HOMOLOGY OF FUNCTION SPECTRA

BY

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1. Introduction

The purpose of this paper is the investigation of the singular homology groups of $F(X, Y)$, the space of base-point preserving maps from X to Y endowed with the compact-open topology [7]. We restrict ourselves to the case where X is compact and Y has the homotopy type of a countable CW-complex.

This problem was investigated by Borsuk [2] who studied the first nonzero Betti number of $F(X, S^m)$. Later, Moore [16] calculated the reduced singular integral homology groups $\tilde{H}_n(F(X, S^m))$ in the stable range. (We suppress notation of the coefficient group in case of integer coefficients.) Moore's result as restated by Spanier [20] in the language of spectra [13] says that $\tilde{H}_{-n}(\mathbf{F}(X, \mathbf{S})) \approx \tilde{H}^n(X)$, where $\mathbf{S}$ is the spectrum of spheres.

The crucial part of Moore's proof is that the homology of $F(X, S^m)$ defines a cohomology theory on X in the stable range. It is shown here that if $\mathbf{E}$ is a spectrum, the groups $\bar{H}^n(X) = \tilde{H}_{-n}(\mathbf{F}(X, \mathbf{E}))$ define a generalized cohomology theory [24] for finite complexes X. From a theorem of Brown [3], it follows that there is a spectrum $\mathbf{F}$ such that $\bar{H}^n(X) \approx \tilde{H}^n(X; \mathbf{F})$, the nth cohomology group of X with coefficients in the spectrum $\mathbf{F}$. This implies that the homotopy groups of $\mathbf{F}$ are isomorphic to the homology groups of $\mathbf{E}$. This suggests that $\mathbf{F}$ might be the infinite symmetric product of $\mathbf{E}$ and this is indeed the case. A final calculation arrives at the formula (Theorem (7.8)):

$$\tilde{H}_n(\mathbf{F}(X, \mathbf{E})) \approx \textstyle\sum_r \tilde{H}^{r-n}(X; \tilde{H}_r(\mathbf{E})).$$

It is assumed that the reader is familiar with the results and notation of Sections 1 through 5 of [24] which present the basic notions of spectra and generalized cohomology theories.

Sections 2 and 3 present elementary results on spectra and generalized cohomology theories. In Section 4 it is proved that the groups $\bar{H}^n(X)$ define a generalized cohomology theory for finite complexes. Section 5 introduces the notion of the infinite symmetric product $SP^\infty\mathbf{E}$ of a spectrum $\mathbf{E}$ and in Section 6 it is proved that $\bar{H}^n(X) \approx \tilde{H}^n(X; SP^\infty\mathbf{E})$. Sections 7 and 8 are devoted to the calculation of $\tilde{H}^n(X; SP^\infty\mathbf{E})$. In the final section, the results are applied to function spaces (rather than function spectra) and to the case where X is an arbitrary compact space.

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2. Preliminaries

In this section we discuss some elementary properties of spectra that will be of use in the sequel.

In this paper, space will mean Hausdorff space with base-point and map will mean continuous, base-point preserving map. In fact, unless otherwise noted, spaces and maps will be assumed to belong to the category $\mathcal{W}_0$ whose objects are spaces having the homotopy type of a countable CW-complex with base-vertex and whose morphisms are all base-point preserving maps. Further, all n-ads will be assumed to have the homotopy type of CW-n-ads. The properties of this category have been discussed in [15]. We will say that a spectrum $\mathbf{E} = \{E_n, \varepsilon_n\}$ is in $\mathcal{W}_0(\mathbf{E} \in \mathcal{W}_0)$ if each $E_n \in \mathcal{W}_0$. The full subcategory of $\mathcal{W}_0$ whose objects are countable CW-complexes is denoted by C_0.

We first sketch a proof of an analogue of the theorem of J. H. C. Whitehead [25] for maps of spectra. This is an unpublished "folk theorem", proved by D. M. Kan among others.

Let $\mathbf{E} = \{E_n, \varepsilon_n\}$ be a spectrum and let $\mathbf{F} = \{F_n, \varphi_n\}$ be a subspectrum of $\mathbf{E}$, that is, (E_n, F_n) is a pair in $\mathcal{W}_0$ and $\varphi_n = \varepsilon_n \mid (S \wedge F_n)$. One has the usual exact homotopy and homology sequences for the pair $(\mathbf{E}, \mathbf{F})$ [23]. The maps ε_n determine maps $\psi_n : S \wedge (E_n/F_n) \to E_{n+1}/F_{n+1}$ which makes $\{E_n/F_n, \psi_n\}$ into a spectrum $\mathbf{E}/\mathbf{F}$. Let $\mathbf{p} : \mathbf{E} \to \mathbf{E}/\mathbf{F}$ and $\mathbf{p}' : (\mathbf{E}, \mathbf{F}) \to \mathbf{E}/\mathbf{F}$ be the maps of spectra induced by the projections $p_k : E_k \to E_k/F_k$ and let $\mathbf{j} : \mathbf{E} \to (\mathbf{E}, \mathbf{F})$ be the inclusion.

PROPOSITION (2.1). *The diagrams*

$$\begin{array}{ccc} \pi_n(\mathbf{E}) & \xrightarrow{\mathbf{j}_\#} & \pi_n(\mathbf{E}, \mathbf{F}) \\ & {\scriptstyle \mathbf{p}_\#} \searrow \quad \swarrow {\scriptstyle \mathbf{p}'_\#}\ \approx & \\ & \pi_n(\mathbf{E}/\mathbf{F}) & \end{array} \qquad \begin{array}{ccc} \tilde{H}_n(\mathbf{E}) & \xrightarrow{\mathbf{j}_*} & \tilde{H}_n(\mathbf{E}, \mathbf{F}) \\ & {\scriptstyle \mathbf{p}_*} \searrow \quad \swarrow {\scriptstyle \mathbf{p}'_*}\ \approx & \\ & \tilde{H}_n(\mathbf{E}/\mathbf{F}) & \end{array}$$

are commutative and both $\mathbf{p}'_\#$ *and* $\mathbf{p}'_*$ *are isomorphisms.*

Proof. Commutativity is obvious as is the fact that $\mathbf{p}'_*$ is an isomorphism. That $\mathbf{p}'$ is an isomorphism can be proved using the Blakers-Massey triad theorem [1] and the technique used in the proof of (4.1) below.

PROPOSITION (2.2). *Let* $\mathbf{E}$ *be a spectrum. Then if* $\pi_n(\mathbf{E}) = 0$ *for all* n, $\tilde{H}_n(\mathbf{E}) = 0$ *for all* n.

Proof. It may be assumed that $\mathbf{E}$ is a semi-simplicial group spectrum [11]. The hypothesis that $\pi_*(\mathbf{E}) = 0$ then implies that E_n has as a deformation retract the subcomplex generated by the base-point. It follows that $\tilde{H}_n(\mathbf{E}) = 0$ for all n.

Remark. The converse of (2.2) holds when $\mathbf{E}$ is convergent [24, p. 242], but not in general. D. M. Kan has supplied a counter-example (unpublished). Hurewicz theorems describing the first non-zero homotopy and homology groups of a spectrum have also been proved with a similar caution about convergence.

PROPOSITION (2.3). *Let* $\mathbf{f} : \mathbf{E} \to \mathbf{F}$ *be a map of spectra in* $\mathcal{W}_0$. *Then if* $\mathbf{f}_{\#} : \pi_n(\mathbf{E}) \to \pi_n(\mathbf{F})$ *is an isomorphism for every* n, $\mathbf{f}_* : \tilde{H}_n(\mathbf{E}) \to \tilde{H}_n(\mathbf{F})$ *is an isomorphism for every* n.

Proof. This follows from (2.1) and (2.2) using standard mapping cylinder techniques [25] as adapted to spectra.

We conclude this section with a discussion of loop spectra. Let $\mathbf{E} = \{E_n, \varepsilon_n\}$ be a spectrum $\in \mathcal{W}_0$. The spectrum E may be described equally well using the adjoint maps [10] $\tilde{\varepsilon}_n : E_n \to \Omega E_{n+1}$ associated to the maps $\varepsilon_n : S \wedge E_n \to E_{n+1}$. $\Omega\mathbf{E} = \mathbf{F}(S, \mathbf{E})$ is the loop-spectrum of $\mathbf{E}$ [24, p. 242]. $\Omega\mathbf{E}$ has maps

$$\tilde{\varphi}_k : F(S, E_k) \to F(S, F(S, E_{k+1})) = \Omega^2 E_{k+1}$$

defined by $\tilde{\varphi}_k(\lambda)(s)(t) = \tilde{\varepsilon}_k(\lambda(t))(s)$, where $\lambda \in F(S, E_k)$ and $s, t \in S$.

Let $g : Y \to Z$. We write $\Omega g = F(1, g) : \Omega Y \to \Omega Z$. The following lemma is easily proved.

LEMMA (2.4). *The following diagrams are anti-commutative*:

$$\begin{array}{ccc}
\pi_{n+k}(E_k) & \xrightarrow{\tilde{\varepsilon}_{k\#}} & \pi_{n+k}(\Omega E_{k+1}) \\
\Big\downarrow{\scriptstyle \tilde{\varepsilon}_{k\#}} & & \Big\downarrow{\scriptstyle (\Omega\tilde{\varepsilon}_{k+1})_{\#}} \\
\pi_{n+k}(\Omega E_{k+1}) & \xrightarrow{\tilde{\varphi}_{k+1\#}} & \pi_{n+k}(\Omega^2 E_{k+2})
\end{array}$$

$$\begin{array}{ccccc}
\tilde{H}_{n+k}(E_k) & \xrightarrow{\tilde{\varepsilon}_{k*}} & \tilde{H}_{n+k}(\Omega E_{k+1}) & \xrightarrow{(\Omega\tilde{\varepsilon}_{k+1})_*} & \tilde{H}_{n+k}(\Omega^2 E_{n+2}) \\
\Big\downarrow{\scriptstyle \tilde{\varepsilon}_{k*}} & & & & \Big\downarrow{\scriptstyle \eta_*} \\
\tilde{H}_{n+k}(\Omega E_{k+1}) & \xrightarrow{\tilde{\varphi}_{k+1*}} & \tilde{H}_{n+k}(\Omega^2 E_{n+2}) & \xrightarrow{\eta_*} & \tilde{H}_{n+k+1}(\Omega E_{n+2})
\end{array}$$

where η_* *is the homology suspension for path fibrations.*

In addition to the usual definition of the groups $\tilde{H}_n(\mathbf{E})$ [24, p. 245], they are also given as the direct limit of sequences

$$\cdots \to \tilde{H}_{n+k}(E_k) \xrightarrow{\tilde{\varepsilon}_{k*}} \tilde{H}_{n+k}(\Omega E_{k+1}) \xrightarrow{\eta_*} \tilde{H}_{n+k+1}(E_{k+1}) \to \cdots.$$

From (2.4) it follows that the maps

$$\psi_k : \tilde{H}_{n+k}(\Omega E_{k+1}) \to \tilde{H}_{n+k}(\Omega E_{k+1}),$$

where $\psi_k(z) = (-1)^{k+1}\cdot z$, define an isomorphism $\sigma : \tilde{H}_n(\mathbf{E}) \approx \tilde{H}_{n-1}(\Omega\mathbf{E})$. σ is also defined by the homomorphisms

$$(-1)^{k+1}\cdot \tilde{\varepsilon}_{k*} : \tilde{H}_{n+k}(E_k) \to \tilde{H}_{n+k}(\Omega E_{k+1}).$$

Similarly, the maps

$$(-1)^{k+1}\cdot \tilde{\varepsilon}_{k\#} : \pi_{n+k}(E_k) \to \pi_{n+k}(\Omega E_{k+1})$$

define an isomorphism $\theta : \pi_n(\mathbf{E}) \approx \pi_{n-1}(\Omega\mathbf{E})$. It is easily shown that θ^{-1} coincides with the isomorphism $\omega : \pi_{n-1}(\Omega\mathbf{E})\pi_n(\mathbf{E})$ defined by G. W. Whitehead [24, p. 245].

3. Cohomology theories

We recall the definition of a generalized cohomology theory [24]. Let $\mathcal{P}_0$ be the category of finite CW-complexes with base-vertex and base-point preserving maps. A generalized cohomology theory $\tilde{\mathcal{H}}^*$ on $\mathcal{P}_0$ consists of a sequence of contravariant functors

$$\tilde{H}^n : \mathcal{P}_0 \to \mathcal{A},$$

where $\mathcal{A}$ is the category of abelian groups and homomorphisms, and a sequence of natural transformations

$$\sigma^n : \tilde{H}^{n+1} \circ S \to \tilde{H}^n,$$

S being the suspension functor on $\mathcal{P}_0$, satisfying the following axioms:

(C_1). If $f_0, f_1 \in \mathcal{P}_0$ are homotopic maps, then

$$\tilde{H}^n(f_0) = \tilde{H}^n(f_1).$$

(C_2). If $X \in \mathcal{P}_0$, then

$$\sigma^n(X) : \tilde{H}^{n+1}(SX) \approx \tilde{H}^n(X).$$

(C_3). If (X, A) is a pair in $\mathcal{P}_0$, $i : A \subset X$, and if $p : X \to X/A$ is the identification map, then the sequence

$$\tilde{H}^n(X/\mathrm{A}) \xrightarrow{\tilde{H}^n(p)} \tilde{H}^n(X) \xrightarrow{\tilde{H}^n(i)} \tilde{H}^n(\mathrm{A})$$

is exact.

Generalized cohomology theories frequently arise as follows: Let $\mathbf{E}$ be a spectrum and let $X \in \mathcal{P}_0$. Define

$$\tilde{H}^n(X; \mathbf{E}) = \pi_{-n}(\mathbf{F}(X, \mathbf{E}))$$

where $\mathbf{F}(X, \mathbf{E})$ is the function spectrum of base-point preserving maps of X into the spaces E_n [24, §4, Ex. 6]. If $f : X \to Y$, then the maps

$$F(f, 1) : F(Y, E_k) \to F(X, E_k)$$

define a map $\mathbf{f}' : \mathbf{F}(Y, \mathbf{E}) \to \mathbf{F}(X, \mathbf{E})$. Define

$$\tilde{H}^n(f) = \mathbf{f}'_{\#} : \pi_{-n}(\mathbf{F}(Y, \mathbf{E})) \to \pi_{-n}(\mathbf{F}(X, \mathbf{E})).$$

For $X \in \mathcal{P}_0$, $\sigma^n(X) : \tilde{H}^{n+1}(SX; \mathbf{E}) \approx \tilde{H}^n(X; \mathbf{E})$ is defined to be the composition

$$\pi_{-n-1}(\mathbf{F}(SX, \mathbf{E})) \xrightarrow{\psi_\#} \pi_{-n-1}(\Omega\mathbf{F}(X, \mathbf{E})) \xrightarrow{\theta^{-1} = \omega} \pi_{-n}(\mathbf{F}(X, \mathbf{E})),$$

where $\psi : \mathbf{F}(SX, \mathbf{E}) \to \Omega\mathbf{F}(X, \mathbf{E})$ is the natural isomorphism [24, §4, Ex. 6].

PROPOSITION (3.1). *$\tilde{\mathcal{H}}^*(E) = \{\tilde{H}^n, \sigma^n\}$ is a generalized cohomology theory on $\mathcal{P}_0$.*

Proof. [24]. We remark that the blanket assumption made in [24] that all the spaces E_k have the homotopy type of a CW-complex was not necessary for the proof of this particular result. The spaces E_k may be arbitrary.

Remark. A theorem of E. H. Brown [3] states that any generalized cohomology theory $\tilde{\mathcal{H}}^*$, for which $\tilde{H}^*(S^0)$ is countable, is naturally equivalent with $\tilde{\mathcal{H}}^*(\mathbf{E})$ for some spectrum $\mathbf{E}$.

The following is a well-known "folk-theorem."

PROPOSITION (3.2). *Let $\tilde{\mathcal{H}}^*$ and $\tilde{\mathcal{G}}^*$ be two generalized cohomology theories on $\mathcal{P}_0$ and let $T : \tilde{\mathcal{H}}^* \to \tilde{\mathcal{G}}^*$ be a natural transformation of cohomology theories. Then, if $T^n(S^0)$ is an isomorphism for every n, T is a natural equivalence.*

Proof. To show that $T(X)$ is an isomorphism, one proceeds by induction on the number of cells in X. The proof is similar to Moore's proof of Theorem 3 in [16].

Example (3.3). Let $\mathbf{E}$ and $\mathbf{E}'$ be spectra and let $\mathbf{g} : \mathbf{E} \to \mathbf{E}'$ be a weakly continuous (w.c.) map of spectra (each g_k is w.c., that is, continuous on compact subsets). Then $\mathbf{g}$ induces a natural transformation of cohomology theories via

$$\mathbf{F}(1, \mathbf{g})_\# : \pi_*(\mathbf{F}(X, \mathbf{E})) \to \pi_*(\mathbf{F}(X, \mathbf{E}'))$$

since w.c. maps induce homomorphisms of homotopy groups. In this case $T(S^0)$ may be identified with $\mathbf{g}_\# : \pi_*(\mathbf{E}) \to \pi_*(\mathbf{E}')$.

PROPOSITION (3.4). *Let $\mathbf{g} : \mathbf{E} \to \mathbf{E}'$ be a map of spectra $\in \mathcal{W}_0$ such that $\mathbf{g}_\# : \pi_*(\mathbf{E}) \approx \pi_*(\mathbf{E}')$. Then*

$$\mathbf{F}(1, \mathbf{g})_* : \tilde{H}_*(\mathbf{F}(X, \mathbf{E})) \approx \tilde{H}_*(\mathbf{F}(X, \mathbf{E}')) \quad \textit{for all } X \in \mathcal{P}_0 .$$

Proof. By (3.2) and (3.3), $\mathbf{F}(1, \mathbf{g})_\# : \pi_*(\mathbf{F}(X, \mathbf{E})) \to \pi_*(\mathbf{F}(X, \mathbf{E}'))$ is an isomorphism. By [15], $\mathbf{F}(X, \mathbf{E})$ and $\mathbf{F}(X, \mathbf{E}')$ are in $\mathcal{W}_0$. The proposition now follows from (2.3).

4. $\bar{H}^*(X)$ as a generalized cohomology theory

We have already made the definition $\bar{H}^n(X) = \tilde{H}_{-n}(\mathbf{F}(X, \mathbf{E}))$. In this section we shall show that the functors $\bar{H}^n$ determine a generalized cohomology theory $\bar{\mathcal{H}}^*(E)$ on $\mathcal{P}_0$.

We assume $\mathbf{E} \in \mathcal{W}_0$. The natural transformations of functors

$$\bar{\sigma}^n : \bar{H}^{n+1} \circ S \to \bar{H}^n$$

are defined by the composites

$$\tilde{H}_{-n-1}(\mathbf{F}(SX, \mathbf{E})) \xrightarrow{\psi_*} \tilde{H}_{-n-1}(\Omega\mathbf{F}(X, \mathbf{E})) \xrightarrow{\sigma^{-1}} H_{-n}(\mathbf{F}(X, \mathbf{E})),$$

where σ is the isomorphism described in Section 2. Naturality of $\bar{\sigma}^n$ is clear.

THEOREM (4.1). *$\bar{\mathcal{H}}^*(\mathbf{E}) = \{\bar{H}^n, \bar{\sigma}^n\}$ is a generalized cohomology theory on $\mathcal{P}_0$.*

Proof. Axioms (C_1) and (C_2) are easily verified. It remains to prove that the exactness axiom (C_3) is satisfied. Let (X, A) be a pair in $\mathcal{P}_0$ and let $i : A \subset X$ and $p : X \to X/A$ be the canonical maps.

LEMMA (4.2). *$\bar{\imath}_k = F(i, 1) : F(X, E_k) \to F(A, E_k)$ is a fibre map in the sense of Serre [19] and $\bar{p}_k = F(p, 1) : F(X/A, E_k) \to F(X, E_k)$ is the inclusion of the fibre into the total space.*

Proof. [16, pp. 200–201].

LEMMA (4.3). *Suppose $d \geq \dim X$ and that Y is $(n - 1)$-connected, where $n > d$. Then $F(X, Y)$ is $(n - d - 1)$-connected.*

Proof. We use the fact that $\pi_k(F(X, Y) \approx \pi_0(F(S^k \wedge X, Y))$. If $(\dim X) + k < n$, then $\pi_k(F(X, Y)) \approx \pi_0(F(S^k \wedge X, Y)) = 0$ by standard obstruction theory.

$\bar{H}^n(i) \circ \bar{H}^n(p) = 0$ since $p \circ i$ is the constant map. To complete the proof of (4.1) it remains to be shown that if $\mathbf{u} \in \tilde{H}_{-n}(\mathbf{F}(X, \mathbf{E}))$ is in the kernel of $\bar{H}^n(i)$, then $\mathbf{u}$ is in the image of $\bar{H}^n(p)$. Let $u \in \tilde{H}_{k-n}(F(X, E_k))$ be a representative of $\mathbf{u}$. We may assume k is chosen so large that $\bar{\imath}_{k*} u = 0$.

Define a new spectrum $\mathbf{E}' = \{E'_r, \varepsilon'_r\}$ by

(4.4) $E'_i = E_i$ for all $i \leq k$,

(4.5) $E'_{k+j} = S^j \wedge E_k$ for all $j > 0$,

(4.6) $\varepsilon'_i = \varepsilon_i$ for all $i < k$,

(4.7) $\varepsilon'_{k+j} =$ identity for all $j \geq 0$.

It is easy to verify that the maps $g_r : E'_r \to E_r$ given by

(4.8) $g_i =$ identity for $i \leq k$

(4.9) g_{k+j} for $j > 0$ is the composite

$$S^j \wedge E_k \xrightarrow{S^{j-1}\varepsilon_k} S^{j-1} \wedge E_{k+1} \to \cdots \to S \wedge E_{k+j-1} \xrightarrow{\varepsilon_{k+j-1}} E_{k+j}$$

determine a map of spectra $\mathbf{g} : \mathbf{E}' \to \mathbf{E}$. The maps $\bar{g} = F(1, g_r)$ define a map of spectra $\bar{\mathbf{g}} : \mathbf{F}(Y, \mathbf{E}') \to \mathbf{F}(Y, \mathbf{E})$.

By (4.4) and (4.8) there is an element $u' \in \tilde{H}_{k-n}(F(X, E'_k))$ representing an element $\mathbf{u}' \in \tilde{H}_{-n}(\mathbf{F}(X, \mathbf{E}'))$ for which $\bar{g}_{k*} u' = u$ and hence, $\bar{\mathbf{g}}_* \mathbf{u}' = \mathbf{u}$. Let v' be the image of u' in $\tilde{H}_{k-n+N}(F(X, E'_{k+N}))$ where we choose

$N > \frac{1}{2}(n + k + 3 + 3d)$, with $d > \dim X$. v' also represents $\mathbf{u}'$. By (4.2), (4.3) and [19, III, Prop. 5] the sequence

$$(4.10)\quad \tilde{H}_{k-n+N}(F(X/A, E'_{k+N})) \xrightarrow{\bar{p}'_{k+N*}} \tilde{H}_{k-n+N}(F(X, E'_{k+N})) \xrightarrow{\bar{i}'_{k+N*}} \tilde{H}_{k-n+N}(F(A, E'_{k+N}))$$

is exact. Since $\bar{i}'_{k*}\, u' = 0$, we have $\bar{i}'_{k+N*}\, v' = 0$. By the exactness of (4.10), there is an element $w' \in \tilde{H}_{k-n+N}(F(X/A,\ E'_{k+N}))$ such that $\bar{p}'_{k+N*}\, w' = v'$. Let $\mathbf{w}'$ be the class of w' in $\tilde{H}_{-n}(\mathbf{F}(X/A,\ \mathbf{E}'))$. Then $\bar{\mathbf{p}}'_{*}\, \mathbf{w}' = \mathbf{u}'$. From the commutativity of

$$\begin{array}{ccc} \tilde{H}_{-n}(\mathbf{F}(X/A, \mathbf{E}')) & \longrightarrow & \tilde{H}_{-n}(\mathbf{F}(X, \mathbf{E}')) \\ \downarrow \bar{\mathbf{g}}_* & & \downarrow \bar{\mathbf{g}}_* \\ \tilde{H}_{-n}(\mathbf{F}(X/A, \mathbf{E})) & \longrightarrow & \tilde{H}_{-n}(\mathbf{F}(X, \mathbf{E})) \end{array}$$

it follows that $\mathbf{w} = \bar{\mathbf{g}}_*\, \mathbf{w}'$ is a class such that $\bar{\mathbf{p}}_*\, \mathbf{w} = \mathbf{u}$. Hence, $\mathbf{u}$ is in the image of $\bar{H}^n(p)$, completing the proof of (4.1).

5. Infinite symmetric products

In this section, we introduce the notion of the infinite symmetric product $SP^\infty\mathbf{E}$ of a spectrum $\mathbf{E}$. We show that the well-known theorem of Dold and Thom [4] that $\tilde{H}_n(X) \approx \pi_n(SP^\infty X)$ for "nice" spaces may be used to obtain an isomorphism $\tilde{H}_n(\mathbf{E}) \approx \pi_n(SP^\infty\mathbf{E})$.

We review the basic material about infinite symmetric products. For details see [4], [21]. Let $E \in \mathcal{W}_0$ and let e_0 be its base-point. The n-fold symmetric product SP^nE, $n > 0$, of E is the identification space E^n/G_n, where E^n is the n-fold cartesian product of E with itself and G_n, the symmetric group on n letters, acts on E^n by permuting coordinates. Thinking of the points of SP^nE as unordered n-tuples $\langle e_1, \cdots, e_n\rangle$ of points of $e_i \in E$, SP^nE may be imbedded as a closed subspace of $SP^{n+1}E$ by identifying $\langle e_1, \cdots, e_n\rangle$ with $\langle e_0, e_1, \cdots, e_n\rangle$. Writing $SP^0E = e_0$, we have

$$e_0 = SP^0E \subset SP^1E \subset \cdots \subset SP^nE \subset SP^{n+1}E \subset \cdots .$$

The union of the SP^nE is called $SP^\infty E$, the infinite symmetric product of E, assigned the base-point $\langle e_0\rangle$ and topologized by calling a set $C \subset SP^\infty E$ closed if and only if $C \cap SP^nE$ is closed for every $n \geq 0$. The multiplication $SP^\infty E \times SP^\infty E \to SP^\infty E$ defined by

$$(\langle e_1, \cdots, e_m\rangle, \langle e_{m+1}, \cdots, e_{m+n}\rangle) \to \langle e_1, \cdots, e_{m+n}\rangle$$

makes $SP^\infty E$ into a weak abelian monoid (WAM), that is, an abelian monoid whose product is weakly continuous. $SP^\infty E$ is free in the sense that if W is a WAM with unit w_0 and $g : E \to W$ such that $f(e_0) = w_0$, then f extends uniquely to w.c. homomorphism $\bar{g} : SP^{\infty}E \to W$. In particular,

$$g : E \to F \subset SP^{\infty}F$$

determines a map

$$SP^{\infty}g : SP^{\infty}E \to SP^{\infty}F,$$

which in this case is continuous.

PROPOSITION (5.1). *SP^{∞} is a functor which preserves homotopy and which takes $\mathcal{W}_0$ to $\mathcal{W}_0$.*

Proof. The only part of (5.1) not proved in [4] is the observation that if $E \in \mathcal{W}_0$, $SP^{\infty}E \in \mathcal{W}_0$. If $E \in \mathcal{W}_0$, then E has the homotopy type of a locally finite simplicial complex K [15]. By [4], [12] $SP^{\infty}E$ can be given the structure of a countable CW-complex. Since SP^{∞} is a functor which preserves homotopy, it follows that $SP^{\infty}E \in \mathcal{W}_0$.

PROPOSITION (5.2). *If $E \in \mathcal{W}_0$ is connected, there is a natural isomorphism*

$$\tau : \tilde{H}_q(E) \approx \pi_q(SP^{\infty}E)$$

for all q.

Proof. Use (5.1) and [21, (7.5)].

We now define the functor SP^{∞} for a spectrum $\mathbf{E} = \{E_k, \varepsilon_k\}$. We define maps

$$\rho_k : SSP^{\infty}E_k \to SP^{\infty}SE_k,$$

$$\tilde{\rho}_k : SP^{\infty}E_k \to \Omega SP^{\infty}SE_k$$

$$\rho_k(t \wedge \langle e_1, \cdots, e_n\rangle) = \langle t \wedge e_1, \cdots, t \wedge e_n\rangle,$$

$$\tilde{\rho}_k(\langle e_1, \cdots, e_n\rangle)(t) = \langle t \wedge e_1, \cdots, t \wedge e_n\rangle.$$

We then define maps

$$\alpha_k : SSP^{\infty}E_k \to SP^{\infty}E_{k+1} \quad \text{and} \quad \tilde{\alpha}_k : SP^{\infty}E_k \to \Omega SP^{\infty}E_{k+1}$$

by the compositions

$$\alpha_k : SSP^{\infty}E_k \xrightarrow{\rho_k} SP^{\infty}SE_k \xrightarrow{SP^{\infty}\varepsilon_k} SP^{\infty}{}_kE_{+1},$$

$$\tilde{\alpha}_k : SP^{\infty}E_k \xrightarrow{\tilde{\rho}_k} \Omega SP^{\infty}SE_k \xrightarrow{\Omega SP^{\infty}\varepsilon_k} \Omega SP^{\infty}E_{k+1}.$$

Then the α_k and $\tilde{\alpha}_k$ are adjoint maps which define a spectrum

$$SP^{\infty}\mathbf{E} = \{SP^{\infty}E_k, \alpha_k\}.$$

If $\mathbf{E}$, $\mathbf{F}$ are spectra, then a map $\mathbf{g} : \mathbf{E} \to \mathbf{F}$ induces a map

$$SP^{\infty}\mathbf{g} : SP^{\infty}\mathbf{E} \to SP^{\infty}\mathbf{F}.$$

LEMMA (5.3). *Let $E \in \mathcal{W}_0$ such that every E_k is connected. Then commtativity holds in the diagrams*

$$\begin{array}{ccccc} \tilde{H}_q(E_k) & & \xrightarrow{\quad\sigma_*\quad} & & \tilde{H}_{q+1}(SE_k) \\ \approx\Big\downarrow\tau & & & & \approx\Big\downarrow\tau \\ \pi_q(SP^\infty E_k) & \xrightarrow{S_*} & \pi_{q+1}(SSP^\infty E_k) & \xrightarrow{\rho_{k\#}} & \pi_{q+1}(SP^\infty SE_k) \end{array}$$

and

$$\begin{array}{ccccc} \tilde{H}_q(E_k) & & \xrightarrow{\quad\sigma_*\quad} & & \tilde{H}_{q+1}(SE_k) \\ \approx\Big\downarrow\tau & & & & \approx\Big\downarrow\tau \\ \pi_q(SP^\infty E_k) & \xrightarrow{k} & \pi_q(\Omega SP^\infty SE_k) & \xrightarrow{\eta_\#} & \pi_{q+1}(SP^\infty SE_k)\,. \end{array}$$

Proof. [21, (10.1)].

Remark (5.4). Let $\mathbf{E}$ be a spectrum and let $\mathbf{E}^0$ be the subspectrum of $\mathbf{E}$ for which E_k^0 is the path-component of the base-point of E_k. Then, since

$$\varepsilon_k(SE_k) \subset E_{k+1}^0 \quad\text{and}\quad \alpha_k(SSP^\infty E_k) \subset SP^\infty E_{k+1}^0,$$

the inclusions $\mathbf{i} : \mathbf{E}^0 \to \mathbf{E}$ and $j : SP^\infty\mathbf{E}^0 \to SP^\infty\mathbf{E}$ induce isomorphisms of homotopy and homology groups.

If $E \in \mathcal{W}_0$ is a spectrum for which each E_k is connected, then the isomorphisms

$$\tau : \tilde{H}_{n+k}(E_k) \approx \pi_{n+k}(SP^\infty E_k)$$

define an isomorphism $\boldsymbol{\tau}' : \tilde{H}_n(\mathbf{E}) \approx \pi_n(SP^\infty\mathbf{E})$. This follows from (5.3) and the commutativity of

$$\begin{array}{ccc} \tilde{H}_{q+1}(SE_k) & \xrightarrow{\quad\varepsilon_{k*}\quad} & \tilde{H}_{q+1}(E_{k+1}) \\ \approx\Big\downarrow\tau & & \approx\Big\downarrow\tau \\ \pi_{q+1}(SP^\infty SE_k) & \xrightarrow{(SP^\infty\varepsilon_k)_\#} & \pi_{q+1}(SP^\infty E_{k+1}) \end{array}$$

which follows from the naturality of τ. For any spectrum $\mathbf{E} \in \mathcal{W}_0$, we define a natural isomorphism $\boldsymbol{\tau} : \tilde{H}_n(\mathbf{E}) \approx \pi_n(SP^\infty\mathbf{E})$ by the composition of isomorphisms

$$\tilde{H}_n(\mathbf{E}) \xrightarrow{\mathbf{i}_*^{-1}} \tilde{H}_n(\mathbf{E}^0) \xrightarrow{\boldsymbol{\tau}'} \pi_n(SP^\infty\mathbf{E}^0) \xrightarrow{\mathbf{j}_\#} \pi_n(SP^\infty\mathbf{E}).$$

THEOREM (5.5). *For any spectrum $\mathbf{E} \in \mathcal{W}_0$ there is a natural isomorphism*

$$\boldsymbol{\tau} : \tilde{H}_n(\mathbf{E}) \approx \pi_n(SP^\infty\mathbf{E}).$$

6. A natural equivalence

In this section, we construct a natural transformation of cohomology theories $SP_{\#} : \mathfrak{H}^*(\mathbf{E}) \to \mathfrak{H}^*(SP^\infty E)$. The observation that $SP_{\#}(S^0)$ is an isomorphism then implies that $SP_{\#}$ is a natural equivalence.

For $X \in \mathcal{P}_0$, $SP_{\#}(X) : \tilde{H}^n(X) \to \tilde{H}^n(X; SP^\infty\mathbf{E})$ is given by the composite

$$\tilde{H}_{-n}(\mathbf{F}(X,\mathbf{E}) \xrightarrow{\tau} \pi_{-n}(SP^\infty\mathbf{F}(X, \mathbf{E})) \xrightarrow{\gamma_{\#}} \pi_{-n}(\mathbf{F}(X, SP^\infty\mathbf{E})) = \tilde{H}^n(X; SP^\infty\mathbf{E})$$

where the map $\gamma : SP^\infty\mathbf{F}(X, \mathbf{E}) \to \mathbf{F}(X, SP^\infty\mathbf{E})$ is defined by

$$\gamma_k\langle f_1, \cdots, f_n\rangle(x) = \langle f_1(x), \cdots, f_n(x)\rangle,$$

where $f_i \in F(X, E_k)$ and $x \in X$. Each γ_k is continuous since it is continuous on each finite symmetric product. It follows from the definitions of the maps involved that the diagrams

$$\begin{array}{ccc} SSP^\infty F(X, E_k) & \xrightarrow{S\gamma_k} & SF(X, SP^\infty E_k) \\ \downarrow \theta_k & & \downarrow \lambda_k \\ SP^\infty F(X, E_{k+1}) & \xrightarrow{\gamma_{k+1}} & F(X, SP^\infty E_{k+1}) \end{array}$$

are strictly commutative (not just homotopy-commutative), where θ_k and λ_k are the maps which define the spectra $SP^\infty\mathbf{F}(X, \mathbf{E})$ and $\mathbf{F}(X, SP^\infty\mathbf{E})$, respectively, thus showing that the γ_k do indeed define a map of spectra

$$\gamma : SP^\infty\mathbf{F}(X, \mathbf{E}) \to \mathbf{F}(X, SP^\infty\mathbf{E}).$$

We now wish to show that $SP_{\#}$ is a natural transformation of cohomology theories. Since τ and $\gamma_{\#}$ are both natural with respect to maps $X \to Y$, it only remains to show that $SP_{\#}$ commutes with suspension. This follows from the commutativity of the diagram

$$\begin{array}{ccccc} \tilde{H}_{k-n}(F(X, E_k) & \underset{\approx}{\xrightarrow{\tau}} & \pi_{k-n}(SP^\infty F(X, E_k)) & \xrightarrow{\gamma_{k\#}} & \pi_{k-n}(F(X, SP^\infty E_k)) \\ \downarrow (\psi_{k+1}^{-1} \circ \tilde{\varepsilon}_k)_* & & \downarrow (SP^\infty(\psi_{k+1}^{-1} \circ \varepsilon_k))_{\#} & & \downarrow (\psi_{k+1}^{-1} \circ \lambda_k))_{\#} \\ \tilde{H}_{k-n}(F(SX, E_{k+1})) & \underset{\approx}{\xrightarrow{\tau}} & \pi_{k-n}(SP^\infty F(SX, E_{k+1})) & \xrightarrow{\gamma'_{k+1\#}} & \pi_{k-n}(F(SX, SP^\infty E_{k+1})) \end{array}$$

where $\gamma' : SP^\infty\mathbf{F}(SX, \mathbf{E}) \to \mathbf{F}(SX, SP^\infty\mathbf{E})$ is the map defining $SP_{\#}(SX)$.

Observe now that if $X = S^0$, γ is an isomorphism of spectra. Since τ is always an isomorphism, it follows that $SP_{\#}(S^0)$ is an isomorphism. Hence, by (3.2) we have

THEOREM (6.1). *$SP_{\#} : \mathfrak{H}^*(\mathbf{E}) \to \mathfrak{H}^*(SP^\infty\mathbf{E})$ is a natural equivalence of cohomology theories.*

7. Calculation of $\tilde{\mathfrak{K}}^*(SP^\infty E)$

Theorem (6.1) has reduced the problem of calculating $\tilde{H}^n(X)$ to the calculation of $\tilde{H}^n(X;\ SP^\infty\mathbf{E})$. The calculation will proceed by showing that $SP^\infty\mathbf{E}$ is "essentially" a product of Eilenberg-MacLane spectra [24, §4, Ex. 6].

It is true that each $SP^\infty E_k$ may be split into a product of Eilenberg-MacLane spaces [17], but there is no guarantee that this splitting will be compatible with the maps $\tilde{\alpha}_k : SP^\infty E_k \to \Omega SP^\infty E_{k+1}$. We will first show that $SP^\infty E$ may be "replaced" by a spectrum $\mathbf{F} = \{F_k, \beta_k\}$ for which

$$\tilde{\beta}_{k\#} : \pi_r(F_k) \xrightarrow{\approx} \pi_r(\Omega F_{k+1})$$

for $r \geq 1$. It will then be possible to use the technique of Dold and Thom [4] to split each F_k° into a product of Eilenberg-MacLane spaces in such a fashion that this splitting is compatible with the maps $\tilde{\beta}_k$.

The method of constructing $\mathbf{F}$ will be analogous to the construction of the "infinite loop-space of the infinite suspension" of a space [5]. We first assume that each $\varepsilon_k : S \wedge E_k \to E_{k+1}$ is an inclusion. (That this assumption causes no problems will be proved in the following section.) Since each ε_k is an inclusion, so is each $\tilde{\rho}_k : SP^\infty E_k \to \Omega SP^\infty E_{k+1}$. Since each $SP^\infty E_k$ is a WAM, we may define a multiplication on $\Omega^r SP^\infty E_k \approx F(S^r,\ SP^\infty E_k)$ by the formula $(f\cdot g)(s) = f(s)\cdot g(s)$ for $f,\ g \in F(S^r,\ SP^\infty E_k)$ and $s \in S^r$. This multiplication makes $\Omega^r SP^\infty E_k$ into a WAM. The following lemma may be verified directly from the definitions:

LEMMA (7.1). *$\Omega^r\tilde{\rho}_k : \Omega^r SP^\infty E_k \to \Omega^{r+1} SP^\infty E_{k+1}$ is a monomorphism.*

Using (7.1), we form the union

$$F_k = SP^\infty E_k \cup_{\tilde{\rho}_k} \Omega SP^\infty E_{k+1} \cup_{\Omega\tilde{\rho}_{k+1}} \Omega^2 SP^\infty E_{k+2} \cup \cdots$$

and give this union the weak topology. It follows from (7.1) and the fact that F_{k+1} has the weak topology that the F_n are WAM's and that ΩF_{k+1} is isomorphic to F_k. Call this isomorphism $\tilde{\beta}_k$. It is possible that $F_k \in \mathcal{W}_0$, but this (if true) is not necessary to our arguments.

LEMMA (7.2). *$\mathbf{F} = \{F_k, \beta_k\}$ is an Ω-spectrum and the inclusion $\mathbf{i} : SP^\infty\mathbf{E} \to \mathbf{F}$ induces an isomorphism of homotopy groups.*

Proof. Let $\alpha \in \pi_r(SP^\infty\mathbf{E})$ be an element such that $\mathbf{i}_\# \alpha = 0$ and let $h : S^{r+k} \to SP^\infty E_k$ represent α, where k is chosen so large that $i_{k\#}[h] = 0$. It follows that the map $i \circ h : S^{r+h} = \dot{D}^{r+k+1} \to F_k$ can be extended to a map $H : D^{r+k+1} \to F_k$, where D^{r+k+1} is the $(r+k+1)$-disc having S^{r+k} as boundary $\dot{D}^{r+k+1}$. Since D^{r+k+1} is compact, $H\ (D^{r+k+1}) \subset \Omega^n SP^\infty E_{k+n}$ for some n. It follows that the image of $[h]$ in $\pi_{r+k+n}(SP^\infty E_{k+n})$ is 0 and hence that $\alpha = 0$. This proves that i is one-one. The proof that $\mathbf{i}_\#$ is onto is similar.

By (3.2) and (3.3), $\mathbf{i}$ induces a natural equivalence of cohomology theories

$T_1 : \tilde{\mathcal{K}}^*(SP^\infty \mathbf{E}) \to \tilde{\mathcal{K}}^*(F)$. We next show that $\mathbf{F}$ may be split into a product of Eilenberg-MacLane spectra.

If G is a countable abelian group, let $\mathcal{L}(G, q)$, where $q > 0$, denote the class of spaces $L \in \mathcal{C}_0$ such that, $H_r(L; Z) = 0$ if $r \neq q$ and $\pi_q(L) \approx H_q(L; Z) \approx G$. $\mathcal{L}(G, q)$ is non-empty [4, p. 278].

Define $G_n = \pi_n(SP^\infty \mathbf{E}) \approx \pi_n(\mathbf{F}) \approx \tilde{H}_n(E; Z)$. We construct spectra $\mathbf{Y}^n = \{Y_k^n, \zeta_k^n\}$ as follows: Let $Y^n \in \mathcal{L}(G_n, 1)$. We set $Y_k^n = S^{n+k+1} \wedge Y^n$. (Recall that S^r = base-point if $r < 0$.) Observe that if $n + k \geq 1$, $Y_k^n \in \mathcal{L}(G_n, n + k)$ and hence $SP^\infty Y_k^n$ is an Eilenberg-MacLane space of type $K(G_n, n + k)$. The maps $\zeta_k^n : S \wedge Y_k^n \to Y_{k+1}^n$ are defined to be the obvious inclusions.

DEFINITION (7.3). The weak cartesian product $\mathbf{P}_{i=q}^\infty X_i$ of the spaces X_i is defined after [4] as the union of the $\prod_{i=q}^n X_i$ with the weak topology, where we identify $\prod_{i=q}^n X_i$ with the subspace $(\prod_{i=q}^n X_i) \times \{x_{n+1}\}$ of $\prod_{i=q}^{n+1} X_i$, x_{n+1} being the base-point of X_{n+1}.

LEMMA (7.4). *Let $W = \mathbf{P}_{i=q}^\infty X_i$ and let K be an arbitrary compact space. Then $F(K, W)$ is naturally homeomorphic with $\mathbf{P}_{i=q}^\infty F(K, X_i)$ and*

$$[K, W] \approx \mathrm{Lim}_\to [K, \textstyle\prod_{i=q}^n X_i].$$

In particular, $\Omega W \approx \mathbf{P}_{i=q}^\infty \Omega X_i$.

Proof. The lemma is an elementary consequence of the observation that any compact subset of W is contained in $\prod_{i=q}^n X_i$ for some n. This is true because W was endowed with the weak topology.

Now consider the spectrum $\mathbf{W}^n = \{W_k^n, \eta_k^n\}$ where $W_k^n = SP^\infty Y_k^n$ and $\tilde{\eta}_k^n$ is defined by the formula

$$\tilde{\eta}_k^n \langle y_1, \cdots, y_r\rangle(t) = \langle \zeta_k^n(t \wedge y_1), \cdots, \zeta_k^n(t \wedge y_r)\rangle.$$

Set $W_k = \mathbf{P}_{n \in z} W_k^n$. Observe that since W_k^n is the base-point for $n < 1 - k$, this definition makes sense. Define maps $\tilde{\eta}_k : W_k\ \Omega\ \ W_{k+1}$ to be $\mathbf{P}_{n \in z} \tilde{\eta}_k^n$ using (7.4). This determines a spectrum $\mathbf{W} = \{W_k, \eta_k\}$.

We now wish to calculate $\tilde{H}^r(X; \mathbf{W})$. Since $\tilde{\eta}_k^n$ is a homotopy equivalence for $n + k \geq 1$ and W_k^n is a space of type $K(G_n, n + k)$, we have

$$\tilde{H}^r(X; \mathbf{W}^n) \approx \tilde{H}^{n+r}(X; G_n).$$

This and (7.4) imply the following:

PROPOSITION (7.5). $\tilde{H}^r(X; \mathbf{W}) \approx \sum_n \tilde{H}^{n+r}(X; \tilde{H}_n(\mathbf{E}))$, *this formula being natural for $X \in \mathcal{P}_0$.*

We now construct a map $\mathbf{W} \to \mathbf{F}$ which induces an isomorphism of homotopy groups. We first define maps $\varphi^n : \mathbf{Y}^n \to \mathbf{F}$. Let $\varphi_{1-n}^n : Y_{1-n}^n \to F_{1-n}$ be a map which induces an isomorphism of fundamental groups. If Y_k^n consists only of the base-point, let φ_k^n be the constant map. Otherwise, ζ_k^n is the identity map and we can define φ_{k+1}^n by requiring that the diagram

$$\begin{array}{ccc} S \wedge Y_k^n & \xrightarrow{S\varphi_k^n} & S \wedge F_k \\ \Big\downarrow \zeta_k^n & & \Big\downarrow \beta_k \\ Y_{k+1}^n & \xrightarrow{\varphi_{k+1}^n} & F_{k+1} \end{array} \tag{7.6}$$

be commutative.

Since F_k is a WAM, φ_k^n extends to a homomorphism

$$\psi_k^n : SP^\infty Y_k^n = W_k^n \to F_k$$

which is a w.c. map.

LEMMA (7.7). *The diagram*

$$\begin{array}{ccc} SP^\infty Y_k^n & \xrightarrow{\psi_k^n} & F_k \\ \Big\downarrow \tilde{\eta}_k^n & & \Big\downarrow \tilde{\beta}_k \\ \Omega SP^\infty Y_{k+1}^n & \xrightarrow{\Omega\psi_{k+1}^n} & \Omega F_{k+1} \end{array}$$

is strictly commutative.

Proof.

$$\begin{aligned} (\Omega\psi_{k+1}^n) \circ \tilde{\eta}_k^n \langle y_1, \cdots, y_n\rangle(t) &= \psi_{k+1}^n \langle \zeta_{k+1}^n(t \wedge y_1), \cdots, \zeta_k^n(t \wedge y_r) \\ &= \langle \varphi_{k+1}^n \zeta_k^n(t \wedge y_1), \cdots, \varphi_{k+1}^n \zeta_k^n(t \wedge y_r)\rangle \\ &= \langle \beta_k(t \wedge \varphi_k^n(y_1)(t), \cdots, \beta_k\varphi_k^n(y_r)(t)\rangle \\ &= \tilde{\beta}_k\langle \varphi_k^n(y_1), \cdots, \varphi_k^n(y_r)\rangle(t) \\ &= \tilde{\beta}_k\, \psi_k^n\langle y_1, \cdots, y_r\rangle(t) \end{aligned}$$

since $\tilde{\beta}_k$ is an isomorphism, Q.E.D.

It follows that the w.c. maps ψ_k^n define a w.c. map of spectra

$$\boldsymbol{\psi}^n : SP^\infty \mathbf{Y}^n \to \mathbf{F}$$

such that

$$\boldsymbol{\psi}_{\#}^n : \pi_n(SP^\infty \mathbf{Y}^n) \xrightarrow{\approx} \pi_n(\mathbf{F}).$$

Define $\boldsymbol{\psi} : \mathbf{W} \to \mathbf{F}$ by

$$\psi_k = \mathbf{P}_{n\in Z}\, \psi_k^n : \mathbf{P}_{n\in Z}\, W_k^n \to F_k .$$

It follows that $\boldsymbol{\psi}$ is a w.c. map of spectra which induces an isomorphism of homotopy groups. By (3.2) and (3.3), $\boldsymbol{\psi}$ induces a natural equivalence

$$T(\boldsymbol{\psi}) : \tilde{\mathfrak{K}}^*(\mathbf{W}) \to \tilde{\mathfrak{K}}^*(\mathbf{F}).$$

Thus the composite

$$\mathfrak{H}^*(\mathbf{E}) \xrightarrow{SP_\#} \mathfrak{H}^*(SP^\infty\mathbf{E}) \xrightarrow{T(\mathbf{i})} \mathfrak{H}^*(\mathbf{F}) \xrightarrow{T(\boldsymbol{\psi})^{-1}} \mathfrak{H}^*(\mathbf{W})$$

is a natural equivalence. This and (7.5) imply the following:

THEOREM (7.8). *There is a natural equivalence*

$$\tilde{H}_n(F(X;\mathbf{E})) \approx \sum_r \tilde{H}^{r-n}(X;\tilde{H}_r(\mathbf{E}))$$

defined for $X \in \mathcal{P}_0$.

8. Alterations of E

In the previous section, it was assumed that $\mathbf{E} = \{E_k, \varepsilon_k\}$ was a spectrum such that each ε_k was an inclusion. In this section we show that from the point of view of $\mathfrak{H}^*(\mathbf{E})$, $\mathbf{E}$ may always be "replaced" by such a spectrum.

"Replacement" of $\mathbf{E}$ by $\mathbf{Q}$ means the exhibiting of a natural equivalence $T : \mathfrak{H}^*(\mathbf{E}) \to \mathfrak{H}^*(\mathbf{Q})$. By (3.4), such a T is given by a map $\mathbf{E} \to \mathbf{Q}$ or $\mathbf{Q} \to \mathbf{E}$ which induces an isomorphism of homotopy groups.

Given $\mathbf{E} = \{E_k, \varepsilon_k\}$, let $\mathbf{F} = \{F_k, \varphi_k\}$ be the subspectrum of $\mathbf{E}$ for which $F_k = E_k$ if $k \geq 0$ and $F_k =$ base-point if $k < 0$. This inclusion $\mathbf{F} \to \mathbf{E}$ induces an isomorphism of homotopy groups. Hence $\mathbf{F}$ "replaces" $\mathbf{E}$. We now "replace" $\mathbf{F}$ by a spectrum $\mathbf{Q} = \{Q_k, \mu_k\}$ for which each μ_k is an inclusion.

If one has a map $f : X \to Y$, the (reduced) mapping cylinder $C(f)$ is defined as follows: Let I^+ be the disjoint union of the unit interval [0, 1] with a point p. $C(f)$ is to be the identification space obtained from $Y \cup (X \wedge I^+)$ via the identifications $\{f(x) \sim (x \wedge 1)\}$. Note that $S \wedge C(f)$ is homeomorphic with $C(Sf)$.

Let

$$(8.1) \qquad X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \to \cdots \to X_{n-1} \xrightarrow{f_{n-1}} X_n$$

be a sequence of spaces and maps. The compound mapping cylinder of the sequence (8.1) is defined as follows: We may view X_{n-1} as a subspace of $C(f_{n-1})$ and f_{n-2} as a map $f'_{n-2} : X_{n-2} \to C(f_{n-1})$. This defines $C(f'_{n-2})$. In general, view f_{n-k} as a map $f'_{n-k} : X_{n-k} \to C(f'_{n-k+1})$. The compound mapping cylinder of (8.1) is defined to be $C(f'_0)$.

Returning to the spectrum $\mathbf{F} = \{F_k, \varphi_k\}$, let Q_n be the compound mapping cylinder of the sequence

$$S^n F_0 \xrightarrow{S^{n-1}\varphi_0} S^{n-1}F_1 \to \cdots \to SF_{n-1} \xrightarrow{\varphi_{n-1}} F_n .$$

We may view $S \wedge Q_n$ as a closed subspace of Q_{n+1} and denote these inclusions by $\mu_n : S \wedge Q_n \subset Q_{n+1}$. F_n is a deformation retract of Q_n for every n. Hence the inclusions $g_n : F_n \subset Q_n$ define a map of spectra $\mathbf{g} : \mathbf{F} \to \mathbf{Q}$ which induces an isomorphism of homotopy groups. Hence, $\mathbf{Q}$ "replaces" $\mathbf{F}$. Since $\mathbf{F}$ "replaces" $\mathbf{E}$, $\mathbf{Q}$ is a spectrum $\{Q_k, \mu_k\}$ for which each μ_k is an inclusion which "replaces" $\mathbf{E}$.

9. Applications

We first show that we may apply Theorem (7.8) for X an arbitrary compact space.

THEOREM (9.1). *Let X be an arbitrary compact space and $\mathbf{E} \in \mathcal{W}_0$ a spectrum then there is a natural equivalence*

$$\tilde{H}_n(\mathbf{F}(X, \mathbf{E})) \approx \sum_r {}^cH^{r-n}(X; \tilde{H}_r(\mathbf{E})),$$

where ${}^cH^p(X; G)$ denotes the pth reduced Čech cohomology group of X with coefficients in G.

Proof. Let $J(X)$ denote the set of finite open coverings of X. If $\alpha \in J$, denote by X_α the nerve of α and by $\varphi_\alpha : X \to X_\alpha$ a projection. Also set

$$\bar{\varphi}_\alpha = F(\varphi_\alpha, 1) : F(X_\alpha, Y) \to F(X, Y),$$

where Y is any space. The maps $\bar{\varphi}_\alpha$ define homomorphisms

$$\Phi_* : \mathrm{Lim}_\to \tilde{H}_n(\mathbf{F}(X_\alpha, \mathbf{E})) \to \tilde{H}_n(\mathbf{F}(X, \mathbf{E})).$$

(9.1) will follow from (7.8) if we can show that Φ_* is an isomorphism. This last is a consequence of the following lemma.

LEMMA (9.2). *Let $Y \in \mathcal{W}_0$. Then*

$$\Phi_* : \mathrm{Lim}_\to \tilde{H}_n(F(X_\alpha, Y)) \to \tilde{H}_n(F(X, Y))$$

is an isomorphism.

Proof. Let $u \in \tilde{H}_n(F(X, Y))$. There is a finite CW-complex K, an element $w \in \tilde{H}_n(K)$ and a map $f : K \to F(X, Y)$ such that $f_*(w) = u$. (For example, take K to be a finite subcomplex of the geometric realization of the singular complex of $F(X, Y)$ [14], where K carries u.) Let $\bar{f} : K \wedge X \to Y$ be the adjoint of f. Since product coverings are cofinal in the set of coverings of $K \times X$ [6], it follows from the "bridge" theorems of Hu [9] that there is a map $\bar{g} : K_\beta \wedge X_\alpha \to Y$ such that the diagram

$$\text{(9.3)} \qquad \begin{array}{ccc} K \wedge X & \xrightarrow{\psi_\beta \wedge \varphi_\alpha} & K_\beta \wedge X_\alpha \\ & \bar{f} \searrow \quad \swarrow \bar{g} & \\ & Y & \end{array}$$

is homotopy-commutative. Here $\beta \in J(K)$, $\alpha \in J(X)$ and $\psi_\beta : K \to K$ is a projection. Let $g : K_\beta \to F(X_\alpha, Y)$ be the adjoint of $\bar{g}$. It follows from the homotopy-commutativity of (9.3) that the diagram

$$\text{(9.4)} \qquad \begin{array}{ccc} K & \xrightarrow{f} & F(X, Y) \\ \downarrow \psi_\beta & & \downarrow \bar{\varphi} \\ K_\beta & \xrightarrow{g} & F(X_\alpha, Y) \end{array}$$

is homotopy-commutative. Write $v = g_*(\psi_{\beta *} w)$. Then $\bar{\varphi}_{\alpha *} v = u$ and Φ_* is onto. The proof that Φ_* is one-one is similar.

The following proposition will show how (7.8) and (9.1) may be used to obtain information about particular function spaces $F(X, Y)$. There is a natural homomorphism

$$\nu : \tilde{H}_j(F(X, Y)) \to \tilde{H}_j(\mathbf{F}(X, \mathbf{S} \wedge Y)) = \mathrm{Lim}_{\to} \tilde{H}_{j+r}(F(X, S^r \wedge Y)).$$

We assume that X has finite dimension $d > 0$ and that Y is $(n - 1)$-connected, $n > d$.

PROPOSITION (9.5). *The homomorphism*

$$\nu : \tilde{H}_j(F(X, Y)) \to \tilde{H}_j(\mathbf{F}(X, \mathbf{S} \wedge Y))$$

is an isomorphism for $j < 2(n - d) - 1$ and onto for $j = 2(n - d) - 1$.

Proof. (9.6) follows from a slight restatement of the discussion on p. 350 of [20].

Note that $F(X, Y)$ is $(n - d - 1)$-connected, so that (9.6) gives the stable homology groups of (9.7). Applying (9.6) to (9.1) gives the following:

COROLLARY (9.6). *Let X and Y be as above. Then, for $j < 2(n - d) - 1$, we have*

$$\tilde{H}_j(F(X, Y)) \approx \textstyle\sum_{r=0}^{d} {}^c H^r(X; \tilde{H}_{r+j}(Y)).$$

Remark (9.7). Using (9.5) and (9.1), it follows from Serre's $\mathcal{C}$-theory [18] that if X is an arbitrary compact space, $Y \in \mathcal{W}_0$ and either ${}^cH^n(X)$ is finite for all n or $\tilde{H}_n(Y)$ is finite for all n, then $\{X, Y\}$, the group of stable homotopy classes of maps $X \to Y$ is finite. This result is essentially due to Thom [22]. Similar results may be derived for p-components of $\{X, Y\}$.

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