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GROUP PROPERTIES CHARACTERISED BY CONFIGURATIONS

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Abstract. J. M. Rosenblatt and G. A. Willis introduced the notion of configurations for finitely generated groups G. They characterised amenability of G in terms of the configuration equations. In this paper we investigate which group properties can be characterised by configurations. It is proved that if G_1 and G_2 are two finitely generated groups having the same configuration sets and G_1 satisfies a semigroup law, then G_2 satisfies the same semigroup law. Furthermore, if G_1 is abelian then G_1 and G_2 are isomorphic.

1. Introduction and definitions

The notion of a configuration for a finitely generated group, G, was introduced in [5]. It was shown in that paper that amenability of G is characterised by its configurations. In this paper we investigate which properties of groups can be characterised by configurations and whether in fact G is determined up to isomorphism by its configurations. The configurations of G are defined in terms of finite generating sets and finite partitions of G.

Definition 1.1. Let G be a finitely generated group. Let $\mathfrak{g} = (g_1, \ldots, g_n)$ be an ordered set of generators for G and $\mathcal{E} = \{E_1, \ldots E_m\}$ be a finite partition of G.

A *configuration* corresponding to this generating sequence and partition is an $(n+1)$-tuple $C = (C_0, C_1, \ldots, C_n)$, where $C_i \in \{1, \ldots, m\}$ for each i, such that there is x in G with

$$x \in E_{C_0} \text{ and } g_i x \in E_{C_i} \text{ for each } i \in \{1, \ldots, n\}.$$

The *set of configurations* corresponding to the generating sequence $\mathfrak{g}$ and partition $\mathcal{E}$ of G will be denoted by $\mathrm{Con}(\mathfrak{g}, \mathcal{E})$. The set of all configuration sets of G is

$$\mathrm{Con}(G) = \{\mathrm{Con}(\mathfrak{g}, \mathcal{E}) : G = \langle \mathfrak{g} \rangle \text{ and } \mathcal{E} \text{ is a finite partition of } G\}. \tag{1.1}$$

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A configuration is thus an $(n+1)$-tuple of positive integers and a configuration set is a finite set of such $(n+1)$-tuples. The configuration set $\mathrm{Con}(\mathfrak{g},\mathcal{E})$ records how the generators in $\mathfrak{g}$ multiply between sets in the partition $\mathcal{E}$.

Configurations are defined in terms of left-translations of G. Hence for any partition $\mathcal{E}$ and element $x \in G$ we have $\mathrm{Con}(\mathfrak{g},\mathcal{E}) = \mathrm{Con}(\mathfrak{g},\mathcal{E}')$, where $\mathcal{E}' = \{E_1x,\ldots,E_mx\}$ is the right-translate of $\mathcal{E}$ by x. This justifies the following remark that is sometimes useful when working with configurations.

Remark 1.2. Let $\mathrm{Con}(\mathfrak{g},\mathcal{E})$ be a configuration set for the group G and let $y \in G$ and $E_i \in \mathcal{E}$. Then it may be supposed that $y \in E_i$.

The configuration $C = (C_0, C_1, \ldots, C_n)$ may be described equivalently as a labelled tree. The tree has one vertex of degree n, labelled by C_0. Emanating from this vertex are edges labelled $1,\ldots,n$, and the other vertex of the ith edge is labelled C_i. When the generators are distinct, this tree is a subgraph of the Cayley graph of the finitely generated group $G = \langle g_1,\ldots,g_n\rangle$. The edge labels indicate which generator gives rise to the edge and the vertex labels show which set of the partition $\mathcal{E}$ the vertex belongs to.

From this perspective the configuration set $\mathrm{Con}(\mathfrak{g},\mathcal{E})$ is a set of rooted trees having height 1; see [6]. This finite set carries information about G and the present paper addresses the question of which properties of G can be recovered from such information.

2. Configurations and amenability

The present paper is motivated by a result from [5] that characterises amenable groups by their configuration sets. For completeness, this section summarises the main ideas.

The statement of the result involves the notion of the system of configuration equations corresponding to a configuration set $\mathrm{Con}(\mathfrak{g},\mathcal{E})$. There are $|\mathrm{Con}(\mathfrak{g},\mathcal{E})|$ variables in the system of configuration equations. They are denoted by f_C, where $C \in \mathrm{Con}(\mathfrak{g},\mathcal{E})$. These are $|\mathcal{E}||\mathfrak{g}| = mn$ equations in the system.

Definition 2.1.

(i) The *configuration equations* corresponding to the configuration set $\mathrm{Con}(\mathfrak{g},\mathcal{E})$ are the equations

$$\sum \{f_C : C \in \mathrm{Con}(\mathfrak{g},\mathcal{E}), C_j = i\} = \sum \{f_C : C \in \mathrm{Con}(\mathfrak{g},\mathcal{E}), C_0 = i\}, \tag{2.1}$$

where $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,n\}$. This system of equations will be denoted $\mathrm{Eq}(\mathfrak{g},\mathcal{E})$.

(ii) A solution to $\mathrm{Eq}(\mathfrak{g},\mathcal{E})$ satisfying $\sum\{f_C : C \in \mathrm{Con}(\mathfrak{g},\mathcal{E})\} = 1$ and $f_C \geq 0$ for all $C \in \mathrm{Con}(\mathfrak{g},\mathcal{E})$ will be called a *normalised solution* of the system.

Amenability is characterised in [5] as follows.

Proposition 2.2. *Let G be a finitely generated group. Then G is amenable if and only if* $\mathrm{Eq}(\mathfrak{g},\mathcal{E})$ *has a normalised solution for every configuration set* $\mathrm{Con}(\mathfrak{g},\mathcal{E})$.

Thus if G is not amenable, then there is a configuration set that witnesses the fact. This link between amenability and normalised solutions of the configuration equations is seen via a refinement of the partition $\mathcal{E}$.

Definition 2.3. Let C be a configuration in $\mathrm{Con}(\mathfrak{g},\mathcal{E})$. Call $x_0 \in G$ a *base point* of C if there is a sequence of elements $x_0, x_1, \dots, x_n$ such that $x_i = g_i\, x_0$ for each $i \in \{1,2,\dots,n\}$ and $x_i \in E_{C_i}$ for each $i \in \{0,1,\dots,n\}$. In this case $x_1,\dots,x_n$ are called *branch points* of C. Define

$$x_0(C) = \{x \in G : x \text{ is a base point of } C\}.$$

That C is a configuration in $\mathrm{Con}(\mathfrak{g},\mathcal{E})$ means that $x_0(C)$ is not empty. Note that for each $E_i \in \mathcal{E}$ we have

$$E_i = \bigcup \{x_0(C) : C \in \mathrm{Con}(\mathfrak{g},\mathcal{E}), C_0 = i\}. \tag{2.2}$$

Thus $\{x_0(C) : C \in \mathrm{Con}(\mathfrak{g},\mathcal{E})\}$ is a refinement of $\mathcal{E}$. Moreover, for each $g_j \in \mathfrak{g}$,

$$g_j^{-1}E_i = \bigcup \{x_0(C) : C \in \mathrm{Con}(\mathfrak{g},\mathcal{E}), C_j = i\}. \tag{2.3}$$

Let M be an invariant mean for G and set $f_C = M(\chi_{x_0(C)})$, for $C \in \mathrm{Con}(\mathfrak{g},\mathcal{E})$. Since M is a mean we have $\sum\{f_C : C \in \mathrm{Con}(\mathfrak{g},\mathcal{E})\} = 1$. Since M is translation invariant, (2.2) and (2.3) imply that

$$\sum\{f_C : C \in \mathrm{Con}(\mathfrak{g},\mathcal{E}), C_j = i\} = \sum\{f_C : C \in \mathrm{Con}(\mathfrak{g},\mathcal{E}), C_0 = i\}$$

because the left hand side equals $M(\chi_{g_j^{-1}E_i})$ and the right hand side equals $M(\chi_{E_i})$. Hence $\{f_C : C \in \mathrm{Con}(\mathfrak{g},\mathcal{E})\}$ is a normalised solution of the configuration equations.

The converse is proved in [5, Proposition 2.4]. Briefly, a normalised solution, f, to $\mathrm{Eq}(\mathfrak{g},\mathcal{E})$ is used to define a probability measure, m_f, on G such that

$$m_f(g_j^{-1}E_i) = m_f(E_i) \quad \text{for each } i,j. \tag{2.4}$$

An invariant mean on $\ell^\infty(G)$ is then produced as a weak*-limit of these probability measures.

Amenability may also be characterised by the non-existence of paradoxical decompositions; see [1]. We consider next how paradoxical decompositions are related to configurations. Let $\mathcal{E} = \{E_1, E_2, \dots, E_r; E_{r+1}, \dots, E_{r+s}\}$ be a partition of a group G and $\mathfrak{g} = (g_1, g_2, \dots, g_r; g_{r+1}, \dots, g_{r+s})$ be a sequence in G giving rise to a paradoxical decomposition, so that $\{g_1^{-1}E_1, g_2^{-1}E_2, \dots, g_r^{-1}E_r\}$

and $\{g_{r+1}^{-1}E_{r+1},\ldots,g_{r+s}^{-1}E_{r+s}\}$ are partitions of G. Suppose that $\mathfrak{g}$ generates G.

The fact that $\{g_1^{-1}E_1, g_2^{-1}E_2,\ldots,g_r^{-1}E_r\}$ is a partition of G implies that for each configuration $C\in\mathrm{Con}(\mathfrak{g},\mathcal{E})$ there is exactly one $j\in\{1,2,\ldots,r\}$ such that $C_j=j$. Hence

$$\sum\{f_C : C\in\mathrm{Con}(\mathfrak{g},\mathcal{E})\}=\sum_{j=1}^{r}\{f_C : C\in\mathrm{Con}(\mathfrak{g},\mathcal{E}),\, C_j=j\}\,.$$

Substituting the appropriate configuration equation from (2.1) into the right hand expression yields

$$\sum_{j=1}^{r}\{f_C : C\in\mathrm{Con}(\mathfrak{g},\mathcal{E}),\, C_0=j\}=\sum\{f_C : C\in\mathrm{Con}(\mathfrak{g},\mathcal{E})\}\,. \tag{2.5}$$

Similarly, that $\{g_{r+1}^{-1}E_{r+1},\ldots,g_{r+s}^{-1}E_{r+s}\}$ is a partition and the configuration equations imply that

$$\sum_{j=r+1}^{r+s}\{f_C : C\in\mathrm{Con}(\mathfrak{g},\mathcal{E}),\, C_0=j\}=\sum\{f_C : C\in\mathrm{Con}(\mathfrak{g},\mathcal{E})\}\,. \tag{2.6}$$

On the other hand, for each configuration C_0 takes exactly one of the values in $\{1,\ldots,r,r+1,\ldots,r+s\}$ and so

$$\sum_{j=1}^{r+s}\{f_C : C\in\mathrm{Con}(\mathfrak{g},\mathcal{E}),\, C_0=j\}=\sum\{f_C : C\in\mathrm{Con}(\mathfrak{g},\mathcal{E})\}\,. \tag{2.7}$$

The three equations (2.5), (2.6) and (2.7) are inconsistent with the existence of a normalised solution to the configuration equations.

It is not shown in [5] how to obtain a paradoxical decomposition directly from a configuration set for which the corresponding configuration equations do not have a normalised solution. This seems to be a more difficult problem than its converse. The partition giving rise to the configuration set need not itself be paradoxical and yet a paradoxical decomposition must be constructed from it.

The criterion for amenability is weakened in [7], where it is shown that for G to be amenable it suffices that there exist a non-zero (possibly discontinuous) translation-invariant functional on $L^\infty(G)$. It may be, therefore, that non-amenable groups have configuration equations for which there is no non-zero solution, let alone a normalised one.

Question 2.4. Can ‘normalised solution’ be replaced by ‘non-zero solution’ in the statement of Proposition 2.2?

Configuration sets for which the configuration equations have no normalised, or non-zero, solutions may provide a useful tool for investigating non-amenable groups because they allow a comparison of the ways in which groups fail to be amenable. Two groups would be non-amenable for different reasons if they have different systems of configuration equations which don't have normalised solutions. Configuration sets may provide a finer invariant for non-amenable groups than the Tarski number.

3. Finiteness properties of groups

A paradoxical decomposition of a group, G, shows that G is infinite in a very strong sense: G is put into one-to-one correspondence with multiple copies of itself by a function which is a piecewise translation. Amenability, that is, the impossibility of such a paradoxical decomposition, is thus a finiteness condition on G. Other characterisations of amenability such as existence of an invariant mean, weak containment of the trivial representation in the regular representation and cohomological characterisations [2], also support the view that it is a finiteness condition.

A single configuration set $\mathrm{Con}(\mathfrak{g}, \mathcal{E})$ will show that $G = \langle \mathfrak{g} \rangle$ is not amenable. It is not clear to us whether the same is true of the property of being finite.

Question 3.1. Is there an infinite finitely generated group G such that for every $\mathrm{Con}(\mathfrak{g}, \mathcal{E}) \in \mathrm{Con}(G)$ there is a finite group, F, with $\mathrm{Con}(\mathfrak{g}, \mathcal{E}) \in \mathrm{Con}(F)$?

Should there be such a group, no single configuration set of the group will show that it is infinite. The group would necessarily be amenable. On the other hand, $\mathrm{Con}(G) \neq \mathrm{Con}(F)$ for any fixed finite group F because, G being infinite, there is a partition $\mathcal{E}$ of G with $|\mathcal{E}| > |F|$. Then, whatever the generating set $\mathfrak{g}$, $\mathrm{Con}(\mathfrak{g}, \mathcal{E}) \notin \mathrm{Con}(F)$. The set of all configuration sets, $\mathrm{Con}(G)$, thus shows that G is not finite.

We now see that various finiteness properties of groups can be characterised by configurations. In each case, a single configuration set $\mathrm{Con}(\mathfrak{g}, \mathcal{E})$ *does* show that $G = \langle \mathfrak{g} \rangle$ does not have the finiteness property.

The first such condition is that of being *periodic*, that is, every element of G has finite order. There are finitely generated, infinite, periodic groups [4, p. 35].

Proposition 3.2. *Let G be a finitely generated group having an element of infinite order. Then there is a partition, $\mathcal{E}$, of G and a generating set $\mathfrak{g} = (g_1, \dots, g_n)$ such that* $\mathrm{Con}(\mathfrak{g}, \mathcal{E})$ *is not a configuration set of any periodic group.*

Proof. Let $g_1 \in G$ have infinite order. Choose further elements $g_2, \dots, g_n$ such that $G = \langle g_1, \dots, g_n \rangle$. Put $E_1 = \{g_1^n : n > 0\}$ and $E_2 = G \setminus E_1$.

Then $\mathrm{Con}(\mathfrak{g},\mathcal{E})$ consists of configurations of three types,

$$\begin{aligned} A &= (2,2,A_2,\ldots), \\ B &= (2,1,B_2,\ldots), \\ \text{and} \quad C &= (1,1,C_2,\ldots). \end{aligned}$$

There are no configurations of the form $(1,2,\ldots)$ because $g_1\,E_1 \subset E_1$.

Let $H = \langle h_1,\ldots,h_n\rangle$ be a group and $\mathcal{F} = \{F_1,F_2\}$ be a partition of H so that $\mathrm{Con}(\mathfrak{g},\mathcal{E}) = \mathrm{Con}(\mathfrak{h},\mathcal{F})$. Since $\mathrm{Con}(\mathfrak{h},\mathcal{F})$ contains a configuration of the form $(2,1,b_2,\ldots)$, there is $y \in H$ such that $y \in F_2$ and $h_1\,y \in F_1$. Since $\mathrm{Con}(\mathfrak{h},\mathcal{F})$ does not contain a configuration of the form $(1,2,\ldots)$, $h_1^n\,y \in F_1$ for all $n \geq 1$. Hence $h_1^n\,y \neq y$ for all $n \geq 1$. Therefore $h_1^n \neq e$ for all $n \geq 1$ and h_1 has infinite order. □

The second finiteness condition is a special case of Theorem 5.1 and a weaker version of Proposition 6.4 but the proof is much shorter in this case.

Proposition 3.3. *Let G be a finitely generated non-abelian group. Then there is a partition, $\mathcal{E}$, of G and a generating set $\{g_1,g_2,\ldots,g_n\}$ such that the corresponding configuration set cannot arise from an abelian group, that is, configurations show that a group is not abelian.*

Proof. Let $g_1,g_2 \in G$ such that $g_1g_2 \neq g_2g_1$. Choose further elements $g_3,\ldots,g_n$ so that $G = \langle g_1,g_2,\ldots,g_n\rangle$. Put $E_1 = \{e\}$, $E_2 = \{g_1\}$, $E_3 = \{g_2\}$, $E_4 = \{g_1g_2\}$, $E_5 = \{g_2g_1\}$ and $E_6 = G\backslash\{e_1\,g_1,g_2,g_1g_2,g_2g_1\}$.

Since E_1, E_2 and E_3 are singletons, there are unique configurations,

$$\begin{aligned} A &= (1,2,3,A_3,\ldots,A_n) \quad \text{with } A_0 = 1, \\ B &= (2,B_1,4,B_3,\ldots,B_n) \quad \text{with } B_0 = 2, \\ \text{and} \quad C &= (3,5,C_2,C_3,\ldots,C_n) \quad \text{with } C_0 = 3. \end{aligned}$$

Let $H = \langle h_1,\ldots,h_n\rangle$ be a group and $\mathcal{F} = \{F_1,F_2,\ldots,F_6\}$ be a partition of H such that $\mathrm{Con}(\mathfrak{g},\mathcal{E}) = \mathrm{Con}(\mathfrak{h},\mathcal{F})$, where $\mathfrak{g} = (g_1,\ldots,g_n)$ and $\mathfrak{h} = (h_1,\ldots,h_n)$. Let $x \in F_1$. Then $h_1x \in F_2$ and $h_2x \in F_3$ because $A = (1,2,3,\ldots)$ is the unique configuration with $A_0 = 1$. Also, $h_2h_1x \in F_4$ because $B = (2,B_1,4,\ldots)$ is the unique configuration with $B_0 = 2$ and $h_1h_2x \in F_5$ because $C = (3,5,C_2,\ldots)$ is the unique configuration with $C_0 = 3$. Hence $h_1\,h_2x \neq h_2\,h_1x$. Therefore $h_1\,h_2 \neq h_2\,h_1$ and H is not abelian. □

Question 3.4. Which other finiteness conditions can be characterised by configuration sets?

4. Groups with the same configuration sets

It is seen in the above examples that configuration sets can distinguish group properties such as being infinite or non-amenable. The remaining sections of the paper begin to examine the question of what it means for two groups to have the same configuration sets. Recall from Definition 1.1 that $\mathrm{Con}(G)$ denotes the set of all configuration sets for G resulting from all finite generating sets and all finite partitions. We begin with some further terminology and notation.

Definition 4.1.

(i) The finitely generated group G is *configuration contained* in the finitely generated group H, written $G \ll H$, if

$$\mathrm{Con}(G) \subset \mathrm{Con}(H).$$

(ii) The groups G and H are *configuration equivalent*, written $G \approx H$, if

$$\mathrm{Con}(G) = \mathrm{Con}(H).$$

If G can be generated by a set of n elements, then it has a configuration set of $(n+1)$-tuples. Hence, if $\mathrm{Con}(G) \subset \mathrm{Con}(H)$ for some H, then H also has a configuration set of $(n+1)$-tuples and it follows that H has a generating set of n elements. This proves the following assertion.

Proposition 4.2. *Let G and H be finitely generated groups with $G \ll H$. If n is the minimum number of elements needed to generate H, then at least n elements are required to generate G.*

The next result and Propositions 6.1 and 6.4 show that in some classes of groups configuration equivalence implies isomorphism.

Proposition 4.3. *Let G be a finite group and suppose that $H \approx G$. Then H is isomorphic to G.*

Proof. The singleton sets partition H into $|H|$ sets. It follows that G has a partition into $|H|$ sets. Hence $|H| \leq |G|$. Similarly $|G| \leq |H|$, so $|G| = |H|$.

Suppose that $|G| = n$ and let $G = \{x_1, \dots, x_n\}$. Then $\mathcal{E} = \{\{x_j\} : 1 \leq j \leq n\}$ is a partition of G and $\mathfrak{g} = (x_1, x_2, \dots, x_n)$ is a generating sequence of G. Clearly the n configurations in $\mathrm{Con}(\mathfrak{g}, \mathcal{E})$ are the rows of the multiplication table for G. There is a generating set $\mathfrak{h}$ for H and partition $\mathcal{F}$ of H such that $\mathrm{Con}(\mathfrak{g}, \mathcal{E}) = \mathrm{Con}(\mathfrak{h}, \mathcal{F})$. The correspondence between $\mathfrak{g}$ and $\mathfrak{h}$ is an isomorphism between G and H. □

The isomorphism results rely on extra properties of groups in the class being considered. It may be that configuration equivalence always implies isomorphism but that seems unlikely.

QUESTION 4.4. Suppose that G_1 and G_2 are finitely generated and that $G_1 \approx G_2$.

(i) Is G_1 isomorphic to G_2?
(ii) Do there exist normal subgroups N_i of G_i $(i = 1, 2)$ such that $G_1 \cong G_2/N_2$ and $G_2 \cong G_1/N_1$?
(iii) Can (i) or (ii) be answered affirmatively if G_1 and G_2 are finitely presented?

Part (ii) of the question is asked as a less optimistic version of part (i) and also because in some circumstances an affirmative answer to part (ii) would imply isomorphism of G_1 and G_2; see Question 5.3. It should not be taken to suggest that G/N is configuration contained in G. Proposition 4.2 shows that this is not possible if G/N can be generated by fewer elements than G.

5. Configurations of groups satisfying semigroup laws

Proposition 3.3 shows that if a group G has the same configuration sets as an abelian group, then G is abelian. The condition for G to be abelian, namely $xy = yx$ for every x and y in G, is an example of a semigroup law. The proposition will be extended to all semigroup laws in this section.

First, we recall the definition of a semigroup law. Let S be the free semigroup on the set $\{x_1, \ldots, x_n\}$, where n is a positive integer. Suppose that $\mu = \mu(x_1, \ldots, x_n)$ and $\nu = \nu(x_1, \ldots, x_n)$ are two elements in S. We say that $\mu = \nu$ is a semigroup law in a group G if for every n-tuple $(g_1, \ldots, g_n)$ of elements of G, we have $\mu(g_1, \ldots, g_n) = \nu(g_1, \ldots, g_n)$.

The main result of this section is:

THEOREM 5.1. *Let G_1 and G_2 be two finitely generated groups with $G_2 \ll G_1$ and suppose that G_1 satisfies the semigroup law $u(x_1, \ldots, x_n) = \nu(x_1, \ldots, x_n)$. Then G_2 satisfies the same law.*

In [3] Neumann and Taylor proved that there exist sequences of words λ_n and ρ_n in the free semigroup on a countable set such that a group G is nilpotent of class c if and only if c is the least positive integer such that G satisfies the semigroup law $\lambda_c = \rho_c$. As a consequence of Theorem 5.1 and this result of Neumann and Taylor we have:

COROLLARY 5.2. *Let G_1 and G_2 be two finitely generated groups with $G_2 \ll G_1$ and suppose that G_1 is nilpotent of class c. Then G_2 is nilpotent of class c.*

Proof of Theorem 5.1. Let

$$\mu(x_1, \ldots, x_n) = x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k} \text{ and } \nu(x_1, \ldots, x_n) = x_{j_1}^{\beta_1} \cdots x_{j_\ell}^{\beta_\ell},$$

where α_i and β_j are positive integers, be two words in the free semigroup and suppose that G_2 does not satisfy the semigroup law $\mu = \nu$. Then there exist n elements $g_1, \ldots, g_n \in G_2$ such that

$$\mu(g_1, \ldots, g_n) \neq \nu(g_1, \ldots, g_n). \tag{5.1}$$

We will show that there exist n elements $h_1, \ldots, h_n \in G_1$ such that $\mu(h_1, \ldots, h_n) \neq \nu(h_1, \ldots, h_n)$ and this will complete the proof.

Set $\alpha = \alpha_1 + \cdots + \alpha_k$. Then each integer $\xi \in \{1, 2, \ldots, \alpha\}$ may be written uniquely as $\xi = r + \sum_{a=s+1}^{k} \alpha_a$, where $r \in \{1, \ldots, \alpha_s\}$ and, if $\xi \leq \alpha_k$, s is taken to be k and the sum to be empty. Define

$$y_\xi = g_{i_s}^r g_{i_{s+1}}^{\alpha_{s+1}} \cdots g_{i_k}^{\alpha_k} \text{ for each } \xi \in \{1, 2, \ldots, \alpha\}.$$

For example, $y_1 = g_{i_k}$ and $y_{\alpha_k+2} = g_{i_{k-1}}^2 g_{i_k}^{\alpha_k}$. Then

$$g_{i_s} y_\xi = y_{\xi+1} \text{ if } \sum_{a=s+1}^{k} \alpha_a \leq \xi < \sum_{a=s}^{k} \alpha_a$$

and so $1, y_1, \ldots, y_\alpha$ is a path of length α in the Cayley graph of G_2 from 1 to $g_{i_1}^{\alpha_1} \cdots g_{i_k}^{\alpha_k}$. Put $Y = \{1, y_1, \ldots, y_\alpha\}$—the set of vertices visited by this path.

Similarly, set $\beta = \beta_1 + \cdots + \beta_\ell$ and define

$$z_\xi = g_{j_s}^r g_{j_{s+1}}^{\beta_{s+1}} \cdots g_{j_\ell}^{\beta_\ell} \text{ for each } \xi \in \{1, 2, \ldots, \beta\},$$

where $\xi = r + \sum_{j=s+1}^{\ell} \beta_j$ and $r \in \{1, \ldots, \beta_s\}$. Then $1, z_1, \ldots, z_\beta$ is a path of length β in the Cayley graph of G_2 from 1 to $g_{j_1}^{\beta_1} \cdots g_{j_\ell}^{\beta_\ell}$. Put $Z = \{1, z_1, \ldots, z_\beta\}$.

Let $w_1 = 1, w_2, \ldots, w_{m-1}$ be an enumeration of the elements of $Y \cup Z$, where $m = |Y \cup Z| + 1$. Define a partition of G_2 by $E_\zeta = \{w_\zeta\}$, $\zeta \in \{1, 2, \ldots, m-1\}$ and $E_m = G_2 \setminus (Y \cup Z)$. Two properties of the configuration set $\mathrm{Con}(\mathfrak{g}, \mathcal{E})$ will be needed for the argument to follow:

(i) since each E_ζ is a singleton when $\zeta \in \{1, 2, \ldots, m-1\}$, for each such ζ there is a unique $C^{(\zeta)} \in \mathrm{Con}(\mathfrak{g}, \mathcal{E})$ with $x_0(C^{(\zeta)}) \subset E_\zeta$; and

(ii) by (5.1) we have $g_{i_1}^{\alpha_1} \cdots g_{i_k}^{\alpha_k} \neq g_{j_1}^{\beta_1} \cdots g_{j_\ell}^{\beta_\ell}$ and it follows that there are distinct $a, b \in \{1, 2, \ldots, m-1\}$ such that $E_a = \{g_{i_1}^{\alpha_1} \cdots g_{i_k}^{\alpha_k}\}$ and $E_b = \{g_{j_1}^{\beta_1} \cdots g_{j_\ell}^{\beta_\ell}\}$.

Since $G_2 \ll G_1$, there is a generating set $\{h_1, \ldots, h_n\}$ of G_1 and a partition $\mathcal{F} = \{F_1, \ldots, F_m\}$ of G_1 such that $\mathrm{Con}(\mathfrak{h}, \mathcal{F}) = \mathrm{Con}(\mathfrak{g}, \mathcal{E})$. We will show that $h_{i_1}^{\alpha_1} \cdots h_{i_k}^{\alpha_k} \neq h_{j_1}^{\beta_1} \cdots h_{j_\ell}^{\beta_\ell}$, so that G_1 does not satisfy the law $\mu = \nu$.

As in Remark 1.2, it may be assumed that $1 \in F_1$. Then, by (i), there is a unique configuration $C^{(1)}$ such that $x_0(C^{(1)}) \subset F_1$. Hence h_{i_k} belongs to the set F_i, where $i = C_{i_k}^{(1)}$. By (i) again, there is a unique configuration $C^{(i)}$ such that $x_0(C^{(i)}) \subset F_i$ and so (in the case when $\alpha_k \geq 2$) $h_{i_k}^2 \in F_j$, where

$j = C^{(i)}_{i_k}$. Continuing in this way we find that 1, h_{i_k}, ..., $h^r_{i_s} h^{\alpha_{s+1}}_{i_{s+1}} \cdots h^{\alpha_k}_{i_k}$, ..., $h^{\alpha_1}_{i_1} \cdots h^{\alpha_k}_{i_k}$ is a path of length α in G_1 that starts in F_1 and ends in F_a, where a is the integer defined in (ii). Similarly, 1, h_{j_ℓ}, ..., $h^r_{j_s} h^{\beta_{s+1}}_{j_{s+1}} \cdots h^{\beta_\ell}_{j_\ell}$, ..., $h^{\beta_1}_{j_1} \cdots h^{\beta_\ell}_{j_\ell}$ is a path that starts in F_1 and ends in F_b. Since, by (ii), $F_a \neq F_b$, we have $F_a \cap F_b = \emptyset$ and so $h^{\alpha_1}_{i_1} \cdots h^{\alpha_k}_{i_k} \neq h^{\beta_1}_{j_1} \cdots h^{\beta_\ell}_{j_\ell}$, as required. □

It is a natural question whether a stronger result can be established.

Question 5.3. Let G_1 and G_2 be finitely generated groups and G_1 be nilpotent. If $G_1 \approx G_2$, does it then follow that $G_1 \cong G_2$?

If the answer to part (ii) of Question 4.4 is positive, then the answer of Question 5.3 is also positive. For by Corollary 5.2, G_2 is also nilpotent, and we know that every finitely generated nilpotent group is finitely presented, hence by hypothesis there exist normal subgroups $N_i \triangleleft G_i$, for $i = 1, 2$, such that $G_1 \cong G_2/N_2$ and $G_2 \cong G_1/N_1$. But G_1 and G_2 are finitely generated residually finite and so they are hopfian (see page 40 of [4]). It follows that $G_1 \cong G_2$.

That the answer to Question 5.3 is positive when G_1 is abelian is shown in the next section.

6. Configuration equivalence and isomorphism

It is shown in this section that configuration equivalence implies isomorphism in the two extreme cases when the groups are free or abelian. In the case of free groups an even stronger result obtains.

Proposition 6.1. *Let $\mathbb{F}_n$ be the free group of rank $n > 0$. If H is a finitely generated group such that $H \ll \mathbb{F}_n$, then $H \cong \mathbb{F}_n$.*

Proof. Suppose that $f_1, \ldots, f_n$ are free generators for $\mathbb{F}_n$ and consider the following subsets of $\mathbb{F}_n$:

$$\begin{aligned} E_1 &= \{1\}, \\ E_{2k} &= \{\text{reduced words starting with } f_k\}, \\ \text{and} \quad E_{2k+1} &= \{\text{reduced words starting with } f_k^{-1}\}, \end{aligned}$$

where $k = 1,\ldots, n$. Clearly $\mathcal{E} = \{E_i : 1 \leq i \leq 2n+1\}$ is a partition for $\mathbb{F}_n$. Let $\mathfrak{f} = (f_1, \ldots, f_n)$. If $x \in \mathbb{F}_n$ and the reduced word for x does not begin with f_k^{-1}, then $f_k x \in E_{2k}$. On the other hand, if the reduced word for x does begin with f_k^{-1}, then $f_k x$ may belong to any E_i except E_{2k}. It follows that $\mathrm{Con}(\mathfrak{f}, \mathcal{E})$ consists of the $1 + n + 2n^2$ configurations:

(1) $(1, 2, 4, \ldots, 2n)$;
(2) $(2k, 2, 4, \ldots, 2n)$, where $k \in \{1, \ldots, n\}$; and

(3) $(2k+1, 2, \ldots, 2k-2, i, 2k+2, \ldots, 2n)$, where $k \in \{1, \ldots, n\}$ and $i \in \{1, 2, \ldots, 2n, 2n+1\} \setminus \{2k\}$.

By hypothesis there exist a generating sequence $\mathfrak{h} = (h_1, \ldots, h_n)$ and a partition $\mathcal{D} = \{D_i \mid 1 \leq i \leq 2n+1\}$ of H such that $\mathrm{Con}(\mathfrak{h}, \mathcal{D}) = \mathrm{Con}(\mathfrak{f}, \mathcal{E})$. We prove that there is no non-trivial relation between $h_1, \ldots, h_n$ to complete the proof.

The following properties of multiplication by h_i's transfer from $\mathbb{F}_n$ to H.

(P1) If α is a positive integer and $x \in H \setminus D_{2k+1}$, where $k \in \{1, \ldots, n\}$, then $h_k^\alpha x \in D_{2k}$. (For this, observe that the only configuration that does not have $2k$ in the kth position is the one that has $2k+1$ in the 0th position.)

(P2) If α is a positive integer and $x \in H \setminus D_{2k}$, where $k \in \{1, \ldots, n\}$, then $h_k^{-\alpha} x \in D_{2k+1}$. (For if $h_k^{-\alpha} x \notin D_{2k+1}$, then, by (P1), $x = h_k^\alpha (h_k^{-\alpha} x) \in D_{2k}$, which is impossible.)

Let $X = h_{i_1}^{\alpha_1} \cdots h_{i_t}^{\alpha_t}$, where $i_k \neq i_{k+1}$ for $k = 1, \ldots, t-1$ and the α_k's are non-zero integers. We must show that $X \neq 1$. By Remark 1.2 it may be supposed that $1 \in D_1$. Then to show that $X \neq 1$ it suffices to show that $X \in D_{2i_1}$ if $\alpha_1 > 0$ and $X \in D_{2i_1+1}$ if $\alpha_1 < 0$. This may be done by induction on t. If $t = 1$, then $X = h_{i_1}^{\alpha_1}$. Since $1 \notin D_s$ for all $s > 1$, (P1) shows that $X \in D_{2i_1}$ if $\alpha_1 > 0$ and (P2) shows that $X \in D_{2i_1+1}$ if $\alpha_1 < 0$. Now suppose inductively that $t > 1$ and the result is true for $t-1$. Thus $Y = h_{i_2}^{\alpha_2} \cdots h_{i_t}^{\alpha_t} \in D_{2i_2}$ if $\alpha_2 > 0$ and $Y \in D_{2i_2+1}$ if $\alpha_2 < 0$. In either case we have $Y \notin D_{2i_1+1}$ because $i_1 \neq i_2$ and so, if $\alpha_1 > 0$, $X = h_{i_1}^{\alpha_1} Y \in D_{2i_1}$ by (P1). Also in either case we have $Y \notin D_{2i_1}$ because $i_1 \neq i_2$ and so, if $\alpha_1 < 0$, $X = h_{i_1}^{\alpha_1} Y \in D_{2i_1+1}$ by (P2). This completes the proof. □

The next couple of lemmas are needed for the proof that configuration equivalence of abelian groups implies isomorphism. However Lemma 6.3 could be used to study configurations of any residually finite group.

Lemma 6.2. *Let G be a finitely generated group and G/N be a quotient group. Suppose that $\{g_1 N, \ldots, g_k N\}$ generates G/N. Then there are elements $h_1, \ldots, h_\ell$ in N such that $G = \langle g_1, \ldots, g_k, h_1, \ldots, h_\ell \rangle$.*

Proof. Let $x_1, \ldots, x_\ell$ be a generating set for G. Then, since N is normal, there is for each $i \in \{1, 2, \ldots, \ell\}$ an element $y_i \in \langle g_1, \ldots, g_k \rangle$ such that $x_i y_i^{-1} \in N$. Put $h_i = x_i y_i^{-1}$. Then $G = \langle g_1, \ldots, g_k, h_1, \ldots, h_\ell \rangle$. □

Lemma 6.3. *Let G_1 and G_2 be finitely generated groups with $G_2 \ll G_1$ and suppose that G_1 has a normal subgroup, N_1, with finite index. Then G_2 has a normal subgroup N_2 with $G_2/N_2 \cong G_1/N_1$.*

Proof. Let the index of N_1 in G_1 be m and let $\mathcal{E} = \{N_1, x_2 N_1, \ldots, x_m N_1\}$ be the partition of G_1 into N_1-cosets. Denote by π_j the permutation of

$\{1, 2, \ldots, m\}$ such that $x_j(x_i N_1) = x_{\pi_j(i)} N_1$. Then $\{\pi_1 = \mathrm{id}, \pi_2, \ldots, \pi_m\}$ is a group of permutations isomorphic to G_1/N_1.

By Lemma 6.2, there are elements $n_1, \ldots, n_\ell$ in N_1 such that

$$\mathfrak{g} = (x_2, \ldots, x_m; n_1, \ldots, n_\ell)$$

is an ordered generating set for G_1. Then $\mathrm{Con}(\mathfrak{g}, \mathcal{E})$ consists of m configurations

$$C^{(i)} = (i, \pi_2(i), \ldots, \pi_m(i); i, \ldots, i), \quad i \in \{1, 2, \ldots m\}.$$

Since $G_2 \ll G_1$, there are a partition $\mathcal{F} = \{F_1, \ldots, F_m\}$ and a generating set $\mathfrak{h} = (y_2, \ldots, y_m; k_1, \ldots, k_\ell)$ of G_2 such that $\mathrm{Con}(\mathfrak{h}, \mathcal{F}) = \mathrm{Con}(\mathfrak{g}, \mathcal{E})$. These configurations imply that for each j we have $y_j F_i \subset F_{\pi_j(i)}, \quad i \in \{1, \ldots, m\}$. Since $\{F_1, \ldots, F_m\}$ is a partition of G_2, it follows that in fact for each $j \in \{1, \ldots, m\}$

$$y_j F_i = F_{\pi_j(i)}, \quad i \in \{1, \ldots, m\}.$$

Similarly, the configurations imply that for each $j \in \{1, \ldots, \ell\}$

$$k_j F_i = F_i, \quad i \in \{1, \ldots, m\}.$$

Thus left multiplication by each of $y_2, \ldots, y_m$; $k_1, \ldots, k_\ell$ permutes $\{F_1, \ldots, F_m\}$. Since these elements generate G_2, we thus obtain a permutation representation, φ, of G_2 on $\{1, \ldots, m\}$. Clearly $\varphi(G_2) = \{\pi_1, \pi_2, \ldots, \pi_m\}$. Hence, setting $N_2 = \ker(\varphi)$, we obtain $G_2/N_2 \cong G_1/N_1$. □

Proposition 6.4. *Let G_1 and G_2 be finitely generated abelian groups such that $G_1 \approx G_2$. Then $G_1 \cong G_2$.*

Proof. We have $G_1 \cong \mathbb{Z}^r \times F_1$ and $G_2 \cong \mathbb{Z}^s \times F_2$, for some finite abelian groups F_1, F_2 and some integers r and s. We show that $r = s$ and $F_1 \cong F_2$, from which the result follows.

Without loss of generality, it may be assumed that $r \geq s$. We may also assume that $G_1 = \mathbb{Z}^r \times F_1$ and $G_2 = \mathbb{Z}^s \times F_2$. Choose $n \in \mathbb{Z}$ so that $\gcd(n, |F_2|) = 1$. Put $H_1 = (n\mathbb{Z})^r \times F_1$, a subgroup with finite index in G_1. By Lemma 6.3, there is a normal subgroup H_2 of G_2 such that

$$G_2/H_2 \cong G_1/H_1 \cong (\mathbb{Z}/n\mathbb{Z})^r.$$

Since $\gcd(n, |F_2|) = 1$, it follows that $F_2 \leq H_2$ and so $\mathbb{Z}^s/(\mathbb{Z}^s \cap H_2) \cong (\mathbb{Z}/n\mathbb{Z})^r$. Since $\mathbb{Z}^s/(\mathbb{Z}^s \cap H_2)$ is generated by s elements and $(\mathbb{Z}/n\mathbb{Z})^r$ can be generated by no fewer than r elements, we have $r = s$.

That $F_1 \cong F_2$ may be shown by a similar method. As before, choose $n \in \mathbb{Z}$ so that $\gcd(n, |F_2|) = 1$ and this time put $H_1 = (n\mathbb{Z})^r \times \{0\}$. By Lemma 6.3, there is a normal subgroup H_2 of G_2 such that

$$G_2/H_2 \cong G_1/H_1 \cong (\mathbb{Z}/n\mathbb{Z})^r \times F_1.$$

Let $\varphi : G_2 \twoheadrightarrow (\mathbb{Z}/n\mathbb{Z})^r \times F_1$ be the surjective homomorphism induced by this isomorphism. Since $\gcd(n, |F_2|) = 1$, it follows that $\varphi^{-1}((\mathbb{Z}/n\mathbb{Z})^r) \cap F_2 = \{0\}$

and so φ induces a surjective homomorphism $\tilde{\varphi} : F_2 \twoheadrightarrow F_1$. By symmetry, there is also a surjective homomorphism $\tilde{\psi} : F_1 \twoheadrightarrow F_2$ and so, since F_1 and F_2 are finite abelian groups, $F_1 \cong F_2$. □

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