THE CHARACTER RING OF A FINITE GROUP

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I. Introduction

Let $\mathfrak{G}$ be a group with finite order g. Denote the conjugacy classes of $\mathfrak{G}$ by $\mathcal{K}_1, \mathcal{K}_2, \cdots, \mathcal{K}_n$, and choose a representative K_j in each class $\mathcal{K}_j$. If $\chi_1, \chi_2, \cdots, \chi_n$ are the (absolutely) irreducible characters of $\mathfrak{G}$, then a matrix of the form

$$M = (\chi_i(K_j))$$

is called a *character table* for $\mathfrak{G}$. (For basic properties of group representations and characters, see one of the many books on the subject; for example, the book by Curtis and Reiner [1].)

We define addition and multiplication of characters as for functions. The set of characters in this way generates a ring X, the *character ring* of $\mathfrak{G}$. The irreducible characters $\chi_1, \chi_2, \cdots, \chi_n$ form a free basis for X over the ring $\mathbf{Z}$ of rational integers. We assume χ_1 is the principal character of $\mathfrak{G}$.

X is isomorphic to the smallest subring of the direct sum of n copies of the complex number field $\mathbf{C}$ which contains the rows of M. Consequently, the character table of $\mathfrak{G}$ determines the character ring X. In this paper we prove the converse; i.e., we shall show that the character table for $\mathfrak{G}$ can be derived directly from X, in an essentially constructive manner.

The key tool used is the ordinary inner product on X. For $\phi, \psi \in X$, this is defined by

$$(1) \qquad f(\phi, \psi) = \frac{1}{g} \sum_{G \in \mathfrak{G}} \phi(G)\overline{\psi(G)}.$$

The irreducible characters are an orthonormal set with respect to f:

$$(2) \qquad f(\chi_i, \chi_j) = \delta_{ij}.$$

These expressions involve knowledge of either group elements or irreducible characters, neither of which can be obtained directly from X. Therefore, we first give an internal characterization of the ordinary inner product among all bilinear forms on X.

In applications, the character ring can frequently be generated by a few accessible characters, for example, characters induced from certain subgroups, and the main theorem applies to all these cases. We give an application of a different nature, a proof for a theorem of G. Higman's on the group rings of finite abelian groups.

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II. Characterization of the ordinary inner product

There is a well-defined conjugation on X. This coincides with conjugation in $\mathbf{C}$; i.e., if $G \in \mathfrak{G}$, $\chi \in X$, then $\bar{\chi}(G) = \overline{\chi(G)}$. From definition (1), we have that f is *conjugate-associative*:

$$f(\theta, \psi\chi) = f(\theta\bar{\psi}, \chi) \qquad \theta, \psi, \chi \in X. \tag{3}$$

If a basis $\phi_1, \phi_2, \cdots, \phi_n$ of X is fixed, then associated with a bilinear form p is a matrix $P = (p(\phi_i, \phi_j))$. We say that p is *unimodular* if there is a basis for X so that the corresponding matrix P is unimodular. Note in particular that a unimodular form is nondegenerate. Equation (2) shows that f is unimodular.

Using an approach of D. G. Higman's [2, p. 500] we get the following connection between bilinear forms on X.

Proposition 1. *Let $p : X \times X \to \mathbf{Z}$ and $q : X \times X \to \mathbf{Z}$ be conjugate-associative, unimodular bilinear forms. There exists an invertible element $\mu \in X$ so that $q(\psi, \theta) = p(\mu\psi, \theta)$ for every $\theta, \psi \in X$.*

Proof. Let $X^* = \mathrm{Hom}_{\mathbf{Z}}(X, \mathbf{Z})$ be the set of all $\mathbf{Z}$-module homomorphisms of X into $\mathbf{Z}$. X^* is a $\mathbf{Z}$-module in a natural way, and can be made into an X-module by defining

$$(\psi \cdot \gamma)(\theta) = \gamma(\bar{\psi}\theta) \qquad \theta, \psi \in X, \gamma \in X^*.$$

Define $\mathfrak{p} : X \to X^*$ by

$$\mathfrak{p}_\psi(\theta) = p(\psi, \theta).$$

It is straightforward to verify that $\mathfrak{p}$ is an X-homomorphism of X, considered as an X-module, into X^*. Furthermore, since p is unimodular, if $p(\psi, X) = 0$, then $\psi = 0$. Thus $\mathfrak{p}$ is injective.

To show that $\mathfrak{p}$ is surjective, let γ be an arbitrary element of X^*. Let $\phi_1, \phi_2, \cdots, \phi_n$ be a basis for X so that $P = (p(\phi_i, \phi_j))$ is unimodular. Let $P^{-1} = (p'_{ij})$. Define $\nu \in X$ by

$$\nu = \textstyle\sum_{k,i} \gamma(\phi_k) p'_{ki} \phi_i .$$

Then $p(\nu, \phi_j) = \gamma(\phi_j)$ for each j, so $\mathfrak{p}_\nu = \gamma$.

Consequently, $\mathfrak{p}$ is an X-isomorphism. Let q be another such form and $\mathfrak{q}$ be the associated X-isomorphism from X onto X^*. Let $\mathfrak{a} = \mathfrak{p}^{-1}\mathfrak{q}$. If $\mathfrak{a}(\chi_1) = \mu$, $\mathfrak{a}^{-1}(\chi_1) = \xi$, then since $\mathfrak{a}$ is an X-isomorphism of X onto itself, $\mathfrak{a}(\theta) = \mu\theta$ for every $\theta \in X$. In particular, $\mu\xi = \chi_1$, so μ is invertible.

For any $\psi \in X$, $(\mathfrak{p}^{-1}\mathfrak{q})(\psi) = \mu\psi$, so that $\mathfrak{q}_\psi = \mathfrak{p}_{\mu\psi}$. Hence for any $\theta \in X$, we have

$$q(\psi, \theta) = p(\mu\psi, \theta)$$

We pass to the complexification $X_c = \mathbf{C} \otimes_{\mathbf{Z}} X$ of X. X_c is isomorphic with, and for the present may be identified with, the *class function ring* of $\mathfrak{G}$, the ring of all complex-valued functions on the set $\{K_1, K_2, \cdots, K_n\}$.

Contained in X_c are the *characteristic functions* $\eta_1, \eta_2, \cdots, \eta_n$, defined by

$$\eta_j = (k_j/g) \sum_{r=1}^{n} \bar{\chi}_r(K_j)\chi_r,$$

where k_j is the number of elements in the class $\mathcal{K}_j$. From the orthogonality relations, we derive at once

$$\begin{aligned} \eta_j(G) &= 1 \quad \text{if} \quad G \in \mathcal{K}_j \\ &= 0 \quad \text{if} \quad G \notin \mathcal{K}_j. \end{aligned} \tag{4}$$

From (4) it follows that

$$\eta_i \eta_j = \delta_{ij} \eta_j \qquad \eta_1 + \eta_2 + \cdots + \eta_n = \chi_1. \tag{5}$$

These functions are characterized as the fundamental idempotents of X_c; i.e., by the equations (5) and the condition that n is the dimension of X_c over $\mathbf{C}$.

Every bilinear form on X has a unique conjugate-bilinear extension to X_c. Note in particular that (1), and therefore (3), holds for $\phi, \psi \in X_c$.

PROPOSITION 2. *For any characteristic function η_j and any $\theta \in X_c$,*

$$f(\theta, \eta_j) = f(\chi_1, \eta_j)\theta(K_j).$$

Proof. From (1) and (4) we have

$$f(\theta, \eta_j) = \frac{1}{g} \sum_{G \in \mathfrak{G}} \theta(G)\bar{\eta}_j(G) = \frac{1}{g} \sum_{G \in \mathcal{K}_j} \theta(G) = \frac{k_j}{g} \cdot \theta(K_j). \tag{6}$$

For the particular case $\theta = \chi_1$,

$$f(\chi_1, \eta_j) = k_j/g. \tag{7}$$

Substituting (7) in (6) gives the desired result.

We remark also that ordinary conjugation can be distinguished internally from all other conjugations on the ring X. If $\zeta_1, \zeta_2, \cdots, \zeta_n$ is a $\mathbf{Z}$-basis for X, any conjugation $\chi \rightarrow \chi^*$ can be extended canonically to X_c by definining

$$(\sum a_i \zeta_i)^* = \sum \bar{a}_i \zeta_i^*, \qquad a_i \in \mathbf{C}.$$

Because of (4), ordinary conjugation, when so extended, satisfies

$$\overline{\sum b_j \eta_j} = \sum \bar{b}_j \eta_j.$$

This equation gives a well-defined conjugation on X_c which can restrict to at most one conjugation on X.

THEOREM 1. *The ordinary inner product is the unique conjugate-associative, unimodular bilinear form $p : X \times X \rightarrow \mathbf{Z}$ for which every $p(\chi_1, \eta_j)$ is a positive rational number.*

Proof. From the previous remarks and equation (7), f does have the required properties. Let $q : X \times X \to \mathbf{Z}$ be another bilinear form satisfying the hypotheses. By Proposition 1, there is an invertible element μ of X so that $q(\psi, \phi) = f(\mu\psi, \phi)$ for every $\psi, \phi \in X$. This relation holds also on X_c, since the form so defined is an extension of q. Applying Proposition 2, we get

$$\mu(K_j) = \frac{q(\chi_1, \eta_j)}{f(\chi_1, \eta_j)}.$$

Thus every value of μ is positive rational. Since $\mu \in X$, every value is also an algebraic integer, and hence a rational integer. Similarly, every value of μ^{-1} must also be a positive rational integer. Consequently, every $\mu(K_j) = +1$, so $\mu = \chi_1$, and $q = f$.

III. The character table

We are now ready to construct the character table for $\mathfrak{G}$. The first step is to choose a $\mathbf{Z}$-basis $\zeta_1, \zeta_2, \cdots, \zeta_n$ for X which is orthonormal with respect to f. It is easily shown that, after suitable permutation, $\zeta_i = \chi_i$ or $\zeta_i = -\chi_i$ for each i; in particular, we can assume $\zeta_1 = \chi_1$.

Next let $\eta_1, \eta_2, \cdots, \eta_n$ be the fundamental idempotents of X_c. Let M be the matrix whose i-j entry is $f(\zeta_i, \eta_j)/f(\zeta_1, \eta_j)$. By Proposition 2, $M = (\zeta_i(K_j))$. So M differs from a character table for $\mathfrak{G}$ only in that some rows may be multiplied through by -1. Therefore the remaining task is to determine coefficients $e_1, e_2, \cdots e_n$, each $e_i = \pm 1$, so that $M_e = (e_i\zeta_i(K_j))$ is a character table.

There is a class, which we may call $\mathcal{K}_1$, with these properties:

$$(9)\quad \begin{array}{ll} \text{(i)} & \zeta_i(K_1) \text{ is real,} \\ \text{(ii)} & |\zeta_i(K_1)| \geq |\zeta_i(K_j)|, \qquad i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, n. \end{array}$$

For example, the class of the identity element E of $\mathfrak{G}$ has these properties. Other classes may have them, too. Let $\mathcal{K}_1$ be one of them, and choose $e_1, e_2, \cdots, e_n$ so that $e_i\,\zeta_i(K_1) > 0$.

Set $M_e = (e_i\,\zeta_i(K_j))$. We claim that M_e is a character table for $\mathfrak{G}$.

PROPOSITION 3. *$\mathcal{K}_1$ contains one central element of $\mathfrak{G}$, whose order is* 1 *or* 2.

Proof. Properties (9) hold for E as well as K_1. Hence for each i there is a $u_i = \pm 1$ so that $u_i\chi_i(E) = \chi_i(K_1)$. By the orthogonality relations,

$$g/k_1 = \textstyle\sum_{r=1}^{n} \chi_r(K_1)\bar{\chi}_r(K_1) = \sum_{r=1}^{n} \chi_r(E)\bar{\chi}_r(E) = g,$$

so $k_1 = 1$ and $\mathcal{K}_1$ contains one element K_1, which must therefore lie in the center of $\mathfrak{G}$.

If Δ_i is an irreducible representation of $\mathfrak{G}$ with character χ_i, then by Schur's Lemma, K_1 is represented by a scalar matrix:

$$\Delta_i(K_1) = w_i(K_1)\cdot I.$$

Taking traces, $(\deg \chi_i)w_i(K_1) = u_i(\deg \chi_i)$, so $w_i(K_1) = \pm 1$, and $\Delta_i(K_1^2) = I$. Since Δ_i is arbitrary, $K_1^2 = E$.

PROPOSITION 4. *Let $N = (\chi_i(K_j))$ be a character table for $\mathfrak{G}$. Then there is a permutation π of the columns of N so that $\pi N = M_e$.*

Proof. For any $G \in \mathfrak{G}$, and any irreducible representation Δ_i of $\mathfrak{G}$, we have $\Delta_i(K_1 G) = w_i(K_1)\Delta_i(G)$, as in the proof of Proposition 3. Hence $\chi_i(K_1 G) = w_i(K_1)\chi_i(G)$. For any K_j,

$$\chi_i(K_j) = \chi_i(K_1^2 K_j) = w_i(K_1)\chi_i(K_1 K_j). \tag{10}$$

In particular, $0 < \chi_i(E) = w_i(K_1)\chi_i(K_1)$, implying

$$w_i(K_1)\chi_i(K_1) = e_i \zeta_i(K_1).$$

Since $\zeta_i = \pm\chi_i$, we have $w_i(K_1)\chi_i = e_i \zeta_i$. From (10), we see that the matrix $M_e = (w_i(K_1)\chi_i(K_j))$ coincides with the matrix $\pi N = (\chi_i(\pi K_j))$, where π is the permutation of classes $\mathcal{K}_j \to K_1 \mathcal{K}_j$.

Since πN is a character table for $\mathfrak{G}$ as well as N, Proposition 4 establishes that M_e is a character table for $\mathfrak{G}$. We have proved the main theorem.

THEOREM 2. *Let $\mathfrak{G}$ be a finite group with character ring X. The full table of character values for $\mathfrak{G}$ is determined by X.*

IV. Isomorphism theorems

We say two finite groups $\mathfrak{G}$ and $\mathfrak{G}'$ *have the same character table* if there is a one-to-one correspondence between the respective conjugacy classes $\mathcal{K}_j \leftrightarrow \mathcal{K}_j'$ and the irreducible characters $\chi_i \leftrightarrow \chi_i'$ so that the matrices $(\chi_i(K_j))$ and $(\chi_i'(K_j'))$ coincide.

PROPOSITION 5. *Let $\Phi : X \to X'$ be an isomorphism between the character rings of the finite groups $\mathfrak{G}$ and $\mathfrak{G}'$. Then $\Phi(\bar{\chi}) = \overline{\Phi(\chi)}$ for every $\chi \in X$.*

Proof. Conjugation in X is uniquely induced from the conjugation in X_c defined by equation (8). Φ extends canonically to a $\mathbf{C}$-isomorphism from X_c to X_c' by defining

$$\Phi(\textstyle\sum a_i \chi_i) = \sum a_i \Phi(\chi_i), \qquad a_i \in \mathbf{C}.$$

If $\chi = \sum b_j \eta_j \in X$, then

$$\Phi(\bar{\chi}) = \Phi(\textstyle\sum \bar{b}_j \eta_j) = \sum \bar{b}_j \Phi(\eta_j) = \overline{\sum b_j \Phi(\eta_j)} = \overline{\Phi(\chi)},$$

the next-to-last equality following since $\Phi(\eta_1), \Phi(\eta_2), \cdots, \Phi(\eta_n)$ are the fundamental idempotents of X_c'.

THEOREM 3. *Let $\mathfrak{G}$* and *$\mathfrak{G}'$ be finite groups with character rings X and X' respectively. Let $\Phi : X \to X'$ be an isomorphism. Then $\mathfrak{G}$ and $\mathfrak{G}'$ have the same character table.*

Proof. Let f' be the ordinary inner product on X' and define $q : X \times X \to \mathbf{Z}$

by $q(\phi, \psi) = f'(\Phi(\phi), \Phi(\psi))$. Then by Proposition 5 and Theorem 1, $q = f$. Hence the isomorphism Φ preserves the ordinary inner product. Consequently, the character table constructed from X as in the last section coincides with the table constructed from X'.

Theorem 3 allows us to give a simple proof for the following theorem, which was first proved by Graham Higman in 1940 [3].

THEOREM 4. *Let $\mathfrak{G}$ and $\mathfrak{G}'$ be finite abelian groups such that their group rings $\mathbf{Z}\mathfrak{G}$ and $\mathbf{Z}\mathfrak{G}'$ over $\mathbf{Z}$ are isomorphic. Then $\mathfrak{G}$ and $\mathfrak{G}'$ are isomorphic.*

Proof. A finite abelian group is isomorphic with its group of characters, so its group ring is isomorphic with its character ring. Hence the hypotheses imply that the respective character rings X and X' are isomorphic. By Theorem 3, $\mathfrak{G}$ and $\mathfrak{G}'$ have the same character table. This means the character groups of $\mathfrak{G}$ and $\mathfrak{G}'$ are isomorphic, so that $\mathfrak{G}$ and $\mathfrak{G}'$ are isomorphic.

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