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CODIMENSION 2 SUBSCHEMES OF PROJECTIVE SPACES WITH STABLE RESTRICTED TANGENT BUNDLE AND LIAISON CLASSES

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Introduction

Let $C \subset \mathbf{P}^n$ be a smooth complex curve. Several papers have considered the problem of the stability of the restricted tangent bundle $T\mathbf{P}^n \mid C$ (e.g., see [HK] and the references quoted there). Using several results and methods from linkage theory we first show how to construct, in any even linkage class, many curves in $\mathbf{P}^3$ such that $T\mathbf{P}^3 \mid C$ is stable. Then a corresponding result is given for even linkage classes of codimension 2 locally Cohen-Macaulay subschemes of $\mathbf{P}^n$. For the required background and main results on linkage theory a very good reading is found in [MP1] for the case $n = 3$ and [Mi] for the general case.

Recall that any even linkage class, $\boldsymbol{L}$, can be decomposed as a disjoint union $\boldsymbol{L} = \boldsymbol{L}_0 \cup \boldsymbol{L}_1 \cup \boldsymbol{L}_2 \cup \cdots$ based on the shift of the various elements of $\boldsymbol{L}$ (see [Mi, Def. 4.3.5]).

Here is our main result.

THEOREM 0.1. *Assume $\boldsymbol{P}^n = \boldsymbol{P}^n_{\boldsymbol{K}}$ where $\boldsymbol{K}$ is an algebraically closed field of characteristic* 0. *Let $\boldsymbol{L} = \boldsymbol{L}_0 \cup \boldsymbol{L}_1 \cup \boldsymbol{L}_2 \cup \cdots$ be an even linkage class of equidimensional codimension* 2 *locally Cohen-Macaulay subschemes of $\boldsymbol{P}^n$ (decomposed in the usual way with respect to shift). Then there is an integer x such that for all integers $t \geq x$ there is an integral locally Cohen-Macaulay subscheme $X_t \in \boldsymbol{L}_t$ with* $\dim(\mathrm{Sing}(X)) \leq n - 6$ *and with $T\boldsymbol{P}^n \mid X_t$ stable.*

In particular, if $n \leq 5$ then any X_t as in the statement of Theorem 0.1 is smooth. First we will prove Theorem 0.1 in the case $n = 3$. The general case will be similar, except for the terminology and except at one point in which we will reduce to the case $n = 3$. A nice feature of the theory of linkage is that we know several interesting examples, in particular in the case of space curves. We will point out these examples in a few remarks.

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1. Proofs and remarks

Recall that we always work over an algebraically closed base field $\mathbf{K}$ with char($\mathbf{K}$) $= 0$. We recall the following results concerning stable sheaves on reduced curves and linkage (or liaison).

Let T be a reduced curve. A torsion free sheaf B on T is said to have constant rank k if for every irreducible component Z of T $B \mid Z := B \otimes \mathbf{O}_Z$ has rank k at the generic point of Z. For every torsion free sheaf B of constant rank k on T, its degree, $\deg(B)$, is defined by the formula $\deg(B) := \chi(B) - k\chi(\mathbf{O}_T)$. According to [HK] a rank n vector bundle E on T is called semistable (resp. stable) if for all integers k with $1 \leq k < n$ and for all subsheaves B of E with constant rank k we have $\deg(B)/k \leq \deg(E)/n$ (resp. $\deg(B)/k < \deg(E)/n$). It was proved in [HK, Th. 2.4] that with these definitions the conditions of stability and semistability are open conditions for vector bundles of constant rank on flat families of reduced curves. We need a measure for the order of stability of a vector bundle on a reduced curve or on one of its irreducible components.

LEMMA 1.1. *Let $Z = U \cup V$ be a reduced curve with U smooth, V with only planar singularities, U intersecting V quasi-transversally, and let E be a rank n vector bundle on Z. Fix an integer k with $1 \leq k < n$. Assume the existence of integers u, v such that for all rank k subsheaves A of $E \mid U$ we have $n(\deg(A)) \leq k(\deg(E \mid U)) - u$ and that for all rank k subsheaves B of $E \mid V$ we have $n(\deg(B)) \leq k(\deg(E \mid V)) + v$. Then for all subsheaves F of E with constant rank k we have $\deg(F)/k \leq deg(E)/n + (v-u)/nk$. In particular if $u > v$ (resp. $u \geq v$) we have $\deg(F)/k < \deg(E)/n$ (resp. $\deg(F)/k \leq \deg(E)/n$); i.e., E is not destabilized by subsheaves with constant rank k.*

Proof. Let D be the scheme $U \cap V$. Set $x := \text{length}(U \cap V) := h^0(D, \mathbf{O}_D)$. Fix a subsheaf F of E with constant rank k and maximal degree. Let F' (resp. F'') be the saturated subsheaves of $E \mid U$ (resp. $E \mid V$) generated by $(F \mid U)/\text{Tors}(F \mid U)$ (resp. $(E \mid V)/\text{Tors}(E \mid V)$). Since F is torsion free we have the Mayer-Vietoris type exact sequence

$$0 \to F \to F' \oplus F'' \to F \otimes \mathbf{O}_D \to 0. \tag{1}$$

By the additivity of the Euler characteristic we have

$$\begin{aligned} n \cdot \deg(F) &= n\chi(F) - nk\chi(\mathbf{O}_Z) \\ &= n\chi(F') + n\chi(F'') - n(\chi(F \otimes \mathbf{O}_D) - nk\chi(\mathbf{O}_U) - nk\chi(\mathbf{O}_V) + nkx \\ &\leq n\chi(F') + n\chi(F'') - nkx - nk\chi(\mathbf{O}_U) - nk\chi(\mathbf{O}_V) + nkx \end{aligned}$$

$$
\begin{aligned}
&= n \cdot \deg(F') + n \cdot \deg(F'') \\
&\leq k \cdot deg(E \mid U) - u + k \cdot \deg(E \mid V) + v \\
&= k \cdot \deg(E) - u + v.
\end{aligned}
$$

Now divide thru by nk and the result follows. □

Let G, G' be vector bundles on $\mathbf{P}^n$. G and G' are said to be stably equivalent if there is an integer c and direct sums of line bundles, P and P', such that $G(c) \oplus P \cong G' \oplus P'$. Let X be a locally Cohen-Macaulay subscheme of $\mathbf{P}^n$ with pure codimension 2. The even linkage classes of such schemes are in bijection with the stable equivalence classes of vector bundles G on $\mathbf{P}^n$ such that $H^{n-1}(\mathbf{P}^n, G(t)) = 0$ for every $t \in \mathbf{Z}$ ([R2] or [Mi, Th. 5.1.3]). The bijection sends any even linkage class represented by a scheme X to the stable equivalence class of a vector bundle G whenever X has a resolution of the form

$$0 \to P \to G \oplus P' \to \boldsymbol{I}_X \to 0 \tag{2}$$

where P and P' are both direct sums of line bundles (a resolution of type N in the terminology of [MP1]). We have also a resolution of $\boldsymbol{I}_X$:

$$0 \to E \to Q \to \boldsymbol{I}_X \to 0 \tag{3}$$

where Q is the direct sum of line bundles and E is a vector bundle with $H^1(\boldsymbol{P}^n, E(t)) = 0$ for every $t \in \mathbf{Z}$ (a resolution of type E with the terminology of [MP1, p. 42]).

Let $R := \mathbf{K}[x_0, \dots, x_n]$ be the coordinate ring of $\boldsymbol{P}^n = \mathrm{Proj}(R)$. For any locally Cohen-Macaulay subscheme X of $\boldsymbol{P}^n$ with pure codimension 2, set $M^i(X) := \oplus_{t \in \mathbf{Z}} H^i(\mathbf{P}^n, \boldsymbol{I}_X(t))$, $1 \leq i \leq n-2$. Each $M^i(X)$ is an R-module of finite length. Furthermore, by the Hartshorne-Schenzel theorem (see [R1] for curves, [Sc, Cor. 5.3] or [Mi, Th. 4.3.1] in general) if X' is obtained from X by a linkage by two hypersurfaces of degree a and b, we have $M^{n-i-1}(X') \cong M^i(X)^*(n+1-a-b)$ as R-modules.

If $n = 3$ the even liaison classes are in natural bijection with the R-modules with finite length, up to shifts: to a locally Cohen-Macaulay space curve X is associated its Hartshorne-Rao module $M^1(X)$ ([R1], or [MP1, I.3], or [Mi]).

To prove Theorem 0.1 for $n = 3$ we will use the following result.

THEOREM 1.2. *Let C be a reduced (possibly not connected) space curve with only planar singularities. Let $\mathbf{M}$ be the Hartshorne-Rao module of C and $\mathbf{L}$ the even linkage class of C. $\mathbf{L}$ decomposes in the usual way as $\mathbf{L} = \mathbf{L}_0 \cup \mathbf{L}_1 \cup \mathbf{L}_2 \cup \cdots$. For each integer i in $\{1, 2\}$ we want to define an integer f_i. If $T\boldsymbol{P}^3 \mid C$ is not destabilized by rank i subsheaves then set $f_i = 0$. If $T\boldsymbol{P}^3 \mid C$ is destabilized by a rank i subsheaf then set f_i equal to the maximal degree of a subsheaf of $T\boldsymbol{P}^3 \mid C$ with constant rank i. Since $T\boldsymbol{P}^3(-1) \mid C$ is spanned we have $f_i \leq \deg(C)$ for all i. Fix integers a, b with $2 \leq a \leq b-3$, $b > f_1$ and $b(2a-3) > 2f_2$. Assume that $H^0(\boldsymbol{P}^3, \boldsymbol{I}_C(a))$ generates $\boldsymbol{I}_C(a)$ and assume $C \in \mathbf{L}_t$. Then there is a curve $Y \in \mathbf{L}_{t+b}$ such that $T\boldsymbol{P}^3 \mid Y$ is stable and such that Y is a basic double link of C via two surfaces of degree a and b.*

Proof. Let W be the linear system of all degree a surfaces containing C. By assumption W has C as its scheme theoretic locus. Hence by Bertini's theorem a general $S \in W$ is smooth outside C. Since C has only planar singularities, for every $P \in C$ there is a Zariski open subset $U(P)$ of W, $U(P) \neq \emptyset$, such that every $F \in U(P)$ is smooth at P. Since $\boldsymbol{I}_C(a)$ is generated by global sections, the twisted normal bundle $N_C(a)$ is generated by the global sections and a general $S \in W$ is smooth. Fix any smooth $S \in W$. Take a general degree b surface F and set $T := S \cap F$. By [Pa, Example 4.1], T has gonality at least $b(a-1)$. To finish off the proof, apply Lemma 1.1 with $U = T$, $V = C$, $Y = Z$ and for $k = 1$ with $u = b$, $v = f_1$ and for $k = 2$ with $u = b(2a-3)$, $v = f_2$. $\square$

Proof of Theorem 0.1 *for* $n = 3$. By [R1], there is a smooth scheme C in the linkage class $\mathbf{L}$. Let Y be any such one and let

$$0 \to E \to Q \to \boldsymbol{I}_C \to 0 \tag{4}$$

be a locally free resolution of $\boldsymbol{I}_C$ with Q a direct sum of line bundles and $H^1(\boldsymbol{P}^3, E(z)) = 0$ for every integer z. We apply to C the construction of the proof of Theorem 1.2, i.e., a basic double link with surfaces of degree a, b with $2 \leq a \leq b-3$. Hence for all integers $a \geq \deg(C)$ we obtain a nodal curve $Y := C \cup T$ which has the following resolution of type E [Mi, p. 101]:

$$0 \to E(-b) \oplus \boldsymbol{O}_{\mathbf{P}^3}(-a-b) \to Q(-a) \oplus \boldsymbol{O}_{\mathbf{P}^3}(-a) \to \boldsymbol{I}_Y \to 0 \tag{5}$$

Note that if b and $b-a$ are large (for fixed E and Q, i.e., for fixed C), the vector bundle $\mathrm{Hom}(E(-b) \oplus \boldsymbol{O}_{\mathbf{P}^3}(-a-b), Q(-a) \oplus \boldsymbol{O}_{\mathbf{P}^3}(-a))$ is spanned by its global sections. Hence by Bertini's theorem [K] a general homomorphism $f\colon E(-b) \oplus \boldsymbol{O}_{\mathbf{P}^3}(-a-b) \to Q(-a) \oplus \boldsymbol{O}_{\mathbf{P}^3}(-a)$ has as cokernel the ideal sheaf of a smooth curve. By the openness of stability in a flat family of reduced curves [HK, Th. 2.4], the general such X is smooth and $T\mathbf{P}^3 \mid X$ is stable. Note that Y and X have a Hartshorne-Rao module shifted by $-b$ with respect to the one of C. Since as b we may take (for fixed large a) every sufficiently large integer, we conclude.

Remark 1.3. Using [Pa, Remark 4.1 and Example 4.2], one can obtain lower bounds on the gonality of a smooth curve linked in one step to another curve. As in the proof of Theorem 1.2 this gives information about and examples of stable restricted tangent bundles for every full (odd and even) equivalence class of liaison.

Remark 1.4. Here we assume $n = 3$. The literature contains descriptions of the type E and type N resolutions for many special families of curves. One would like to find the minimal shift x in the statement of Theorem 0.1 from which the construction outlined in the proof of Theorem 0.1 gives smooth curves with stable restricted tangent bundle. For this type of results the filtered form of Bertini's theorem proved in [Ch] is extremely useful. To find examples and criteria for the existence of smooth space curves at a given shift of an even linkage class $\boldsymbol{L}$, see [MP2]. It is very interesting to know when there are such examples at the minimal shift because the curves in the minimal shift produce, up to basic double links and deformations, every curve in the even linkage class $\boldsymbol{L}$ (see [BBM] or [MP1, Ch. IV, Th. 4.5 and 5.1], or [Mi]). For such smooth examples at the minimal shift in the case of arithmetically Buchsbaum curves the Hartshorne-Rao module associated to $\boldsymbol{L}$ must have diameter less than 3 (see [BM, §2]). For the case of curves with non-special hyperplane section, see [LR].

1.5. We will recast the notion of Liaison Addition due to P. Schwartau (see [Mi, 3.2], [SW] or [GM] for much more) in the set up of the proof of Theorem 1.2. We start with two space curves C' and C'' and with two surfaces S' and S'' (say $S' := \{f' = 0\}$, $S'' := \{f'' = 0\}$ with $\deg(f') = a'$, $\deg(f'') = a''$) with $\dim(S' \cap S'') = 1$, $C' \subset S''$, $C'' \subset S'$. Let Z be the 1-dimensional locally Cohen-Macaulay subscheme of $\mathbf{P}^3$ with ideal sheaf $f'\boldsymbol{I}_{C'} + f''\boldsymbol{I}_{C''}$ (hence $Z_{\text{red}} = C' \cup C'' \cup (S' \cap S'')$). Then Z has as Hartshorne-Rao module $M^1(Z) := M^1(C')(-a') \oplus M^1(C'')(-a'')$. Hence, if C' and C'' are smooth and a' and a'' are sufficiently large, we may take S' and S'' transversal and apply the proof of Theorem 1.2. This construction applies in particular (see [SW] or [Mi, 3.2]) to the construction of arithmetically Buchsbaum curves.

Now we consider the case of the codimension 2 subschemes of $\mathbf{P}^n$ with $n > 3$. Let X be a projective scheme on which is defined a notion of degree, rank and stability for vector bundles on X. One such case is when X is an integral variety with a choice of a polarization H. The degree of line bundles and torsion free sheaves can be defined with respect to H and in this way we can also define stability. If Z is a reduced subscheme of $\mathbf{P}^n$ with pure codimension 2 and D is a Weil divisor on Z then we will let $\deg(D)$ denote the intersection product of D with $n-3$ copies of the class of the Cartier divisor $\boldsymbol{O}_Z(1)$. In this way we may define the degree of the determinant of a subsheaf F of $T\mathbf{P}^n \mid Z$ when F has the same rank on all irreducible components of Z. From this definition of degree we can then give an appropriate definition of stability.

Proof of 0.1 *for* $n > 3$. We fix the even linkage class $\boldsymbol{L}$. By [R2, 1.10], there is an integral locally Cohen-Macaulay scheme C in the class of $\boldsymbol{L}$ with $\dim(\mathrm{Sing}(C)) \leq n-6$. We fix any such C and we choose integers a, b such that $3 \leq a \leq b-3$, $\deg(C) < b(a-2)$ and such that $H^0(\mathbf{P}^n, \boldsymbol{I}_C(a))$ generates $\boldsymbol{I}_C(a)$. As in the case $n = 3$, we take a general hypersurface S of degree a, a general hypersurface F of degree b and we make the basic double link with respect to S and F. We obtain a pure codimension 2 reduced subvariety $Y = C \cup T$ with $T := S \cap F$ a complete intersection. By [PS, 4.1], we may assume $\dim(\mathrm{Sing}(T)) \leq n-6$.

Claim. We have $\deg(L) \geq b(a-1)$ for every non trivial rank 1 torsion free sheaf L on T which is locally free outside a set of dimension $n-4$ and which is spanned by its global sections.

Proof of the claim. Take a general linear section $T \cap M$ of T with a projective space M with $\dim(M) = 3$. As in the case $n = 3$ apply [Pa, Example 4.1], to $T \cap M$.

Fix an integer k with $1 \leq k \leq n-1$. Let F be a saturated subsheaf of $T\mathbf{P}^n(-1) \mid Y$ of constant rank k and with $\deg(F)$ maximal. Set $Q := (T\mathbf{P}^n(-1) \mid Y)/F$. Since Q is spanned, by the claim we have $\deg((Q \mid T)/\operatorname{Tors}(Q \mid T)) \geq b(a-1)$. Since $T\mathbf{P}^n(-1) \mid C$ is spanned, we have $\deg((Q \mid C)/\operatorname{Tors}(Q \mid C)) \geq 0$. Hence we have the conclusion because $\deg(C) < b(a-2)$.

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