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UNIVALENT FUNCTIONS, HARDY SPACES AND SPACES OF DIRICHLET TYPE

ALBERT BAERNSTEIN II, DANIEL GIRELA, AND JOSÉ ÁNGEL PELÁEZ

ABSTRACT. We prove that for $p \in (0,\infty)$ an analytic univalent function in the unit disk belongs to the Hardy space H^p if and only if it belongs to the Dirichlet type space $\mathcal{D}^p_{p-1}$.

1. Introduction

For $0 < p < \infty$ and $\alpha > -1$, the (standard) weighted Bergman space A^p_α is the set of all analytic functions f in the unit disk Δ such that

$$\int_\Delta (1-|z|)^\alpha |f(z)|^p \, dA(z) < \infty,$$

where $dA(z) = dx\,dy = r\,dr\,d\theta$ is the Lebesgue area measure. The standard unweighted Bergman space A^p_0 is simply denoted by A^p.

For $0 < p < \infty$, the Dirichlet type space $\mathcal{D}^p_{p-1}$ consists of all functions analytic in Δ whose derivative belongs to A^p_{p-1}. The $\mathcal{D}^p_{p-1}$ spaces are closely related to the Hardy spaces. Indeed, a direct calculation with power series shows that $H^2 = \mathcal{D}^2_1$. A classical result of Littlewood and Paley [19] (see also [20]) asserts that

$$H^p \subset \mathcal{D}^p_{p-1}, \quad 2 \le p < \infty. \tag{1}$$

On the other hand, it is not difficult to show that

$$\mathcal{D}^p_{p-1} \subset H^p, \quad 0 < p \le 2. \tag{2}$$

The inclusion (1) can be proved by Riesz-Thorin interpolation. The same method gives (2) for $1 \le p \le 2$, since the inclusion $\mathcal{D}^1_0 \subset H^1$ is trivial. Vinogradov ([29, Lemma 1.4]) extended (2) to the range $0 < p \le 2$. However, we will see that

$$H^p \ne \mathcal{D}^p_{p-1}, \quad \text{if } p \ne 2. \tag{3}$$

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In this paper we are primarily interested in characterizing the univalent functions which belong to the spaces $\mathcal{D}^p_{p-1}$. Our main result asserts that, in spite of (3), for every $p \in (0, \infty)$ the univalent functions in $\mathcal{D}^p_{p-1}$ and those in H^p are the same. It is well known (see, e.g., Theorem 3.16 of [5]) that any univalent function belongs to H^p for all $p < 1/2$. Hence, bearing in mind (2), our result improves this as it implies that any univalent function belongs to $\mathcal{D}^p_{p-1}$ for any $p < 1/2$.

By a theorem of Hardy and Littlewood [16] (see also Theorem 5.6 of [5], or [30] for a simple proof), for every p, the Hardy space H^p is contained in the Bergman space A^{2p} and the exponent $2p$ cannot be improved. Using (2) we deduce that, if $0 < p \le 2$, then $\mathcal{D}^p_{p-1} \subset A^{2p}$. Actually, this is also true for $p > 2$. Thus,

$$\mathcal{D}^p_{p-1} \subset A^{2p}, \quad 0 < p < \infty. \tag{4}$$

This is a particular case of Theorem 2.1 of [3] and follows from the work of Flett [10] and [11]. In view of (1), (2) and (4), we have

$$\begin{aligned} &\mathcal{D}^p_{p-1} \subset H^p \subset A^{2p}, && 0 < p \le 2, \\ &H^p \subset \mathcal{D}^p_{p-1} \subset A^{2p}, && 2 \le p < \infty. \end{aligned} \tag{5}$$

It is natural to ask whether or not the univalent functions in A^{2p} and those in $\mathcal{D}^p_{p-1}$ (or equivalently in H^p) are the same. This is certainly true if $p < 1/2$ because any univalent function belongs $\mathcal{D}^p_{p-1}$ if p is in this range. However, we will show that for any $p \ge 1/2$ there exists a univalent function f which belongs to A^{2p} but not to H^p. Also, we shall obtain a sharp geometric condition on the image of the unit disc under a univalent function f in A^{2p} which implies that f belongs to H^p.

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2. Background

In this section we define all function spaces and classes which will be studied later, as well as certain concepts, and fix the notation.

Throughout the paper, the letter Ω will be used to denote a planar domain and $\partial\Omega$ its boundary. $\Delta = \{z : |z| < 1\}$ will stand for the unit disc in the complex plane $\mathbb{C}$, $\partial\Delta$ will be the unit circle, and $dA(z) = r dr d\theta = dx dy$ the Lebesgue area measure.

If f is a function which is analytic in Δ and $0 < r < 1$, we set

$$\begin{aligned} M_p(r,f) &= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{it})|^p\,dt\right)^{1/p}, && 0<p<\infty,\\ I_p(r,f) &= M_p^p(r,f), && 0<p<\infty,\\ M_\infty(r,f) &= \sup_{|z|=r} |f(z)|. \end{aligned}$$

For $0 < p \le \infty$, the Hardy space H^p is defined to be the set of all analytic functions f in the disc for which

$$\|f\|_{H^p} \stackrel{\text{def}}{=} \sup_{0<r<1} M_p(r,f) < \infty.$$

We refer the reader to [5] and [12] for the theory of Hardy spaces.

As noted at the beginning of this paper, if $0 < p < \infty$ and $\alpha > -1$, we let A^p_α denote the (standard) *weighted Bergman space*, that is, the set of analytic functions f in Δ such that

$$\int_\Delta (1-|z|)^\alpha |f(z)|^p\,dA(z) < \infty.$$

The standard unweighted Bergman space A^p_0 is simply denoted by A^p. More information about Bergman spaces can be found in the recently published books [18] and [7].

The *Bloch space* $\mathcal{B}$ consists of all analytic functions f in Δ with bounded invariant derivative:

$$\|f\|_{\mathcal{B}} \stackrel{\text{def}}{=} |f(0)| + \sup_{z\in\Delta} (1-|z|^2)\,|f'(z)| < \infty\,.$$

A classical source for Bloch functions is [1]; see also [23] or [26].

Finally, for $p > 0$ we let $\mathcal{D}^p_{p-1}$ denote the space of all functions f which are analytic in Δ and satisfy $f' \in A^p_{p-1}$. Hence, if f is analytic in Δ then $f \in \mathcal{D}^p_{p-1}$ if and only if

$$\int_\Delta (1-|z|)^{p-1}|f'(z)|^p\,dA(z) = 2\pi\int_0^1 r(1-r)^{p-1} M_p^p(r,f')\,dr < \infty.$$

As noted in the Introduction we have:

$$\begin{aligned} \mathcal{D}^p_{p-1} &\subset H^p, \quad 0<p\le 2,\\ H^p &\subset \mathcal{D}^p_{p-1}, \quad 2\le p<\infty, \end{aligned}$$

and these inclusions are strict if $p \ne 2$. The strictness can be seen in several ways:

(i) Using Proposition 2.1 of [3] we easily see that if f is given by a power series with Hadamard gaps,

$$f(z)=\sum_{k=1}^{\infty} a_k z^{n_k} \ (z\in\Delta) \text{ with } n_{k+1}\geq \lambda n_k \text{ for all } k \ (\lambda>1),$$

then, for every $p\in(0,\infty)$,

$$f\in\mathcal{D}^p_{p-1} \quad \Longleftrightarrow \quad \sum_{k=1}^{\infty}|a_k|^p<\infty.$$

Since for Hadamard gap series we have, for $0<p<\infty$,

$$f\in H^p \quad \Longleftrightarrow \quad \sum_{k=1}^{\infty}|a_k|^2<\infty,$$

we immediately deduce that $\mathcal{D}^p_{p-1}\neq H^p$ if $p\neq 2$.

(ii) Rudin proved in [28] that there are Blaschke products which do not lie in $\mathcal{D}^1_0$ (see also [21]). Vinogradov, in Theorem 3.11 of [29], extended this result by showing that for any $p\in(0,2)$ there are Blaschke products which do not belong to $\mathcal{D}^p_{p-1}$. This yields $\mathcal{D}^p_{p-1}\neq H^p$, if $0<p<2$.

The strict inclusion can also be deduced from results in [4] or [13]. More information about the spaces $\mathcal{D}^p_{p-1}$, $2<p<\infty$, can be obtained in [14].

A complex-valued function defined in Δ is said to be *univalent* if it is analytic and one-to-one there. We refer to [6] and [23] for the theory of these functions. Throughout the paper, $\mathcal{U}$ will stand for the class of all univalent functions in Δ. Sometimes it is useful to consider certain normalized subclasses of $\mathcal{U}$ such as the class S and the class S_0:

$$S=\{f\in\mathcal{U}: f(0)=0, f'(0)=1\},$$
$$S_0=\{f\in\mathcal{U}: f \text{ is zero-free in } \Delta, f(0)=1\}.$$

We shall also make use of the following results.

Theorem A. *Suppose that* $0<p<\infty$*.*

(a) *If* $f\in H^p$ *then* $\int_0^1 M^p_\infty(r,f)\,dr\leq\pi\|f\|^p_{H_p}$*.*

(b) *If* $f\in\mathcal{U}$ *and* $\int_0^1 M^p_\infty(r,f)\,dr<\infty$*, then* $f\in H^p$*.*

In particular, if f is univalent then $\int_0^1 M^p_\infty(r,f)\,dr<\infty$ if and only if $f\in H^p$.

Part (a) does not require univalence of f. With an unspecified constant instead of π, the inequality is due to Hardy and Littlewood. See page 411 of [16]. On p. 127 of [23] the inequality is stated with constant π and attributed to Hardy and Littlewood, but without reference. We have not found a proof in the works of Hardy and Littlewood, but it is easy enough to supply one:

If there is a fixed θ such that $M_\infty(r,f) = |f(re^{i\theta})|$ for every $r \in (0,1)$ then the inequality follows from the Fejér-Riesz Theorem. See [5], for example. For general f, let h be the symmetric decreasing rearrangement of $\log|f|$ on the unit circle, let H be an analytic function in Δ with real part the Poisson integral of h, and let $F = e^H$. Well known inequalities for rearrangements imply that $|F|$ achieves its maximum modulus on the positive real axis, that F and f have the same H^p norm, and that $M_\infty(r,f) \le M_\infty(r,F)$.

Part (b) may be deduced from a theorem of Prawitz [27]. See [23, p. 17].

3. The main result

In this section we shall prove our main result, which is stated as Theorem 1.

Theorem 1. *Let f be a univalent function in Δ and $0 < p < \infty$. Then*

$$f \in \mathcal{D}^p_{p-1} \iff f \in H^p.$$

In other words,

$$\mathcal{U} \cap \mathcal{D}^p_{p-1} = \mathcal{U} \cap H^p, \tag{6}$$

for every $p \in (0,\infty)$.

For $p = 1$ this was proved by Pommerenke in Satz 1 of [22]. Actually, in this theorem Pommerenke gave another characterization of the univalent functions in H^p for $0 < p < 2$, as follows:

Theorem B. *If $f \in \mathcal{U}$ and $0 < p < 2$, then*

$$f \in H^p \iff \int_0^1 M_1^p(r,f')\,dr < \infty.$$

To see how Pommerenke's characterization is related to ours, we use the following result, due essentially to Hardy and Littlewood, which can be proved by modifying the proof of Theorem 5.9 in [5].

For $0 < p < q \le \infty$, there exists a constant C depending only on p and q such that for each analytic function f in Δ and each $r \in (0,1)$ we have

$$M_q(r,f) \le C M_p\left(\frac{1+r}{2}, f\right)(1-r)^{1/q-1/p}. \tag{7}$$

Suppose that $0 < p < 1$. Take $q = 1$ and replace f by f'. Then (7) gives

$$M_1^p(r,f') \le C I_p\left(\frac{1+r}{2}, f'\right)(1-r)^{p-1}.$$

On the other hand, if $p > 1$ then (7) leads to

$$I_p(r,f') \le C M_1^p\left(\frac{1+r}{2}, f'\right)(1-r)^{1-p}. \tag{8}$$

When $0 < p < 2$ the significant contents of Theorems 1 and B are the assertions that if $f \in \mathcal{U} \cap H^p$ then f' satisfies the respective integrability conditions. The last two inequalities above, together with a change of variable, show then that for $1 < p < 2$ Theorem B is stronger than Theorem 1, but for $0 < p < 1$ Theorem 1 is stronger than Theorem B.

The proof of Theorem 1 is organized as follows. First, in §3.1 we shall handle the case $0 < p \leq 1$ using results obtained by the first author in [2]. A chief ingredient of [2] is a theorem of Hayman [17] which asserts that a univalent function cannot be too large at too many widely scattered points. Our proof of the case $p = 1$ of Theorem 1 thus provides a new proof of Pommerenke's Theorem B for $p = 1$. For $1/2 \leq p < 1$ we know of no way to prove Theorem 1 other than the one presented here. But for $0 < p < 1/2$, when Theorem 1 reduces to saying that $\mathcal{U} \subset \mathcal{D}^p_{p-1}$, the result also follows from estimates on the growth of the integral means $M_p(r, f')$, $f \in \mathcal{U}$, due to Feng and MacGregor [9]. This is the content of §3.2. Next, in §3.3 we will prove the case $1 < p \leq 2$. As noted above, this case follows easily from inequality (8) and Theorem B. Finally, in §3.4, the case $2 < p < \infty$ will be handled using the Hardy-Littlewood inequality (7) together with Theorem A and another Hardy-Littlewood inequality relating growth of the integral means of a function f analytic in Δ with those of its derivative.

Before we properly start with the proof, let us say that from now on we shall be using the convention that C denotes a positive constant (which may depend on p, f, ε, but not on r, ρ or E) which can change from line to line.

3.1. The case $\mathbf{0 < p \leq 1}$. Since $\mathcal{D}^p_{p-1} \subset H^p$ for $0 < p \leq 1$, we have to prove that

$$\mathcal{U} \cap H^p \subset \mathcal{D}^p_{p-1}, \quad 0 < p \leq 1,$$

and, clearly, it suffices to show that

$$S_0 \cap H^p \subset \mathcal{D}^p_{p-1}, \quad 0 < p \leq 1. \tag{9}$$

Proof of (9). Take $p \in (0, 1]$ and $f \in S_0 \cap H^p$. Let

$$\delta = \frac{1}{40}, \quad \beta = \frac{1 - \delta + 10\delta^2}{2 - \delta} = \frac{1}{2} - \frac{1}{316}, \quad \gamma_1 = \frac{\delta}{2 - \delta}, \quad \gamma_2 = 1 - \gamma_1.$$

For simplicity, we shall write $M(r)$ for $M_\infty(r, f)$ $(0 \leq r < 1)$. Then $r \mapsto M(r)$ is a continuous and strictly increasing function from $[0, 1)$ onto $[1, K)$, where $K = \sup_{z \in \Delta} |f(z)|$. We shall denote by M^{-1} its inverse function.

If E is a measurable subset of the unit circle, we set

$$M(r, E) = \sup_{e^{i\theta} \in E} |f(re^{i\theta})|, \quad 0 \leq r < 1.$$

Using Theorem 3 of [2] we deduce that there exists a constant $C > 0$ such that

$$\int_E |f'(re^{i\theta})|\,d\theta \le C\left(\frac{1-\rho}{1-r}\right)^{\beta} |f(\rho e^{i\phi})|^{\gamma_1} M(r,E)^{\gamma_2}, \tag{10}$$

whenever $0 \le \rho < r < 1$, I is a subarc of the unit circle with center $e^{i\phi}$ and length $|I| \le 2\pi(1-\rho)$, and $E \subset I$.

If $0 \le r < 1$ and $j \ge 0$ is an integer, we set

$$A_j(r) = \{\theta \in [0, 2\pi] : 2^{-j-1}M(r) < |f(re^{i\theta})| \le 2^{-j}M(r)\}$$

and we define $\rho = \rho(j,r)$ as follows: If $M(r) \le e^2 2^{j+1}$ then $\rho = 0$. If $e^2 2^{j+1} < M(r) < \infty$, then

$$\rho(j,r) = M^{-1}(2^{-j-1}e^{-2}M(r)). \tag{11}$$

Note that $\rho(j,r) < r$.

Let $k = k(r)$ be the smallest $j \ge 0$ such that $\rho(j,r) \le 1/2$, and let

$$B(r) = \bigcup_{j=k(r)}^{\infty} A_j(r).$$

Then

$$|f(re^{i\theta})| \le 2e^2 M(1/2), \quad \theta \in B(r).$$

Assume that $1/2 < r < 1$.

Write the unit circle as the union of two disjoint intervals I_1 and I_2 of length π. Apply (10) with $\rho = 1/2$ and $E = B(r) \cap I_j$ $(j = 1, 2)$. We obtain

$$\int_{B(r)} |f'(re^{i\theta})|\,d\theta \le CM(1/2)(1-r)^{-\beta} \le C(1-r)^{-\beta}, \quad \frac{1}{2} < r < 1. \tag{12}$$

Take j with $0 \le j < k(r)$. Write $\rho = \rho(j,r)$. Then $1/2 < \rho < r$. If I is an arc of length at most $2\pi(1-\rho)$, then (10) and (11) give

$$\begin{aligned}\int_{A_j(r)\cap I} |f'(re^{i\theta})|\,d\theta &\le C\left(\frac{1-\rho}{1-r}\right)^{\beta}\left(\frac{M(\rho)}{M(r)2^{-j}}\right)^{\gamma_1} 2^{-j}M(r) \\ &\le C\left(\frac{1-\rho}{1-r}\right)^{\beta} 2^{-j}M(r).\end{aligned} \tag{13}$$

Applying Hölder's inequality with exponents $1/p$ and $1/(1-p)$, we get

$$\int_{A_j(r)\cap I} |f'(re^{i\theta})|^p\,d\theta \le C\left(\left(\frac{1-\rho}{1-r}\right)^{\beta} 2^{-j}M(r)\right)^p |I|^{1-p}.$$

Since $|I| \le 2\pi(1-\rho)$, it follows that

$$\int_{A_j(r)\cap I} |f'(re^{i\theta})|^p (1-r)^{p-1}\,d\theta \le C2^{-jp}M^p(r)\left(\frac{1-\rho}{1-r}\right)^{\beta p+1-p}. \tag{14}$$

Write the unit circle as the union of m nonoverlapping intervals $I_1, I_2, \ldots, I_m$ of length $2\pi/m$, where m is the smallest integer such that $m \geq (1-\rho)^{-1}$. Then $2\pi/m \in (\pi(1-\rho), 2\pi(1-\rho]$. Let $\nu(j,r) = \nu$ be the number of the intervals I_l $(1 \leq l \leq m)$ which intersect $A_j(r)$. From (14), we have

$$\int_{A_j(r)} |f'(re^{i\theta})|^p (1-r)^{p-1}\, d\theta \leq C\nu(j,r)2^{-jp} M^p(r) \left(\frac{1-\rho}{1-r}\right)^{\beta p+1-p}, \quad 0 \leq j < k(r). \tag{15}$$

For each interval I_l which meets $A_j(r)$ take $\theta \in I_l \cap A_j(r)$. Arrange these θ's in ascending order and denote them by $0 \leq \theta_0 < \cdots < \theta_{\nu-1} < 2\pi$. Let $Q = e^{10}$, and let n be the largest integer such that $nQ \leq \nu$. Then $(n+1)Q > \nu$. If $n \leq 1$ then $\nu < 2Q$. Assume $n \geq 2$. Let $S = \{\theta_0, \theta_Q, \ldots, \theta_{(n-1)Q}\}$. If $\theta, \theta' \in S$ then $e^{i\theta}$ and $e^{i\theta'}$ are separated by at least $Q-1$ intervals of length $2\pi/m > \pi(1-\rho)$, so

$$|e^{i\theta} - e^{i\theta'}| > \frac{2}{\pi}(Q-1)\pi(1-\rho) = 2(Q-1)(1-\rho).$$

Since $\rho > 1/2$, we have $|\rho e^{i\theta} - \rho e^{i\theta'}| > (Q-1)(1-\rho)$. Then the discs with centers $\rho e^{i\theta}, \rho e^{i\theta'}$, radius $\frac{1}{2}(Q-1)(1-\rho)$ are disjoint, hence so are the discs with radius $1-\rho$.

Now $|f(re^{i\theta})| > 2^{-j-1}M(r)$ for $\theta \in S$ and $|f(\rho e^{i\theta})| \leq M(\rho) = M(r)2^{-j-1}e^{-2}$. So f satisfies the conditions of Theorem 2.4 of [17] with $R_2 = 2^{-1-j}M(r)$, $R_1 = 2e^{-2}R_2$, $\delta_i = (1-r)/(1-\rho)$ for $i = 0, \ldots, n-1$. So

$$n < 2\left(\log \frac{R_2}{R_1} - 1\right)^{-1} \log\left[C\left(\frac{1-r}{1-\rho}\right)^{-1}\right] \leq 8\left(\log\left(\frac{1-\rho}{1-r}\right) + C\right).$$

Thus,

$$\nu(j,r) < 2nQ < 16Q\left(\log\left(\frac{1-\rho}{1-r}\right) + C\right). \tag{16}$$

Recall that $\beta = 1/2 - 1/316$. Take $\eta > 0$ so small that

$$\gamma(p) \equiv \beta p + 1 - p + \eta < 1. \tag{17}$$

By (16), there is a constant C such that

$$\nu(j,r) < C\left(\frac{1-\rho}{1-r}\right)^{\eta}. \tag{18}$$

From (15) and (18), we obtain

$$\int_{A_j(r)} |f'(re^{i\theta})|^p (1-r)^{p-1}\,d\theta \leq CM^p(r)2^{-jp}\left(\frac{1-\rho}{1-r}\right)^{\gamma(p)}, \quad r \in (1/2,1), \quad 0 \leq j < k(r), \tag{19}$$

where $\rho = \rho(j,r)$.

Let $\Delta(1/2) = \{z \in \Delta : 0 \leq |z| \leq 1/2\}$, and

$$A = \left\{ re^{i\theta} \in \Delta : r > \frac{1}{2},\, \theta \in \bigcup_{j=0}^{k(r)-1} A_j(r) \right\},$$
$$B = \left\{ re^{i\theta} \in \Delta : r > \frac{1}{2},\, \theta \in B(r) \right\}.$$

Then

$$\int_\Delta (1-|z|)^{p-1}|f'(z)|^p\,dA(z) = \int_{\Delta(1/2)} (1-|z|)^{p-1}|f'(z)|^p\,dA(z) + \int_B (1-|z|)^{p-1}|f'(z)|^p\,dA(z) + \int_A (1-|z|)^{p-1}|f'(z)|^p\,dA(z). \tag{20}$$

Clearly,

$$\int_{\Delta(1/2)} (1-|z|)^{p-1}|f'(z)|^p\,dA(z) \leq C. \tag{21}$$

Applying (12) and Hölder's inequality, we obtain

$$\int_{B(r)} |f'(re^{i\theta})|^p\,d\theta \leq C\left(\int_{B(r)} |f'(re^{i\theta})|\,d\theta\right)^p \leq C(1-r)^{-\beta p}, \quad \frac{1}{2} < r < 1. \tag{22}$$

So

$$\int_B |f'(z)|^p(1-|z|)^{p-1}\,dA(z) \leq C\int_{1/2}^1 (1-r)^{(-\beta p+p-1)}\,dr. \tag{23}$$

Since $p - \beta p > 0$, the integral on the right converges, and we see that

$$\int_B (1-|z|)^{p-1}|f'(z)|^p\,dA(z) \leq C. \tag{24}$$

Recall that $K = \sup\{M(r) : 0 < r < 1\}$. To investigate the third integral on the right side of (20), for $j \geq 0$ define $r_j \in (0,1)$ by

$$r_j = M^{-1}\left(2^{j+1}e^2 M\left(\frac{1}{2}\right)\right) \tag{25}$$

when $2^{j+1}e^2M(1/2) < K$. Define $r_j = 1$ when $2^{j+1}e^2M(1/2) \geq K$. Then $\rho(j, r_j) = 1/2$ when $r_j < 1$. Moreover, $r_0 > 1/2$, and for each $j \geq 0$, $j \leq k(r) - 1$ iff $r_j < r < 1$. Thus, for $j \geq 0$ and $r \in (1/2, 1)$, the set $A_j(r)$ is contained in A if $r_j < r < 1$ and $A_j(r)$ is contained in B if $r \in [1/2, r_j]$. Writing $\rho = \rho(j, r)$, it follows from (19) and (11) that

$$
\begin{aligned}
(26) \qquad & \int_A (1-|z|)^{p-1}|f'(z)|^p \, dA(z) \\
&= \int_{1/2}^1 r(1-r)^{p-1} \sum_{j=0}^{k(r)-1} \int_{A_j(r)} |f'(re^{i\theta})|^p \, d\theta \, dr \\
&\leq C \int_{1/2}^1 \sum_{j=0}^{k(r)-1} \left(\frac{1-\rho}{1-r}\right)^{\gamma(p)} M^p(r) 2^{-jp} \, dr \\
&\leq C \int_{1/2}^1 \sum_{j=0}^{k(r)-1} M^p(\rho) \left(\frac{1-\rho}{1-r}\right)^{\gamma(p)} dr \\
&\leq C \sum_{j=0}^{\infty} \int_{r_j}^1 M^p(\rho) \left(\frac{1-\rho}{1-r}\right)^{\gamma(p)} dr.
\end{aligned}
$$

Since $f \in S_0$, $\log f \in \mathcal{B}$. Indeed, the Koebe one-quarter theorem (see p. 31 of [6]) implies that $\|\log f\|_{\mathcal{B}} \leq 4$. Then it is a simple exercise to show that there exists a positive number b slightly less than 1 such that

$$
(27) \qquad M(1-b^{m+1}) < 2M(1-b^m), \quad m = 0, 1, 2, \ldots
$$

Fix such a b. Define

$$
(28) \qquad J_m = [1-b^m, 1-b^{m+1}), \quad E_{jm} = \{r \in (0,1) : \rho(j,r) \in J_m\}.
$$

Then

$$
(29) \qquad r \in E_{jm} \quad \text{iff} \quad M(1-b^m) \leq 2^{-j-1}e^{-2}M(r) < M(1-b^{m+1}).
$$

Let m_0 be the integer such that $1 - b^{m_0} \leq 1/2 < 1 - b^{m_0+1}$. For fixed j, the E_{jm} with $m \geq m_0$ form a partition of an interval which contains $(r_j, 1)$, so

$$
\begin{aligned}
(30) \qquad \int_{r_j}^1 M^p(\rho) \left(\frac{1-\rho}{1-r}\right)^{\gamma(p)} dr &\leq \sum_{m=m_0}^{\infty} \int_{E_{jm}} M^p(\rho) \left(\frac{1-\rho}{1-r}\right)^{\gamma(p)} dr \\
&\leq C \sum_{m=m_0}^{\infty} M^p(1-b^m) b^{m\gamma(p)} \int_{E_{jm}} (1-r)^{-\gamma(p)} \, dr.
\end{aligned}
$$

From (27) and (29), it follows that for fixed $m \geq m_0$ the E_{jm} with $j \geq 0$ are pairwise disjoint. Moreover, their union over j is contained in $(s, 1)$, where

$s = M^{-1}(2e^2M(1-b^m)) > 1-b^m$. With (26) and (30), this gives

$$
\begin{aligned}
(31)\quad & \int_A (1-|z|)^{p-1}|f'(z)|^p\,dA(z) \\
&\le C\sum_{j=0}^{\infty}\sum_{m=m_0}^{\infty} M^p(1-b^m)b^{m\gamma(p)}\int_{E_{jm}}(1-r)^{-\gamma(p)}\,dr \\
&\le C\sum_{m=m_0}^{\infty} M^p(1-b^m)b^{m\gamma(p)}\int_{1-b^m}^{1}(1-r)^{-\gamma(p)}\,dr \\
&= C\sum_{m=m_0}^{\infty} M^p(1-b^m)b^m.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(32)\qquad \int_{1-b^m}^{1-b^{m+1}} M^p(r)\,dr &\ge M^p(1-b^m)(b^m-b^{m+1}) \\
&= M^p(1-b^m)b^m(1-b).
\end{aligned}
$$

From (31) and (32), it follows that

$$
(33)\qquad \int_A (1-|z|)^{p-1}|f'(z)|^p\,dA(z) \le C\int_0^1 M^p(r)\,dr,
$$

and then, since $f \in H^p$, from Theorem A we deduce that

$$
\int_A (1-|z|)^{p-1}|f'(z)|^p\,dA(z) < \infty,
$$

which, together with (20), (21) and (24), gives $f \in \mathcal{D}^p_{p-1}$. This finishes the proof of Theorem 1 when $0 < p \le 1$. □

3.2. Another proof for the case $\mathbf{0 < p < 1/2}$. Since $\mathcal{U} \subset H^p$ for $0 < p < 1/2$, Theorem 1 for these values of p reduces to the following:

$$
(34)\qquad \mathcal{U} \subset \mathcal{D}^p_{p-1}, \quad 0 < p < 1/2.
$$

Even though we proved this in the previous subsection, here we present an alternative proof as a consequence of estimates on the growth of the integral means $M_p(r, f')$, $f \in \mathcal{U}$.

Proof. Feng and MacGregor proved in [9] that if $f \in \mathcal{U}$ and $p > 2/5$, then

$$
I_p(r, f') = O\left((1-r)^{-(3p-1)}\right), \quad \text{as } r \to 1.
$$

Consequently, if $f \in \mathcal{U}$ and $2/5 < p < 1/2$, we have

$$\begin{aligned}\int_0^1 (1-r)^{p-1} I_p(r,f')\,dr &\le C \int_0^1 (1-r)^{p-1} \frac{1}{(1-r)^{3p-1}}\,dr \\ &\le C \int_0^1 \frac{1}{(1-r)^{2p}}\,dr < \infty,\end{aligned}$$

and hence $f \in \mathcal{D}^p_{p-1}$.

Following Pommerenke [25], [26], for any real number p, we let $\beta(p)$ denote the smallest number such that

$$I_p(r,f') = \mathrm{O}\left(\frac{1}{(1-r)^{\beta(p)+\epsilon}}\right), \quad \text{as } r \to 1, \text{ for every } \epsilon > 0,$$

for any $f \in \mathcal{U}$. Feng and MacGregor [9] (see also [25]) also proved that

$$\beta(p) \le \frac{p}{2}, \quad 0 < p \le 2/5. \tag{35}$$

Thus, if $f \in \mathcal{U}$ and $0 < p \le 2/5$, then, taking ε with $0 < \varepsilon < p/2$, we obtain

$$\begin{aligned}\int_0^1 (1-r)^{p-1} I_p(r,f')\,dr &\le C \int_0^1 (1-r)^{p-1} \frac{1}{(1-r)^{\beta(p)+\varepsilon}}\,dr \\ &\le C \int_0^1 (1-r)^{p-1} \frac{1}{(1-r)^{p/2+\varepsilon}}\,dr \\ &\le C \int_0^1 (1-r)^{p/2-\varepsilon-1}\,dr < \infty,\end{aligned}$$

which implies that $f \in \mathcal{D}^p_{p-1}$. □

3.3. The case $1 < p \le 2$. The equality $\mathcal{U} \cap H^2 = \mathcal{U} \cap \mathcal{D}^2_1$ is trivially true because $H^2 = \mathcal{D}^2_1$.

Since $\mathcal{D}^p_{p-1} \subset H^p$ for $1 < p < 2$, it suffices to prove that

$$\mathcal{U} \cap H^p \subset \mathcal{U} \cap \mathcal{D}^p_{p-1}, \quad 1 < p < 2.$$

Proof. Take $p \in (1,2)$ and $f \in \mathcal{U} \cap H^p$. From (8), there follows the existence of a positive constant C such that

$$I_p(r,f') \le C M_1^p\left(\frac{1+r}{2}, f'\right)(1-r)^{1-p}, \quad 0 < r < 1.$$

Hence, making the change of variable $\rho = (1+r)/2$, we obtain

$$\int_0^1 (1-r)^{p-1} I_p(r,f')\,dr \le C \int_0^1 M_1^p\left(\frac{1+r}{2}, f'\right) dr \le C \int_0^1 M_1^p(\rho, f')\,d\rho,$$

which, using Theorem B, implies that $f \in \mathcal{D}^p_{p-1}$. □

3.4. The case $\mathbf{2 < p < \infty}$. Since $H^p \subset \mathcal{D}^p_{p-1}$, $2 < p < \infty$, it suffices to show that

$$\mathcal{D}^p_{p-1} \cap \mathcal{U} \subset H^p \cap \mathcal{U}, \quad 2 < p < \infty.$$

Proof. Let $2 < p < \infty$ and suppose that $f \in \mathcal{D}^p_{p-1} \cap \mathcal{U}$. From (7) we obtain

$$M_\infty(r,f) \le C \frac{M_{2p}\left(\frac{1+r}{2}, f\right)}{\left(1 - \frac{1+r}{2}\right)^{1/(2p)}}, \quad 0 < r < 1,$$

or, equivalently,

$$M^p_\infty(r,f) \le C \frac{M^p_{2p}\left(\frac{1+r}{2}, f\right)}{\left(1 - \frac{1+r}{2}\right)^{1/2}}, \quad 0 < r < 1.$$

Thus, integrating and making the change of variable $\rho = (1+r)/2$, we obtain

$$\begin{aligned} \int_0^1 M^p_\infty(r,f)\,dr &\le C \int_0^1 \frac{M^p_{2p}(\frac{1+r}{2}, f)}{\left(1 - \frac{1+r}{2}\right)^{1/2}}\,dr \\ &\le C \int_0^1 \frac{M^p_{2p}(\rho, f)}{(1-\rho)^{1/2}}\,d\rho. \end{aligned} \tag{36}$$

On the other hand, using Theorem 5.6 of [5], we see that

$$\int_0^1 (1-r)^{-1/2} M^p_{2p}(r,f)\,dr \le C \left\{ \int_0^1 (1-r)^{p-1/2} M^p_{2p}(r,f')\,dr + |f(0)|^p \right\}. \tag{37}$$

Furthermore, using (7) once more we deduce that

$$M_{2p}(r,f') \le C \frac{M_p(\frac{1+r}{2}, f')}{\left(1 - \frac{1+r}{2}\right)^{1/p - 1/(2p)}} = C \frac{M_p(\frac{1+r}{2}, f')}{\left(1 - \frac{1+r}{2}\right)^{1/(2p)}}, \quad 0 < r < 1,$$

and, hence,

$$M^p_{2p}(r,f') \le C \frac{I_p(\frac{1+r}{2}, f')}{\left(1 - \frac{1+r}{2}\right)^{1/2}}, \quad 0 < r < 1. \tag{38}$$

Using (36), (38) and the fact that $f \in \mathcal{D}^p_{p-1}$, we deduce after a change of variable that

$$\begin{aligned} \int_0^1 M^p_\infty(r,f) &\le C \left\{ \int_0^1 (1-r)^{p-1/2} M^p_{2p}(r,f')\,dr + |f(0)|^p \right\} \\ &\le C \left\{ \int_0^1 (1-r)^{p-1/2} \frac{I_p(\frac{1+r}{2}, f')}{\left(1 - \frac{1+r}{2}\right)^{1/2}}\,dr + |f(0)|^p \right\} \\ &\le C \left\{ \int_0^1 (1-\rho)^{p-1} I_p(\rho, f')\,d\rho + |f(0)|^p \right\} < \infty, \end{aligned} \tag{39}$$

which, by Theorem A, implies that $f \in H^p$. The proof of Theorem 1 is complete. $\square$

In the proof above the constants C relating the H^p and $\mathcal{D}^p_{p-1}$ norms were permitted to depend on f. However, inspection of the proof and some simple considerations show that under appropriate normalization of f the norms are essentially equivalent, with constants which depend only on p. Here, for the record, is one such statement:

For $0 < p < \infty$ there exist constants C_1, C_2, C_3, depending only on p, such that for every $f \in \mathcal{U}$ with $f(0) = 0$ we have

$$\begin{aligned} \int_\Delta (1-|z|)^{p-1}|f'(z)|^p\,dA(z) &\le C_1 \int_0^{2\pi} |f(e^{i\theta})|^p\,d\theta \\ &\le C_2 \int_0^1 M_\infty^p(r,f)\,dr \\ &\le C_3 \int_\Delta (1-|z|)^{p-1}|f'(z)|^p\,dA(z). \end{aligned} \tag{40}$$

4. Univalent functions and Bergman spaces

Since $\mathcal{U} \subset H^p$ if $0 < p < 1/2$ and $H^p \subset A^{2p}$ for all p, we immediately deduce

$$\mathcal{U} \subset A^p, \quad 0 < p < 1. \tag{41}$$

In the following theorem we give a characterization of the univalent functions which belong to the Bergman space A^p $(0 < p < \infty)$. It is the analogue of Theorems A and B for the spaces A^p.

THEOREM 2. *Let $f \in \mathcal{U}$ and $0 < p < \infty$. Then the following two conditions are equivalent:*

(i) $f \in A^p$.

(ii) $\int_0^1 \int_0^r M_\infty^p(\rho,f)\,d\rho\,dr < \infty$.

Furthermore, if $0 < p < 2$ then these conditions are also equivalent to

(iii) $\int_0^1 \int_0^r M_1^p(\rho,f')\,d\rho\,dr < \infty$.

Proof. (i)$\Rightarrow$(ii). This implication holds even if we do not assume that f is univalent. Indeed, take $p > 0$ and $f \in A^p$. By part (a) of Theorem A,

$$\int_0^r M_\infty^p(\rho,f)\,d\rho \le \pi I_p(r,f), \quad 0 \le r < 1. \tag{42}$$

Thus

$$\int_0^1 \int_0^r r M_\infty^p(\rho,f)\,d\rho\,dr \le \pi \int_0^1 r I_p(r,f)\,dr = \frac{1}{2}\int_\Delta |f(z)|^p\,dA(z) < \infty, \tag{43}$$

and (ii) follows.

(ii)⇒(i). Take a univalent function f which satisfies (ii). An identity of Hardy [15] (see p. 126 of [23]) gives

$$\frac{d}{dr}\left[rI_p'(r,f)\right] = \frac{p^2 r}{2\pi}\int_{-\pi}^{\pi} |f(re^{it})|^{p-2}|f'(re^{it})|^2\,dt.$$

Integrating and making the change of variable $w = f(z)$, we obtain

$$\begin{aligned}\rho I_p'(\rho,f) &= \frac{p^2}{2\pi}\int_{|z|<\rho} |f(z)|^{p-2}|f'(z)|^2\,dA(z)\\ &\le \frac{p^2}{2\pi}\int_{|w|\le M_\infty(\rho,f)} |w|^{p-2}\,dA(w)\\ &= p^2\int_0^{M_\infty(\rho,f)} t^{p-1}\,dt = p\,M_\infty^p(\rho,f).\end{aligned}$$

Integrating, we obtain for $1/2 \le r < 1$,

$$\begin{aligned}I_p(r,f) - I_p\left(\frac{1}{2},f\right) &= \int_{1/2}^{r} I_p'(\rho,f)\,d\rho\\ &\le p\int_{1/2}^{r} \rho^{-1}M_\infty^p(\rho,f)\,d\rho \le 2p\int_{1/2}^{r} M_\infty^p(\rho,f)\,d\rho,\end{aligned}$$

and then it follows that

$$\begin{aligned}\int_{1/2}^{1} rI_p(r,f)\,dr &\le \frac{1}{2}I_p\left(\frac{1}{2},f\right) + 2p\int_{1/2}^{1} r\int_{1/2}^{r} M_\infty^p(\rho,f)\,d\rho\,dr\\ &\le \frac{1}{2}I_p\left(\frac{1}{2},f\right) + \int_0^1\int_0^r M_\infty^p(\rho,f)\,d\rho\,dr < \infty.\end{aligned}$$

Clearly, this implies that $f \in A^p$.

The implication (iii)⇒(ii) holds for any $p > 0$. Indeed, take $p \in (0,\infty)$ and suppose that f is a univalent function which satisfies (iii). Assume without loss of generality that $f \in S$. For $0 < r < 1$, let C_r be the image of the circle $\{|z| = r\}$ under f. Then C_r is a Jordan curve with 0 in its inner domain. Also, $2\pi r M_1(r,f')$ is the length of C_r. Hence

$$M_\infty(r,f) \le 2\pi r M_1(r,f'), \quad 0 < r < 1,$$

and then (ii) follows.

It remains to prove that if $0 < p < 2$ then (ii) implies (iii). Thus, suppose that $p \in (0,2)$ and take a function $f \in \mathcal{U}$ which satisfies (ii). Assume, without loss of generality, that $f \in S$. Using Satz 4 of [22] and Theorem 5.1 on p. 127

of [23], we deduce that

$$\int_0^r M_1^p(\rho, f')\, d\rho \le C I_p(r, f) \le C \int_0^r M_\infty^p(\rho, f)\, \frac{d\rho}{\rho}, \quad 0 < r < 1,$$

which, with (ii), clearly implies (iii). $\square$

Now we can state our main result in this section.

THEOREM 3. *Given p with $1/2 \le p < \infty$ there exists a univalent function f which belongs to $A^{2p} \setminus H^p$.*

The following lemmas play a basic role in the proof of Theorem 3.

LEMMA 4. *Define*

$$Q(z) = \frac{1}{(1-z)\log \frac{2e}{1-z}}, \quad z \in \Delta. \tag{44}$$

Then:

$$\operatorname{Re} Q(z) > 0, \quad z \in \Delta. \tag{45}$$

$$Q \in \mathcal{U}. \tag{46}$$

$$M_\infty(r, Q) = \frac{1}{(1-r)\log \frac{2e}{1-r}}, \quad 0 < r < 1. \tag{47}$$

Proof. The assertion (45) is part of Theorem 7 of [8]. Next, set

$$F(z) = \log \frac{2e}{1-z}, \quad z \in \Delta.$$

Then F is a conformal mapping from Δ onto a domain D contained in $\{z \in \mathbb{C} : \operatorname{Re} z > 1, |\operatorname{Im} z| < \pi/2\}$ and it follows that

$$\operatorname{Re} \frac{1}{F(z)} > 0. \tag{48}$$

Also, $|1/F(z)| < 1$ $(z \in \Delta)$, which implies

$$\operatorname{Re}\left(1 - \frac{1}{F(z)}\right) > 0, \quad z \in \Delta. \tag{49}$$

Notice also that

$$\operatorname{Im}\left(\frac{1}{F(z)}\right) = -\operatorname{Im}\left(1 - \frac{1}{F(z)}\right), \quad z \in \Delta. \tag{50}$$

A simple computation gives

$$Q'(z) = \frac{1}{(1-z)^2}\, G(z) = \frac{k(z)}{z}\, G(z), \quad z \in \Delta, \tag{51}$$

where $k(z) = z(1-z)^{-2}$ ($z \in \Delta$) is the Koebe function and

$$G(z) = \frac{1}{F(z)}\left(1 - \frac{1}{F(z)}\right), \quad z \in \Delta. \tag{52}$$

Now, (48), (49) and (50) imply that

$$\operatorname{Re} G(z) > 0, \quad z \in \Delta.$$

In other words, we have

$$\operatorname{Re}\left(z\,\frac{Q'(z)}{k(z)}\right) > 0, \quad z \in \Delta.$$

Using the notation and terminology of Section 2.2 and Section 2.3 of [23], it follows that, since the Koebe function is starlike, the function $Q - Q(0)$ is close-to-convex and, hence, univalent (see Theorem 2.11 on p. 51 of [23]). Consequently, (46) follows.

It remains to prove (47). Clearly,

$$M_\infty(r, Q) \geq Q(r) = \frac{1}{(1-r)\log\frac{2e}{1-r}}, \quad 0 < r < 1. \tag{53}$$

Now, bearing in mind that the function $x \mapsto x\,\log(2e/x)$ is increasing in $(0, 2)$, we have

$$\left|(1-z)\log\frac{2e}{1-z}\right| \geq |1-z|\log\frac{2e}{|1-z|} \geq (1-r)\log\frac{2e}{1-r}, \quad |z| = r,$$

for all $r \in (0, 1)$. This implies

$$M_\infty(r, Q) \leq \frac{1}{(1-r)\log\frac{2e}{1-r}}, \quad 0 < r < 1.$$

This and (53) give (47). □

Lemma 5. *Let γ and α be two positive constants. Then*

$$\int_0^r (1-s)^{-(1+\gamma)}\left(\log\frac{2e}{1-s}\right)^{-\alpha} ds \approx (1-r)^{-\gamma}\left(\log\frac{2e}{1-r}\right)^{-\alpha}, \quad \textit{as} \quad r \to 1^-.$$

The proof of this Lemma is elementary and will be omitted.

Proof of Theorem 3. Take p with $1/2 \leq p < \infty$. Let Q be the function defined in Lemma 4 and set $f = Q^{1/p}$. Since $1/p \leq 2$, (45) and (46) imply that $f \in \mathcal{U}$.

Now, (47) implies that

$$M_\infty(r, f) = \left[\frac{1}{(1-r)\log\frac{2e}{1-r}}\right]^{1/p}, \quad 0 < r < 1, \tag{54}$$

and then, using Lemma 5, it follows that

$$
\begin{aligned}
(55)\quad \int_0^1 \left(\int_0^r M_\infty^{2p}(\rho, f)\, d\rho \right) dr &= \int_0^1 \left(\int_0^r (1-\rho)^{-2} \Big(\log \frac{2e}{1-\rho} \Big)^{-2} d\rho \right) dr \\
&\le C \int_0^1 (1-r)^{-1} \Big(\log \frac{2e}{1-r} \Big)^{-2} dr < \infty.
\end{aligned}
$$

We note that (54) also implies

$$
(56) \qquad \int_0^1 M_\infty^p(r, f)\, dr \ge C \int_0^1 (1-r)^{-1} \Big(\log \frac{2e}{1-r} \Big)^{-1} dr = \infty.
$$

Using Theorem 2 and (55) we deduce that $f \in A^{2p}$. On the other hand, by part (a) of Theorem A, (56) implies that $f \notin H^p$. This finishes the proof. □

Next we shall use Theorem 1 to find geometric conditions on the image domain of a function $f \in \mathcal{U}$ which imply its membership in H^p. For simplicity, we shall assume that $0 \in f(\Delta)$.

Given a domain Ω in the plane and a point w in Ω, we shall write $d_\Omega(w)$ to denote the (Euclidean) distance from w to the boundary $\partial\Omega$. The following statement is well known (see, e.g., [26, Corollary 1.4]).

If Ω is a simply connected proper subdomain of $\mathbb{C}$ and F is a conformal mapping from Δ onto Ω then we have

$$
(57) \qquad d_\Omega(F(z)) \le |F'(z)|(1-|z|^2) \le 4 d_\Omega(F(z)), \quad z \in \Delta.
$$

Now we can prove the following result.

Theorem 6. *Suppose that $f \in \mathcal{U}$ and let $\Omega = f(\Delta)$. We have:*

(1) *If there exists α with $0 < \alpha < 1$ such that*

$$
(58) \qquad \int_\Omega \frac{d_\Omega(w)^{2p-2}}{|w|^\alpha}\, dA(w) < \infty,
$$

then $f \in H^p$.

(2) *Suppose that $0 \in \Omega$. For $\varepsilon > 0$, set $\Omega_\varepsilon = \{w \in \Omega : |w| > \varepsilon\}$. If there exists $\alpha \ge 1$ such that $f \in A^\alpha$ and*

$$
(59) \qquad \int_{\Omega_\varepsilon} \frac{d_\Omega(w)^{2p-2}}{|w|^\alpha}\, dA(w) < \infty,
$$

for all sufficiently small $\varepsilon > 0$, then $f \in H^p$.

Proof. Take $f \in \mathcal{U}$, $p > 0$, $\alpha > 0$. Let Ω^* be a subdomain of Ω to be specified later and set $\Delta^* = f^{-1}(\Omega^*)$. Using the Cauchy-Schwarz inequality,

(57) and making the change of variable $w = f(z)$, we obtain

$$
\begin{aligned}
(60)\quad & \int_{\Delta^*} (1-|z|^2)^{p-1}|f'(z)|^p \, dA(z) \\
&= \int_{\Delta^*} (1-|z|^2)^{p-1} \frac{|f'(z)|^p}{|f(z)|^{\alpha/2}} |f(z)|^{\alpha/2} \, dA(z) \\
&\le \left\{ \int_{\Delta^*} (1-|z|^2)^{2p-2} \frac{|f'(z)|^{2p}}{|f(z)|^{\alpha}} \, dA(z) \right\}^{1/2} \left\{ \int_{\Delta} |f(z)|^{\alpha} \, dA(z) \right\}^{1/2} \\
&= \left\{ \int_{\Delta^*} (1-|z|^2)^{2p-2} \frac{|f'(z)|^{2p-2}}{|f(z)|^{\alpha}} |f'(z)|^2 \, dA(z) \right\}^{1/2} \times \\
&\qquad \times \left\{ \int_{\Delta} |f(z)|^{\alpha} \, dA(z) \right\}^{1/2} \\
&\le C \left\{ \int_{\Omega^*} \frac{d_\Omega(w)^{2p-2}}{|w|^{\alpha}} \, dA(w) \right\}^{1/2} \left\{ \int_{\Delta} |f(z)|^{\alpha} \, dA(z) \right\}^{1/2}.
\end{aligned}
$$

If $0 < \alpha < 1$, we set $\Omega^* = \Omega$ (then $\Delta^* = \Delta$). Then, bearing in mind that $f \in H^{\alpha/2} \subset A^{\alpha}$, (58) and (60) imply that $f \in \mathcal{D}^p_{p-1}$, and hence $f \in H^p$. This finishes the proof of the first case.

Suppose now that $\alpha \ge 1$, $f \in \mathcal{U} \cap A^{\alpha}$ and $0 \in \Omega$. Take $\eta > 0$ such that $\{|w| < \eta\} \subset \Omega$ and take ε with $0 < \epsilon < \eta$. Set $\Omega^* = \Omega_\varepsilon$. Then (59), (60) and the assumption $f \in A^{\alpha}$ give

$$\int_{\Delta^*} (1-|z|^2)^{p-1}|f'(z)|^p \, dA(z) < \infty.$$

Clearly, this implies that f belongs to $\mathcal{D}^p_{p-1}$ and, thus, to H^p. So, the proof of the second case is also finished. □

Since $H^p \subset A^{2p}$, for all p, the most interesting case of Theorem 6 is the following.

Corollary 7. *Suppose that $1/2 \le p < \infty$, and $f \in A^{2p} \cap \mathcal{U}$. Set $\Omega = f(\Delta)$ and suppose that $0 \in \Omega$. If*

$$\int_{\Omega_\varepsilon} \frac{d_\Omega(w)^{2p-2}}{|w|^{2p}} \, dA(w) < \infty,$$

for all sufficiently small $\varepsilon > 0$, then $f \in H^p$.

We finish by showing that Corollary 7 is sharp.

Theorem 8. *If $1/2 < p < \infty$ then there exists a univalent function $g \in A^{2p} \setminus H^p$ with $g(0) = 0$ and such that, setting $\Omega = g(\Delta)$ and $\Omega_\varepsilon = \{w \in \Omega :$*

$|w| > \varepsilon\}$,

$$\iint_{\Omega_\varepsilon} \frac{d_\Omega(w)^{2p-2}}{|w|^{2p+\kappa}}\, dA(w) < \infty, \quad \varepsilon > 0, \tag{61}$$

for every $\kappa > 0$.

Proof. Take $p \in (1/2, \infty)$ and let f be the function defined in the proof of Theorem 3, that is,

$$f(z) = \left[\frac{1}{(1-z)\log\frac{2e}{1-z}}\right]^{1/p}, \qquad z \in \Delta.$$

Set

$$g(z) = f(z) - f(0), \quad z \in \Delta.$$

Then g is univalent, $g(0) = 0$ and $g \in A^{2p} \setminus H^p$. Hence, it remains to prove that (61) holds for every $\kappa > 0$.

Take $\varepsilon > 0$ and $\kappa > 0$. Since $g(0) = 0$, there exists η with $0 < \eta < 1$ such that

$$g^{-1}(\Omega_\varepsilon) \subset \Delta_\eta \stackrel{\text{def}}{=} \{z \in \Delta : |z| > \eta\}.$$

A simple calculation shows that

$$g'(z) = \frac{1}{p(1-z)^{1+1/p}}\left[\left(\frac{1}{\log\frac{2e}{1-z}}\right)^{1/p}\left(1 - \frac{1}{\log\frac{2e}{1-z}}\right)\right], \quad z \in \Delta, \tag{62}$$

and that there exists a positive constant C such that

$$|g(z)| \geq C\left|\frac{1}{(1-z)\log\frac{2e}{1-z}}\right|^{1/p} \qquad z \in \Delta_\eta. \tag{63}$$

Assume, without loss of generality, that $0 < \kappa < p(2p-1)$ (or, equivalently, that $2p - \kappa/p > 1$). Using (57), making the change of variable $w = g(z)$, and using (62) and (63), we obtain

$$
\begin{aligned}
(64)\quad & \int_{\Omega_\varepsilon} \frac{d_\Omega(w)^{2p-2}}{|w|^{2p+\kappa}}\, dA(w) \\
&\leq C \int_{\Delta_\eta} (1-|z|^2)^{2p-2} \frac{|g'(z)|^{2p}}{|g(z)|^{2p+\kappa}}\, dA(z) \\
&\leq C \int_{\Delta_\eta} (1-|z|^2)^{2p-2} \frac{\left|(1-z)\log \frac{2e}{1-z}\right|^{2+\kappa/p}}{|1-z|^{2p+2}\left|\log\frac{2e}{1-z}\right|^2} \left|1-\frac{1}{\log\frac{2e}{1-z}}\right|^{2p} dA(z) \\
&\leq C \int_{\Delta} (1-|z|^2)^{2p-2} \frac{\left|\log\frac{2e}{1-z}\right|^{\kappa/p}}{|1-z|^{2p-\kappa/p}}\, dA(z) \\
&\leq C \int_0^1 (1-r)^{2p-2} \left(\log\frac{2e}{1-r}\right)^{\kappa/p} \int_{-\pi}^{\pi} \frac{1}{|1-re^{it}|^{2p-\kappa/p}}\, dt dr \\
&\leq C \int_0^1 (1-r)^{2p-2} \left(\log\frac{2e}{1-r}\right)^{\kappa/p} \frac{1}{(1-r)^{2p-1-\kappa/p}}\, dr \\
&\leq C \int_0^1 (1-r)^{-1+\kappa/p} \left(\log\frac{2e}{1-r}\right)^{\kappa/p} dr < \infty.
\end{aligned}
$$

This finishes the proof. □

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A. Baernstein II, Department of Mathematics, Washington University, St. Louis, MO 63130, USA

E-mail address: al@math.wustl.edu

D. Girela, Departamento de Análisis Matemático, Universidad de Málaga, Campus de Teatinos, 29071 Málaga, Spain

E-mail address: girela@uma.es

J. A. Peláez, Departamento de Análisis Matemático, Universidad de Málaga, Campus de Teatinos, 29071 Málaga, Spain

E-mail address: pelaez@anamat.cie.uma.es