z_1, r)|}\int_{B(z_1,r)} m^k_{S+}g(\zeta)d\zeta. \tag{2.8}$$

To prove this inequality, it is enough to show that the integral on the right is greater than or equal to the integral on the left. With $B_j = B(z_j, r)$, $j = 1, 2$, the integral on the right can be expressed as the sum of two integrals over the two disjoint sets given by the partition

$$B(z_1, r) = B_1 \cap B_2 \quad \cup \quad B_1 \cap B_2^c.$$

Analogously, the integral on the left can be split into two integrals corresponding to the partition

$$B(z_2, r) = B_1 \cap B_2 \quad \cup \quad B_1^c \cap B_2.$$

If $\zeta \in B_1 \cap B_2^c$, then ζ is below the horizontal line generated by z_1 and z_2. Moreover, the mapping $\zeta \mapsto \tilde{\zeta} : B_1 \cap B_2^c \to B_1^c \cap B_2$ is one-to-one and onto. So by (2.6) and (2.7), the integral over $B_1^c \cap B_2$ is less than or equal to the integral over $B_1 \cap B_2^c$. Consequently, the inequality (2.8) holds. Since $R_{z_1} \leq$

R_{z_2}, taking the infimum of both sides of (2.8) with respect to $r \in (0, R_{z_1})$ gives $m_{S^+}^{k+1} g(z_1)$ on the right and an upper bound to $m_{S^+}^{k+1} g(z_2)$ on the left. Therefore, (2.5) holds for all nonnegative integers, and by (2.3),

$$W_x(z_2) \le W_x(z_1). \tag{2.9}$$

Since W_x is harmonic, W_{xy} exists and (2.9) leads at once to $W_{xy} \le 0$ on Ω, which proves Lemma 2.5. □

3. Proof of Lemma 2.1

(a) By definition, $U(x,y) = V(x,y)$ if $|y| \ge 1$. Also $W(0,0) = K_p^{-1}$. Thus, if $|y| < 1$, then

$$U(x,y) = 1 - K_p W(x,y) = -K_p[W(x,y) - W(0,0)]. \tag{3.1}$$

Hence the property (a) follows if $-K_p|x|^p \le -K_p[W(x,y) - W(0,0)]$ on S. By the symmetry of W, it suffices to show

$$E(x,y) \le 0 \text{ if } (x,y) \in S^+,$$

where $E(x,y) = W(x,y) - W(0,0) - |x|^p$. Since $E(0,y) \le 0$, it suffices to show

$$E_x(x,y) = W_x(x,y) - px^{p-1} \le 0$$

in S^+. But W_x is the harmonic drop in S^+ of the superharmonic function px^{p-1}. This is a consequence of Lemma 2.4. Hence $W_x(x,y) - px^{p-1} \le 0$ in S^+.

(b) By (3.1), property (b) becomes

$$W(u,v) \text{ is subharmonic on } D.$$

First observe that

$$W(x,y) \ge |x|^p \text{ if } |y| < 1. \tag{3.2}$$

Let $w = W(u,v)$ on D. When $|v| > 1$, $w = |u|^p$ is subharmonic since u is harmonic. When $|v| < 1$,

$$\begin{aligned}\Delta w &= W_{xx}|\nabla u|^2 + W_{yy}|\nabla v|^2 + 2W_{xy}\nabla u \cdot \nabla v + W_x \Delta u + W_y \Delta v \\ &= W_{xx}(|\nabla u|^2 - |\nabla v|^2) \ge 0,\end{aligned}$$

using the assumptions (i) and (ii) of Theorem 1.1, Lemma 2.3 and the harmonicity of u and v. For $|v| = 1$ we have at $\eta \in D$ and for all $r > 0$ small enough,

$$\operatorname{Avg}(w; \eta, r) \ge \operatorname{Avg}((|u|^p; \eta, r) \ge |u(\eta)|^p = w(\eta),$$

where the first inequality follows by (3.2). The function w is therefore subharmonic at η and on D.

(c) By (3.1), property (c) of U follows from

$$W(x,y) \ge W(0,0) \text{ if } |x| \ge |y|. \tag{3.3}$$

Let $I_0 = [0, \infty)$ and for $-1 \le a < 0$ let $I_a = [0, -1/a)$. Define $\Phi_a(t) = W(t, at)$ for $t \in I_a$. Then for t in the interior of I_a,

$$\Phi_a'(t) = W_x(t, at) + aW_y(t, at)$$

and

$$\begin{aligned}\Phi_a''(t) &= W_{xx}(t, at) + a^2 W_{yy}(t, at) + 2aW_{xy}(t, at)\\ &= (1 - a^2)W_{xx}(t, at) + 2aW_{xy}(t, at)\\ &\ge 0,\end{aligned}$$

because W is harmonic, $W_{xx}(t, at) \ge 0$ by Lemma 2.3 and $W_{xy}(t, at) \le 0$ by Lemma 2.5. Observe that $\Phi_a'(0) = W_x(0, 0) + aW_y(0, 0) = 0$ by Lemma 2.2. Hence $\Phi_a(t) \ge \Phi_a(0)$ for $t \in I_a$. Thus $W(t, at) \ge W(0, 0)$ if $-1 \le a \le 0$ and $t \in I_a$. But $\{(x, y) : x \ge -y \text{ and } -1 \le y \le 0\} = \{(t, at) : -1 \le a \le 0 \text{ and } t \in I_a\}$. Using the symmetry of W, we have

$$W(x, y) \ge W(0, 0) \text{ if } |x| \ge |y| \text{ and } |y| < 1.$$

Also, if $|x| \ge |y|$ and $|y| \ge 1$, then

$$W(x, y) = |x|^p \ge 1 \ge \frac{1}{K_p} = W(0, 0).$$

This proves (3.3) and hence (c). $\square$

To see that $1/K_p \le 1$, note that

$$K_p^{-1/p} = \left(\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\left|\frac{2}{\pi}\log|t|\right|^p}{t^2+1}\,dt\right)^{1/p}$$

is the L^p norm of the function $\frac{2}{\pi}\log|t|$ on a probability measure space and hence is increasing with respect to p. When $p = 2$, $W(0, 0) = 1$ since the harmonic lift of $|x|^2$ in S is $W(x, y) = x^2 - y^2 + 1$.

4. The case $p > 2$

The proof given above does not work for $p > 2$ since it is no longer true that $W(x, y) - |x|^p \le W(0, 0)$. It can be verified that $W(x, 0) - W(0, 0) \ge |x|^p$ for all x. In fact, the constant is not equal to K_p for $p > 2$. To see this, first observe that if $p > 2$, then

$$\begin{aligned}
K_p^{-1} &= \frac{2}{\pi}\int_0^\infty \frac{\left|\frac{2}{\pi}\log t\right|^p}{t^2+1}dt \\
&= \left(\frac{2}{\pi}\right)^{p+1}\int_{-\infty}^\infty \frac{|s|^p e^s}{e^{2s}+1}ds \\
&= 2\left(\frac{2}{\pi}\right)^{p+1}\int_0^\infty s^p e^{-s}\sum_{k=0}^\infty \left(-e^{-2s}\right)^k ds \\
&= 2\left(\frac{2}{\pi}\right)^{p+1}\sum_{k=0}^\infty \frac{(-1)^k}{(1+2k)^{p+1}}\int_0^\infty s^p e^{-s} ds \\
&= 2\left(\frac{2}{\pi}\right)^{p+1}\Gamma(p+1)\sum_{k=0}^\infty \frac{(-1)^k}{(1+2k)^{p+1}}.
\end{aligned}$$

Therefore

$$K_p^{-1} \geq 2\left(\frac{2}{\pi}\right)^{p+1}\Gamma(p+1)(1-3^{-(p+1)})$$

and

$$K_p \leq c\left(\frac{\pi}{2}\right)^p (\Gamma(p+1))^{-1}.$$

Now if $\mu_\xi(|v| \geq 1) \leq K_p\|u\|_p^p$, then $\mu_\xi(|v| \geq 1)^{1/p} \leq K_p^{1/p}\|u\|_p$. As $p \to \infty$, the left hand side goes to 1 whenever $\mu_\xi(|v| \geq 1) > 0$, whereas for all u with $\|u\|_\infty < \infty$, the right hand side goes to 0. This suggests that for all u with $\|u\|_\infty < \infty$, $\|v\|_\infty \leq 1$, which is false. Hence the constant K_p is too small for the inequality to hold.

Let C_p be the correct weak–type (p,p) constant for $p > 2$. Then by Theorem (7.1) in Suh [15],

$$C_p \leq \frac{p^{p-1}}{2}. \tag{4.1}$$

A lower bound is given below by considering the Hilbert transform. The Hilbert transform on $L^1(\mathbb{R})$ is defined by

$$Hf(x) = \text{p.v.}\frac{1}{\pi}\int_{-\infty}^\infty \frac{f(x-t)}{t}dt.$$

By standard arguments (see Zygmund [17], Vol.2, p. 256), it can be shown that H satisfies

$$\sup_{\lambda>0}\lambda^p m(x \in \mathbb{R} : |Hf(x)| \geq \lambda) \leq C_p\|f\|_p^p,$$

where m is the Lebesgue measure on $\mathbb{R}$ and C_p is the same constant as in our case. Let $f(x) = \chi_E(x)$ be the characteristic function of a measurable set E.

Stein and Weiss [13] proved that

$$\lambda^p m(x \in \mathbb{R} : |Hf(x)| \geq \lambda) = \frac{4\lambda^p m(E)}{e^{\pi\lambda} - e^{-\pi\lambda}}.$$

Putting $\lambda = p/\pi$ shows that $C_p \geq 4\,(p/\pi)^p\, e^{-p}$. This and (4.1) give

$$\frac{1}{e\pi} \leq C_p^{1/p} p^{-1} \leq 1. \tag{4.2}$$

So in general, C_p behaves like $(cp)^p$. This suggests that the best constant in the orthogonal case may also be $p^{p-1}/2$.

5. Proof of Theorem 1.2

Recall that X and Y are real-valued orthogonal martingales such that for the covariation process, $[X,Y]_t = 0$ for all $t > 0$ and $[X]_t - [Y]_t$ is non-increasing process with respect to t. Theorem 1.2 states that for $1 \leq p \leq 2$ and $t > 0$,

$$P(|Y_t| \geq 1) \leq K_p \|X_t\|_p^p,$$

where K_p is the constant given by (1.6). Moreover this inequality is sharp in general.

Let V and U be the functions defined before. It suffices to show $EV(X_t, Y_t)$ ≤ 0, and since $V \leq U$ by Lemma 2.1, this follows from

$$EU(X_t, Y_t) \leq EU(X_0, Y_0).$$

The second term is less than or equal to 0 since $|Y_0| \leq |X_0|$ (which follows from Lemma 2.1 in [3]).

Define the stopping time as

$$T = \inf\{t \geq 0 : |Y_t| \geq 1\},$$

and let $Z = \{Z_t\}_{t \geq 0} = \{U(X_{t\wedge T}, Y_{t\wedge T})\}_{t\geq 0}$. Then it is enough to show

$$EU(X_t, Y_t) \leq EZ_t \tag{5.1}$$

and that

$$Z_t \text{ is a supermartingale.} \tag{5.2}$$

To prove (5.1), note that $EU(X_t, Y_t) = E[U(X_t, Y_t)I_{t\geq T}] + E[U(X_t, Y_t)I_{t<T}]$. By Lemma 2.3, $W(X_t, Y_t) \geq |X_t|^p$. Since X is a martingale, $|X_t|^p$ is a submartingale. Thus,

$$\begin{aligned} E[|X_t|^p I_{t\geq T}] &= E[E(|X_t|^p | F_T) I_{t\geq T}] \\ &\geq E[|X_T|^p I_{t\geq T}]. \end{aligned}$$

Consequently,

$$\begin{aligned}
E[U(X_t,Y_t)I_{t\leq T}] &= E[(1-K_pW(X_t,Y_t))I_{t\geq T}]\\
&\leq E[(1-K_p|X_t|^p)I_{t\geq T}]\\
&\leq E[(1-K_p|X_T|^p)I_{t\geq T}]\\
&= E[(1-K_pW(X_T,Y_T))I_{t\geq T}]\\
&= E[U(X_T,Y_T)I_{t\geq T}],
\end{aligned}$$

where we used the fact that $|Y_T| = 1$ on $\{t \geq T\}$, since Y has continuous paths. It remains to prove (5.2).

By Itô's formula,

$$\begin{aligned}
W(X_{t\wedge T},Y_{t\wedge T}) - W(X_0,Y_0) &= \int_0^{t\wedge T} \nabla W(X_s,Y_s)\cdot d(X_s,Y_s)\\
&\quad + \frac{1}{2}\int_0^{t\wedge T} \frac{\partial^2}{\partial x^2}W(X_s,Y_s)d[X]_s\\
&\quad + \frac{1}{2}\int_0^{t\wedge T} \frac{\partial^2}{\partial y^2}W(X_s,Y_s)d[Y]_s\\
&\quad + \int_0^{t\wedge T} \frac{\partial^2}{\partial x\partial y}W(X_s,Y_s)d[X,Y]_s.
\end{aligned}$$

Since $[X,Y]=0$ and W is harmonic on S, it follows that

$$\begin{aligned}
W(X_{t\wedge T},Y_{t\wedge T}) - W(X_0,Y_0) &= \int_0^{t\wedge T} \nabla W(X_s,Y_s)\cdot d(X_s,Y_s)\\
&\quad + \frac{1}{2}\int_0^{t\wedge T} \frac{\partial^2}{\partial x^2}W(X_s,Y_s)d([X]-[Y])_s.
\end{aligned}$$

The first term is a martingale. The second is an increasing process since $W_{xx}\geq 0$ by Lemma 2.3 and since $[X]_s - [Y]_s$ is an non-decreasing process by the assumption of differential subordination. Therefore $W(X_{t\wedge T},Y_{t\wedge T})$ is a submartingale, and Z_t is a supermartingale. □

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