ADJOINT FUNCTORS AND TRIPLES[1]

BY

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A *triple* $\mathbf{F} = (F, \eta, \mu)$ in a category $\mathfrak{A}$ consists of a functor $F : \mathfrak{A} \to \mathfrak{A}$ and morphisms $\eta : 1_{\mathfrak{A}} \to F$, $\mu : F^2 \to F$ satisfying some identities (see §2, (T.1)–(T.3)) analogous to those satisfied in a monoid. Cotriples are defined dually.

It has been recognized by Huber [4] that whenever one has a pair of adjoint functors $T : \mathfrak{A} \to \mathfrak{B}$, $S : \mathfrak{B} \to \mathfrak{A}$ (see §1), then the functor TS (with appropriate morphisms resulting from the adjointness relation) constitutes a triple in $\mathfrak{B}$ and similarly ST yields a cotriple in $\mathfrak{A}$.

The main objective of this paper is to show that this relation between adjointness and triples is in some sense reversible. Given a triple $\mathbf{F}$ in $\mathfrak{A}$ we define a new category $\mathfrak{A}^F$ and adjoint functors $T : \mathfrak{A}^F \to \mathfrak{A}$, $S : \mathfrak{A} \to \mathfrak{A}^F$ such that the triple given by TS coincides with $\mathbf{F}$. There may be many adjoint pairs which in this way generate the triple $\mathbf{F}$, but among those there is a universal one (which therefore is in a sense the "best possible one") and for this one the functor T is faithful (Theorem 2.2). This construction can best be illustrated by an example. Let $\mathfrak{A}$ be the category of modules over a commutative ring K and let Λ be a K-algebra. The functor $F = \Lambda\otimes$ together with morphisms η and μ resulting from the morphisms $K \to \Lambda$, $\Lambda \otimes \Lambda \to \Lambda$ given by the K-algebra structure of Λ, yield then a triple $\mathbf{F}$ in $\mathfrak{A}$. The category $\mathfrak{A}^F$ is then precisely the category of Λ-modules. The general construction of $\mathfrak{A}^F$ closely resembles this example. As another example, let $\mathfrak{A}$ be the category of sets and let F be the functor which to each set A assigns the underlying set of the free group generated by A. There results a triple $\mathbf{F}$ in $\mathfrak{A}$ and $\mathfrak{A}^F$ is the category of groups.

Let $\mathbf{G} = (\delta, \varepsilon, G)$ be a cotriple in a category A. It has been recognized by Godement [3] and Huber [4], that the iterates G^n of G together with face and degeneracy morphisms

$$G^{n+1} \to G^n, \qquad G^n \to G^{n+1}$$

defined using ε and δ yield a simplicial structure which can be used to define homology and cohomology.

Now if $\mathbf{F}$ is a triple in $\mathfrak{A}$, then one has an adjoint pair $T : \mathfrak{A}^F \to \mathfrak{A}$, $S : \mathfrak{A} \to \mathfrak{A}^F$ and therefore one has an associated cotriple $\mathbf{G}$ in $\mathfrak{A}^F$. This in turn yields a simplicial complex for every object in $\mathfrak{A}^F$, thus paving the way for homology and cohomology in $\mathfrak{A}^F$. In §4 we show that under suitable

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conditions this complex is a projective resolution in a suitable relative sense as developed by us in [2].

For some further developments of the ideas presented here see a forthcoming dissertation of Jon M. Beck.

1. Review of adjoint functors

Given a category $\mathcal{A}$ we use the symbol $\mathcal{A}(A, A')$ to denote the set of all morphisms $A \to A'$ in $\mathcal{A}$ where A, A' are objects of $\mathcal{A}$.

We shall use the notation

$$a : S \dashv T : (\mathcal{A}, \mathcal{B}) \tag{1.1}$$

whenever $T : \mathcal{A} \to \mathcal{B}$ and $S : \mathcal{B} \to \mathcal{A}$ are functors and a is an isomorphism

$$a : \mathcal{A}(S,\ \ ) \to \mathcal{B}(\ \ , T)$$

of functors. Explicitly for each pair $A \in \mathcal{A}$, $B \in \mathcal{B}$ a yields a bijection

$$a : \mathcal{A}(S(B), A) \to \mathcal{B}(A, T(A))$$

satisfying

$$a(gfS(h)) = T(g)a(f)h \tag{1.2}$$

for

$$h : B' \to B, \qquad f : S(B) \to A, \qquad g : A \to A'.$$

Under the relation (1.1) the functor S is said to be the coadjoint of T, and T is said to be the adjoint of S.

Setting

$$\alpha(A) = a^{-1}(1_{T(A)}) : ST(A) \to A \tag{1.3}$$

$$\beta(B) = a(1_{T(B)}) : B \to TS(B) \tag{1.4}$$

we obtain morphisms of functors

$$\alpha : ST \to 1_{\mathcal{A}}, \qquad \beta : 1_{\mathcal{B}} \to TS$$

such that the compositions

$$S \xrightarrow{S\beta} STS \xrightarrow{\alpha S} S, \qquad T \xrightarrow{\beta T} TST \xrightarrow{T\alpha} T$$

are identities. Conversely we have:

$$a(f) = T(f)\beta(B) \quad \text{for} \quad f : S(B) \to A \tag{1.5}$$

$$a^{-1}(g) = \alpha(A)S(g) \quad \text{for} \quad g : \quad B \to T(A). \tag{1.6}$$

We shall write $a \sim (\alpha, \beta)$.

Given

$$a : S \dashv T : (\mathcal{A}, \mathcal{B})$$

$$c : R \dashv Q : (\mathcal{B}, \mathcal{C})$$

we have

$$ca : SR \dashv QT : (\mathcal{A}, \mathcal{C})$$

where ca is the composition

$$\mathcal{A}(SR,\ \ ) \xrightarrow{a} \mathcal{B}(R, T) \xrightarrow{c} \mathcal{C}(\ \ , QT).$$

If $a \sim (\alpha, \beta)$, $c \sim (\sigma, \tau)$ then

$$ca \sim (\sigma(Q\alpha R), (S\tau T)\beta).$$

Given

(1.7) $$a : S \dashv T : (\mathcal{A}, \mathcal{B})$$

(1.8) $$a' : S' \dashv T' : (\mathcal{A}, \mathcal{B})$$

and given morphisms $\varphi : S' \to S$, $\psi : T \to T'$ we write

(1.9) $$\varphi \dashv \psi$$

if the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{A}(S,\ \ ) \arrow[r, "a"] \arrow[d, "{\mathcal{A}(\varphi,\ \ )}"'] & \mathcal{B}(\ \ , T) \arrow[d, "{B(\ \ , \psi)}"] \\
\mathcal{A}(S',\ \ ) \arrow[r, "a'"'] & \mathcal{B}(\ \ , T')
\end{tikzcd}$$

We note the following properties of adjointness of morphisms:

(1.10) If in addition to the above we also have $a'' : S'' \dashv T'' : (\mathcal{A}, \mathcal{B})$ and $\varphi' \dashv \psi'$ for $\varphi' : S'' \to S'$, $\psi' : T' \to T''$ then $\varphi\varphi' \dashv \psi\psi'$

(1.11) If (1.7) and (1.8) hold and $\varphi : S' \to S$ then there exists a unique $\psi : T \to T'$ such that $\varphi \dashv \psi$. Further ψ is an isomorphism (or identity) if and only if φ is.

(1.12) If (1.7), (1.8) and (1.9) hold and if $c : R \dashv Q : (\mathcal{B}, \mathcal{C})$ then $\varphi R \dashv Q\psi$ relative to the adjointness relations ca, ca'.

2. Triples

Let $\mathcal{A}$ be a category. *A* triple $\mathbf{F} = (F, \eta, \mu)$ in $\mathcal{A}$ consists of a functor $F : \mathcal{A} \to \mathcal{A}$ and of morphisms

$$\eta : 1_A \to F, \qquad \mu : F^2 \to F$$

such that

(T.1) the composition $F \xrightarrow{\eta F} F^2 \xrightarrow{\mu} F$ is the identity,

(T.2) the composition $F \xrightarrow{F\eta} F^2 \xrightarrow{\mu} F$ is the identity, (T.3) the diagram

$$\begin{tikzcd} F^3 \arrow[r, "F\mu"] \arrow[d, "\mu F"'] & F^2 \arrow[d, "\mu"] \\ F^2 \arrow[r, "\mu"'] & F \end{tikzcd}$$

is commutative.

Dually a cotriple (δ, ε, F) in $\mathcal{A}$ is given by a triple $(F^*, \varepsilon^*, \delta^*)$ in the dual category $\mathcal{A}^*$.

PROPOSITION 2.1. *Let*

(2.1) $$a : S \dashv T : (\mathcal{A}, \mathcal{B})$$

with $a \sim (\alpha, \beta)$. *Then*

$$\nabla(a) = (TS, \beta, T\alpha S)$$

is a triple in $\mathcal{B}$. *Dually*

$$\triangle(a) = (S\beta T, \alpha, ST)$$

is a cotriple in $\mathcal{A}$.

We say that $\nabla(a)$ *is generated by* (2.1) *and* $\triangle(a)$ *is cogenerated by* (2.1).

Proof. Since $(T\alpha)(\beta T) = 1_T$ we have $(ST\alpha)(S\beta T) = 1_{ST}$ and (T.1) holds. Since $(\alpha S)(S\beta) = 1_S$ we have $(\alpha ST)(S\beta T) = 1_{ST}$ so that (T.2) holds. Relation (T.3) follows from the commutative diagram

$$\begin{tikzcd} STST \arrow[r, "\alpha ST"] \arrow[d, "ST\alpha"'] & ST \arrow[d, "\alpha"] \\ ST \arrow[r, "\alpha"'] & 1_A . \end{tikzcd}$$

THEOREM 2.2. *Every triple* $\mathbf{F} = (F, \eta, \mu)$ *in a category* $\mathcal{A}$ *admits a generator*

(2.2) $$a : S \dashv T : (\mathcal{B}, \mathcal{A}).$$

Moreover, there exists a universal generator

(2.3) $$a^F : S^F \dashv T^F : (\mathcal{A}^F, \mathcal{A})$$

of $\mathbf{F}$ *such that for any generator* (2.2) *of* $\mathbf{F}$ *there exists a unique functor* $L : \mathcal{B} \to \mathcal{A}^F$ *such that*

(2.4) $$LS = S^F, \qquad L\alpha = \alpha^F L.$$

These relations imply

(2.5) $$T^F L = T.$$

In addition the functor T^F is faithful.

Proof. We define the objects of $\mathcal{A}^F$ to be the pairs (A, φ) where A is an object of $\mathcal{A}$ and $\varphi : F(A) \to A$ is a morphism in A satisfying

(2.6) $$\varphi\eta(A) = 1_A, \qquad \varphi F(\varphi) = \varphi\mu(A).$$

A morphism $[f] : (A, \varphi) \to (A', \varphi')$ in $\mathcal{A}^F$ is given by a morphism $f : A \to A'$ in $\mathcal{A}$ such that

(2.7) $$f\varphi = \varphi' F(f).$$

If $[g] : (A', \varphi') \to (A'', \varphi'')$ then we define $[g]\,[f] = [gf]$. This defines the category $\mathcal{A}^F$. The functor $T^F : \mathcal{A}^F \to \mathcal{A}$ is given by

$$T^F(A, \varphi) = A, \qquad T^F[f] = f.$$

Clearly T^F is faithful.

The functor $S^F : \mathcal{A} \to \mathcal{A}^F$ is defined by

$$S^F(A) = (F(A), \mu(A)), \qquad S^F(f) = [F(f)]$$

for $f : A \to A'$ in $\mathcal{A}$.

Since $T^F S^F = F$ we define

$$\beta^F = \eta : 1_A \to F = T^F S^F.$$

Since $[\varphi] : (F(A), \mu(A)) \to (A, \varphi)$ is a morphism in $\mathcal{A}^F$ we define

$$\alpha^F : S^F T^F \to 1_{\mathcal{A}^F}, \qquad \alpha^F(A, \varphi) = [\varphi].$$

For each A in $\mathcal{A}$, the composition

$$S^F(A) \xrightarrow{S^F\beta^F} S^F T^F S^F(A) \xrightarrow{\alpha^F S^F} S^F(A)$$

becomes the composition

$$(F(A), \mu(A)) \xrightarrow{[F\eta(A)]} (F^2(A), \mu F(A)) \xrightarrow{[\mu(A)]} (F(A), \mu(A))$$

which is the identity. Similarly the composition

$$T^F(A, \varphi) \xrightarrow{\beta^F T^F} T^F S^F T^F(A, \varphi) \xrightarrow{T^F \alpha^F} T^F(A, \varphi)$$

becomes the composition

$$A \xrightarrow{\eta(A)} F(A) \xrightarrow{\varphi} A$$

which again is the identity. This yields (2.3) with $a^F \sim (\alpha^F, \beta^F)$. Since

$$T^F\beta^F S^F(A) = T^F\beta^F(F(A), \mu(A)) = T^F[\mu(A)] = \mu(A)$$

we have $\nabla(a^F) = F$ so that (2.3) is a generator for $\mathbf{F} = (F, \eta, \mu)$.

To show that (2.3) has the universal property consider an arbitrary generator (2.2) of $\mathbf{F}$.

Given an object B in $\mathcal{B}$ we have $\alpha(B) : ST(B) \to B$ and therefore

$$T\alpha(B) : FT(B) = TST(B) \to T(B).$$

We assert that $(T(B), T\alpha(B))$ is an object in $\mathcal{A}^F$. Firstly, the composition

$$T(B) \xrightarrow{\eta T} FT(B) \xrightarrow{T\alpha} T(B)$$

is the identity since $\eta = \beta$ and $(T\alpha)(\beta T) = 1_T$. Secondly, from the commutative diagram

$$\begin{tikzcd} STST \arrow[r, "ST\alpha"] \arrow[d, "\alpha ST"'] & ST \arrow[d, "\alpha"] \\ ST \arrow[r, "\alpha"'] & 1_B \end{tikzcd}$$

we deduce

$$(T\alpha)(FT\alpha) = (T\alpha)(TST\alpha) = (T\alpha)(T\alpha ST) = (T\alpha)(\mu T).$$

Thus we may define

$$L(B) = (T(B), T\alpha(B)).$$

If $f : B \to B'$ in $\mathcal{B}$ then

$$\begin{aligned} T(f)T\alpha(B) &= T(f\alpha(B)) = T(\alpha(B')ST(f)) = (T\alpha(B'))(TST(f)) \\ &= T\alpha(B')FT(f). \end{aligned}$$

Thus (2.7) holds and $[T(f)] : L(B) \to L(B')$ is a morphism in $\mathcal{A}^F$. Thus setting $L(f) = [T(f)]$ we obtain a functor $L : \mathcal{B} \to \mathcal{A}^F$. Clearly

$$LS(A) = (TS(A), T\alpha S(A)) = (F(A), \mu(A)) = S^F(A)$$

$$LS(f) = [TS(f)] = [F(f)] = S^F(f)$$

so that $LS = S^F$. Also

$$T^F L(B) = T^F(T(B), T\alpha(B)) = T(B)$$

$$T^F L(f) = T^F[T(f)] = T(f)$$

so that $T^F L = T$. Further

$$\alpha^F L(B) = \alpha^F(T(B), T\alpha(B)) = [T\alpha(B)] = L\alpha(B)$$

so that $\alpha^F L = L\alpha$. Thus (2.4) and (2.5) hold.

To show that L is unique consider another functor $L' : \mathcal{B} \to \mathcal{A}^F$ satisfying (2.4). Let $B \in \mathcal{B}$ and let $L'(B) = (A, \varphi)$. Then

$$A = T^F(A, \varphi) = T^F L'(B) = T(B)$$

$$\varphi = T^F[\varphi] = T^F \alpha^F(A, \varphi) = T^F \alpha^F L'(B) = T^F L'\alpha(B) = T\alpha(B)$$

and thus $L'(B) = L(B)$. If $f : B \to B'$ then both $L(f)$ and $L'(f)$ are morphisms $L(B) \to L(B')$. Since $T^F L = T = T^F L'$ it follows that $T^F L(f) = T^F L'(f)$. The functor T^F being faithful it follows that $L(f) = L'(f)$. Thus $L = L'$. Since the uniqueness proof uses only (2.4) while L satisfies (2.5), it follows that (2.5) is a consequence of (2.4). This concludes the proof of 2.2.

PROPOSITION 2.3. *Let*

$$a : S \dashv T : (\mathcal{A}, \mathcal{B})$$

with $a \sim (\alpha, \beta)$ *and let* $\mathbf{F} = (F, \eta, \mu)$ *be a triple in* $\mathcal{A}$. *Then*

$$\mathbf{F}' = (TFS, (T\eta S)\beta, (T\mu S)(TF\alpha FS))$$

is a triple in $\mathcal{B}$.

Proof. A purely computational proof was given by Huber [4, p. 10]. The following proof is somewhat more conceptual. Let

$$c : R \dashv Q : (\mathcal{C}, \mathcal{A})$$

with $c \sim (\sigma, \tau)$ be a generator of $\mathbf{F}$. Thus $\mathbf{F} = (QR, \tau, Q\sigma R)$. We then have by §1 the adjoint relationship

$$ac : RS \dashv TQ : (\mathcal{C}, \mathcal{B})$$

with

$$ac \sim (\sigma(R\alpha Q), (T\tau S)\beta).$$

Then by 2.1, ac generates the triple

$$\begin{aligned} \nabla(ac) &= (TQRS, (T\tau S)\beta, (TQ\sigma RS)(TQR\alpha QRS)) \\ &= (TFS, (T\eta S)\beta, (T\mu S)(TF\alpha FS)) = \mathbf{F}'. \end{aligned}$$

3. Adjoint triples

Let

$$(3.1) \qquad a : F \dashv G : (\mathcal{A}, \mathcal{A}).$$

We define

$$a^n : F^n \dashv G^n : (\mathcal{A}, \mathcal{A})$$

for $n = 0, 1, \cdots$ inductively as follows. For $n = 0$, $F^0 = G^0 = 1$ and a^0 is the identity. For $n > 0$, a^n is the composition

$$\mathcal{A}(F^n, \ ) \xrightarrow{a} \mathcal{A}(F^{n-1}, G) \xrightarrow{a^{n-1}} \mathcal{A}(\ , G^n).$$

In particular $a^1 = a$.

If $\mathbf{F} = (F, \eta, \mu)$ is a triple in $\mathcal{A}$ and $\mathbf{G} = (\delta, \varepsilon, G)$ is a cotriple in $\mathcal{A}$, then we write

$$a : \mathbf{F} \dashv \mathbf{G}$$

if (3.1) holds and if

$$\eta \dashv \varepsilon, \qquad \mu \dashv \delta$$

under a^0, a^1 and a^2.

PROPOSITION 3.1. *If* $\mathbf{F} = (F, \eta, \mu)$ *is a triple in* $\mathcal{A}$ *and if* $a : F \dashv G : (\mathcal{A}, \mathcal{A})$ *then there exists a unique cotriple* $\mathbf{G} = (\delta, \varepsilon, G)$ *such that* $a : \mathbf{F} \dashv \mathbf{G}$.

Proof. We must define ε and δ by the conditions $\mu \dashv \delta$, $\eta \dashv \varepsilon$. Then by (1.12), $\eta F \dashv G\varepsilon$ and by (1.10) $\mu(\eta F) \dashv (G\varepsilon)\delta$. Since $\delta(\varepsilon F) = 1_F$ it follows that $(G\varepsilon)\delta = 1_G$. Similarly we show that $(\varepsilon G)\delta = 1_G$ and that $\delta(G\delta) = \delta(\delta G)$. Thus $G = (\delta, \varepsilon, G)$ is a cotriple as required.

PROPOSITION 3.2. *Let*

$$a : S \dashv T : (\mathcal{A}, \mathcal{B})$$

$$c : R \dashv S : (\mathcal{B}, \mathcal{A})$$

Then in $\mathcal{A}$ *we have*

$$ca : \nabla(c) \dashv \Delta(a).$$

Proof. Let $a \sim (\alpha, \beta)$, $c \sim (\sigma, \tau)$. Then by definition

$$\Delta(a) = (S\beta T, \alpha, ST), \qquad \nabla(c) = (SR, \tau, S\sigma R)$$

By §1 we have

$$ca : SR \dashv ST.$$

Thus it suffices to show that

$$\tau \dashv \alpha, \qquad S\sigma R \dashv S\beta T.$$

The relation $\tau \dashv \alpha$ is the commutativity relation in the diagram

$$\begin{tikzcd}
\mathcal{A}(SR,\ \ ) \arrow[r, "a"] \arrow[dr, "\mathcal{A}(\tau,\ \ )"'] & \mathcal{B}(R, T) \arrow[r, "c"] & \mathcal{A}(\ \ , ST) \arrow[dl, "\mathcal{A}(\ \ ,\alpha)"] \\
& \mathcal{A}(\ \ ,\ \ ) &
\end{tikzcd}$$

which follows from the commutative diagram

$$\begin{tikzcd}
& \mathcal{B}(R, T) \arrow[dl, "a^{-1}"'] \arrow[d, "S"] \arrow[dr, "c"] & \\
\mathcal{A}(SR,\ \ ) \arrow[dr, "\mathcal{A}(\tau,\ \ )"'] & \mathcal{A}(SR, ST) \arrow[l, "\mathcal{A}(SR,\alpha)"] \arrow[r, "\mathcal{A}(\tau, ST)"'] \arrow[d, "\mathcal{A}(\tau,\alpha)"] & \mathcal{A}(\ \ , ST). \arrow[dl, "\mathcal{A}(\ \ ,\alpha)"] \\
& \mathcal{A}(\ \ ,\ \ ) &
\end{tikzcd}$$

The relation $S\sigma R \dashv S\beta T$ follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{A}(SR,\ ) \arrow[r, "a"] \arrow[d, "\mathcal{A}(S\sigma R,\ )"'] & \mathcal{B}(R,T) \arrow[r, "c"] \arrow[d, "S"] & \mathcal{A}(\ ,ST) \arrow[d, "\mathcal{A}(\ ,S\beta T)"] \\
\mathcal{A}(SRSR,\ ) \arrow[r, "ca"] & \mathcal{A}(SR,ST) \arrow[r, "ca"] & \mathcal{A}(\ ,STST).
\end{tikzcd}$$

The commutativity in the left square follows from the commutative diagram

$$\begin{tikzcd}
\mathcal{A}(SR,\ ) \arrow[r, "a"] \arrow[d, "\mathcal{A}(S\sigma R,\ )"'] & \mathcal{B}(R,T) \arrow[r, "1"] \arrow[d, "\mathcal{B}(\sigma R,T)"'] & \mathcal{B}(R,T) \arrow[d, "S"] \\
\mathcal{A}(SRSR,\ ) \arrow[r, "a"] & \mathcal{B}(RSR,T) \arrow[r, "c"'] \arrow[ur, "B(R\tau,T)"] & \mathcal{A}(SR,ST)
\end{tikzcd}$$

and the commutativity of the right square is shown dually.

PROPOSITION 3.3. *Let* $\mathbf{F} = (F, \eta, \mu)$ *be a triple in* $\mathcal{A}$ *with a universal generator*

(3.2) $$c : R \dashv S : (\mathcal{B}, \mathcal{A}).$$

Then the following properties are equivalent:

(i) *F has an adjoint $F \dashv G : (\mathcal{A}, \mathcal{A})$*,

(ii) *The triple* $\mathbf{F}$ *has an adjoint cotriple* $\mathbf{F} \dashv \mathbf{G}$,

(iii) *The functor S has an adjoint*

(3.3) $$a : S \dashv T : (\mathcal{A}, \mathcal{B}).$$

If the above is the case, then (3.3) *is a universal cogenerator for* $\mathbf{G}$.

Proof. The implication (i) $\Rightarrow$ (ii) follows from 3.1 while the implication (iii) $\Rightarrow$ (i) follows from 3.2. There remains to prove the implication (ii) $\Rightarrow$ (iii) and the last statement.

Let

(3.4) $$a' : S' \dashv T' : (\mathcal{B}', \mathcal{A})$$

be a universal cogenerator for $\mathbf{G}$. Assuming that $e : \mathbf{F} \dashv \mathbf{G}$ we shall construct an isomorphism $L : B \to B'$ of categories such that $S'L = S$. Once this is done we replace S', T', B' by S, $T = T'L$, B so that (3.4) becomes

(3.5) $$a : S \dashv T : (\mathcal{B}, \mathcal{A})$$

which is still a universal cogenerator for $\mathbf{G}$.

In order to construct L we take for (3.2) and (3.4) the explicit constructions as given in §2. Thus an object of $\mathcal{B}$ is a pair (A, φ) with $\varphi : F(A) \to A$, satisfying

$$\varphi\eta(A) = 1_A, \qquad \varphi F(\varphi) = \varphi\eta(A).$$

Dually, an object of $\mathcal{B}'$ is a pair (ψ, A) with $\psi : A \to G(A)$ such that

$$\varepsilon(A)\psi = 1_A, \qquad G(\psi)\psi = \delta(A)\psi.$$

Now given $(A, \varphi) \in \mathcal{B}$, let $L(A, \varphi) = (e\varphi, A)$. Since $\eta \dashv \varepsilon$ we have the commutative diagram

$$\begin{tikzcd} \mathcal{A}(F(A),A) \arrow[rr, "e"] \arrow[dr, "\mathcal{A}(\eta(A){,}A)"'] & & \mathcal{A}(A,G(A)) \arrow[dl, "\mathcal{A}(A{,}\ \varepsilon(A))"] \\ & \mathcal{A}(A,A) & \end{tikzcd}$$

so that

$$\varepsilon(A)e\varphi = \varphi\eta(A) = 1_A .$$

Since $\mu \dashv \delta$ we have the commutative diagram

$$\begin{tikzcd} \mathcal{A}(F(A),A) \arrow[r, "e"] \arrow[d, "\mathcal{A}(\mu(A){,}A)"'] & \mathcal{A}(A,G(A)) \arrow[d, "\mathcal{A}(A{,}\delta(A))"] \\ \mathcal{A}(F^2(A),A) \arrow[r, "e^2"] & \mathcal{A}(A,G^2(A)) \end{tikzcd}$$

so that

$$e^2(\varphi\eta(A)) = \delta(A)e\varphi.$$

On the other hand

$$e^2(\varphi F(\varphi)) = e[e(\varphi)\varphi] = G(e\varphi)e\varphi.$$

Thus $G(e\varphi)e\varphi = \delta(A)e\varphi$ so that $L(A, \varphi) \in \mathcal{B}'$.

Let $[f] : (A, \varphi) \to (A', \varphi')$ be a morphism in $\mathcal{B}$. Then by definition $f : A \to A'$ and $\varphi' F(f) = f\varphi$. Then

$$G(f)e\varphi = e(f\varphi) = e(\varphi' F(f)) = (e\varphi')f$$

so that $[f] : (e\varphi, A) \to (e\varphi', A')$ is a morphism in $\mathcal{B}'$. This yields the required functor L.

4. Relations with projective classes

We rapidly review some of the notions discussed in [2] and needed here.

Let $\mathcal{A}$ be a pointed category. A *sequence* in $\mathcal{A}$ is a diagram

$$(*) \qquad A' \xrightarrow{f} A \xrightarrow{g} A''$$

such that $gf = 0$. The sequence $(*)$ is *exact* if f admits a factorization $f = kl$ where k is a kernel of g and l is an epimorphism.

Let $\mathcal{E}$ be a class of sequences $(*)$ in $\mathcal{A}$. An object P of $\mathcal{A}$ is $\mathcal{E}$-projective if for every sequence $(*)$ in $\mathcal{E}$ the sequence

$$(**) \qquad A(P, A') \to A(P, A) \to A(P, A'')$$

is exact in the category of pointed sets. The class $\mathcal{E}$ is called *projective* if the following two conditions hold: (1) If $(*)$ is sequence in $\mathcal{A}$ such that $(**)$ is exact for every $\mathcal{E}$-projective object P, then $(*)$ is in $\mathcal{E}$; (2) For every $g : A \to A''$ in A there exists a sequence $(*)$ in $\mathcal{E}$ in which A' is $\mathcal{E}$-projective.

If the class of all exact sequences in $\mathcal{A}$ is projective then $\mathcal{A}$ is called *projectively perfect.*

Let $\mathcal{A}$ be a pointed category with kernels and $\mathbf{G} = (\delta, \varepsilon, G)$ a cotriple in A, where the functor $G : A \to A$ is pointed.

We verify that for any $A, B \in \mathcal{A}$ the morphism

$$\mathcal{A}(G(B), \varepsilon(A)) : \mathcal{A}(G(B), G(A)) \to \mathcal{A}(G(B), A) \tag{4.1}$$

is surjective. Indeed, let $\varphi : G(B) \to A$. Then

$$\varepsilon(A)G(\varphi)\delta(B) = \varphi\varepsilon(G(B))\delta(B) = \varphi.$$

By [2, Ch. I, §6] this implies that the cotriple $\mathbf{G}$ determines a projective class $\mathcal{G}$ in $\mathcal{A}$ as follows. A sequence

$$A' \xrightarrow{f} A \xrightarrow{g} A''$$

(with $gf = 0$) is in $\mathcal{G}$ provided the sequence

$$\mathcal{A}(G(B), A') \to \mathcal{A}(G(B), A) \to \mathcal{A}(G(B), A'')$$

is exact for every $B \in \mathcal{A}$. The $\mathcal{G}$-projective objects are the objects $G(B)$, $B \in \mathcal{A}$ and their retracts. The class $\mathcal{G}$ is exact if and only if $\varepsilon(A)$ is an epimorphism for every $A \in \mathcal{A}$.

The kernel functor of ε

$$0 \to K \xrightarrow{\kappa} G \xrightarrow{\varepsilon} 1_{\mathcal{A}}$$

leads to a sequence

$$\cdots \to GK^3 \xrightarrow{d_3} GK^2 \xrightarrow{d_2} GK \xrightarrow{d_1} G \xrightarrow{\varepsilon} 1_{\mathcal{A}} \to 0 \tag{4.2}$$

where $K^0 = 1_{\mathcal{A}}$, $K^n = KK^{n-1}$ and $d_n : GK^n \to GK^{n-1}$ is the composition

$$GK^n \xrightarrow{\varepsilon K^n} K^n \xrightarrow{\kappa K^{n-1}} GK^{n-1}.$$

Applied to any object A in $\mathcal{A}$ the sequence (4.2) yields a sequence in $\mathcal{G}$ and since $GK^n(A)$ is $\mathcal{G}$-projective, there results a $\mathcal{G}$-projective resolution of A, called the *canonical resolution* (relative to the cotriple $\mathbf{G}$).

Now consider the morphisms

$$\varepsilon^i : G^{n+1} \to G^n, \qquad \delta^i : G^{n+1} \to G^{n+2} \qquad i = 0, 1, \cdots, n$$

defined as follows

$$\varepsilon^i = G^i\varepsilon G^{n-i}, \qquad \delta^i = G^i\delta G^{n-i}.$$

The "face" operators ε^i and the "degeneracy" operators δ^i satisfy the usual simplicial identities so that there results a simplicial functor $\bar{G}$ with G^{n+1} in degree n. If the category $\mathcal{A}$ is preadditive then we may construct the boundary operator

$$\partial_n : G^{n+1} \to G^n$$

by setting

$$\partial_n = \textstyle\sum_{i=0}^{n} (-1)^i \varepsilon^i.$$

There results a complex

$$\cdots \to G^4 \xrightarrow{\partial_3} G^3 \xrightarrow{\partial_2} G^2 \xrightarrow{\partial_1} G^1 \xrightarrow{\varepsilon} 1_A \to 0 \tag{4.3}$$

with ε as augmentation. This is the *standard complex* of the cotriple **G**.

PROPOSITION 4.1. *Let $\mathcal{A}$ be a preadditive category with kernels, and let* $\mathbf{G} = (\delta, \varepsilon, G)$ *be a cotriple in $\mathcal{A}$ with G an additive functor. If for each $A \in \mathcal{A}$ the morphism $G\kappa(A)$ is a kernel of $G\varepsilon(A)$, then for each $A \in \mathcal{A}$, the sequence* (4.3) *applied to A is a $\mathcal{G}$-projective resolution of A.*

Proof. Since $(G\varepsilon)\delta = 1_G$ it follows from the hypothesis that

$$0 \to GK \xrightarrow{G\kappa} G^2 \xrightarrow{G\varepsilon} G \to 0$$

is a split exact sequence. Since the functor G is additive, it follows that

$$0 \to G^nK \xrightarrow{G^n\kappa} G^{n+1} \xrightarrow{G^n\varepsilon} G^n \to 0$$

is split exact.

We define morphisms

$$\tau_n : GK^n \to G^{n+1}$$

by induction as follows: $\tau_0 = 1_G : G \to G$; for $n > 0$, τ_n is the composition

$$GK^n \xrightarrow{\tau_{n-1}K} G^nK \xrightarrow{G^n\kappa} G^{n+1}.$$

We verify that for $n > 0$

$$\tau_{n-1}\, d_n = \varepsilon^0 \tau_n , \qquad \varepsilon^i \tau_n = 0 \quad \text{if} \quad i > 0$$

There results a commutative diagram

$$\begin{array}{ccccccccc} \cdots \to & GK^3 & \xrightarrow{d_3} & GK^2 & \xrightarrow{d^2} & GK & \xrightarrow{d_1} & G & \\ & \downarrow{\scriptstyle \tau_3} & & \downarrow{\scriptstyle \tau_2} & & \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_0} & . \\ \cdots \to & G^4 & \xrightarrow[\partial_3]{} & G^3 & \xrightarrow[\partial_2]{} & G^2 & \xrightarrow[\partial_1]{} & G & \end{array} \tag{4.4}$$

The upper row is the canonical resolution. To show that the lower row also is a resolution (with augmentation $\varepsilon : G \to 1_{\mathcal{A}}$) it suffices to show that the morphism (4.4) is a homotopy equivalence of complexes. To verify this it suffices to show that the upper row of (4.4) is the normalized subcomplex of the simplicial complex $\bar{G}$ with τ as inclusion (for a neat exposition see [1, §3]). To do this it suffices to show that $\tau_n(A) : GK^n(A) \to G^{n+1}(A)$ is the simultaneous kernel of $\varepsilon^i(A) : G^{n+1}(A) \to G^n(A)$ for $i = 1, 2, \cdots, n$. This means that if $f : C \to G^{n+1}(A)$ is such that $\varepsilon^i(A)f = 0$ for $i = 1, 2, \cdots, n$, then there exists a unique $g : C \to GK^n(A)$ such that $\tau_n g = f$. For $n = 0$ this is clear. We now assume that $n > 0$ and proceed by induction.

Since $G^n\kappa : G^nK \to G^{n+1}$ is the kernel of $\varepsilon^n = G^n\varepsilon : G^{n+1} \to G^n$, therefore, f admits a unique factorization

$$C \xrightarrow{f'} G^nK(A) \xrightarrow{G^n\kappa} G^{n+1}(A). \tag{4.5}$$

Let $B = K(A)$ and consider the commutative diagrams for $i = 1, 2, \cdots, n-1$:

$$\begin{tikzcd} C \arrow[r, "f'"] \arrow[dr, "f"'] & G^n(B) \arrow[r, "\varepsilon^i(B)"] \arrow[d, "G^n\kappa"] & G^{n-1}(B) \arrow[d, "G^{n-1}\kappa"] \\ & G^{n+1}(A) \arrow[r, "\varepsilon^i(A)"'] & G^n(A). \end{tikzcd}$$

We have $(G^{n-1}\kappa)(\varepsilon^i(B))f' = 0$ and since $G^{n-1}\kappa$ is a kernel it follows that $\varepsilon^i(B)f' = 0$ for $i = 1, 2, \cdots, n-1$. Thus, by the inductive hypothesis, f' admits a unique factorization

$$C \xrightarrow{g} GK^{n-1}(B) \xrightarrow{\tau_{n-1}(B)} G^n(B).$$

Combining this with the factorization (4.5) of f we obtain a unique factorization

$$C \xrightarrow{g} GK^n(A) \xrightarrow{\tau_n(A)} G^{n+1}(A)$$

of f, as required.

An alternative proof of 4.1 may be given as follows. Denote by $\hat{G}$ the complex (4.3) with the augmentation included (i.e. with 1_A in degree -1 and with $\partial_0 = \varepsilon$). Next show that the complex $\hat{G}(G(A))$ is contractible (i.e. is split exact). This is done by defining $s_n : G^{n+1} \to G^{n+2}$ by $s_{-1} = 0$, $s_n = (-1)^{n-1}G^n\delta$ for $n > 0$. Then a calculation shows that

$$\partial_{n+1} s_n + s_{n-1} \partial_n = 1_{G^{n+1}}.$$

It follows that for every $B \in \mathfrak{A}$ we have

$$H\mathfrak{A}(G(B), \hat{G}(G(A))) = 0. \tag{4.6}$$

Next observe that the sequence

$$0 \to K(A) \to G(A) \to A \to 0$$

is in $\mathcal{G}$ while the sequences

$$0 \to G^nK(A) \to G^{n+1}(A) \to G^n(A) \to 0$$

are split exact for $n > 0$. There results an exact sequence of complexes

$$0 \to \mathfrak{A}(G(B), \hat{G}K(A)) \to \mathfrak{A}(G(B), \hat{G}G(A)) \to \mathfrak{A}(G(B), \hat{G}(A)) \to 0.$$

Thus (4.6) implies an isomorphism

$$H_n\,\mathcal{A}(G(B),\,\hat{G}(A)) \approx H_{n-1}\,\mathcal{A}(G(B),\,\hat{G}K(A)).$$

Since $H_{-2}\,\mathcal{A}(G(B),\,\hat{G}(A)) = 0$ for all A, $B \in \mathcal{A}$, it follows inductively that $H_n\,\mathcal{A}(G(B),\,\hat{G}(A)) = 0$ for all n. Thus $\hat{G}(A)$ is in $\mathcal{G}$, as required.

This proof has the disadvantage of not exhibiting the canonical complex as the normalized standard complex.

5. Properties of universal generators

Let $\mathbf{F} = (F,\,\eta,\,\mu)$ be a triple in a category $\mathcal{A}$ and let

$$(5.1) \qquad a^F : S^F \dashv T^F : (\mathcal{A}^F,\,\mathcal{A}), \qquad a^F \sim (\alpha^F,\,\beta^F),$$

be the universal generator of $\mathbf{F}$. From the explicit construction given in §2 it is clear that if $\mathcal{A}$ is a pointed (or additive) category and if F is a pointed (or additive) functor, then $\mathcal{A}^F$ is a pointed (or additive) category and the functors S^F and T^F are pointed (or additive).

PROPOSITION 5.1. *If $\mathcal{A}$ is a pointed category with kernels and if the functor F is pointed, then $\mathcal{A}^F$ is a pointed category with kernels and the functor T^F preserves and reflects kernels (i.e. g is a kernel of f in $\mathcal{A}^F$ if and only if T^Fg is a kernel of T^Ff in $\mathcal{A}$).*

Proof. Let $[f] : (A,\varphi) \to (A'',\varphi'')$ be a morphism in $\mathcal{A}^F$ and let $g : A' \to A$ be a kernel of f in $\mathcal{A}$. Since

$$f\varphi F(g) = \varphi'' F(f)F(g) = \varphi'' F(fg) = 0,$$

it follows that there exists a unique $\varphi' : F(A') \to A'$ such that $g\varphi' = \varphi F(g)$. Since

$$g\varphi'\eta(A') = \varphi F(g)\eta(A') = \varphi\eta(A)g = g$$

and since g is a monomorphism, it follows that $\varphi'\eta(A') = 1_{A'}$. Similarly since

$$\begin{aligned} g\varphi' F(\varphi') &= \varphi F(g)F(\varphi') = \varphi F(\varphi)F^2(g) = \varphi\mu(A)F^2(g) \\ &= \varphi F(g)\mu(A') = g\varphi'\mu(A') \end{aligned}$$

we have $\varphi' F(\varphi') = \varphi'\mu(A')$. Thus $(A',\,\varphi')$ is in $\mathcal{A}^F$ and

$$[g] : (A',\,\varphi') \to (A,\,\varphi).$$

Now let $[h] : (A_1,\,\varphi_1) \to (A,\,\varphi)$ be a morphism in B such that $[f]\,[h] = 0$. Then $fh = 0$ and there is a unique morphism $k : A_1 \to A'$ such that $h = gk$. Then

$$gk\varphi_1 = h\varphi_1 = \varphi F(h) = \varphi F(g)F(k) = g\varphi' F(k)$$

and therefore $k\varphi_1 = \varphi' F(k)$. Thus $[k] : (A_1,\,\varphi_1) \to (A',\,\varphi')$, and $[h] = [g]\,[k]$. Thus $[g]$ is a kernel of $[f]$ and the proposition is established.

PROPOSITION 5.2. *If $\mathcal{A}$ is a pointed category with cokernels and the functor F*

is pointed and preserves cokernels, then the category $\mathfrak{A}^F$ *has cokernels and the functor* T^F *preserves and reflects cokernels.*

Proof. Let $[g] : (A', \varphi') \to (A, \varphi)$ and let $f : A \to A''$ be a cokernel of g in $\mathfrak{A}$. Then $F(f)$ is a cokernel of $F(g)$. Since $f\varphi F(g) = fg\varphi' = 0$ it follows that there exists a unique $\varphi'' : F(A'') \to A''$ such that $f\varphi = \varphi'' F(f)$. Since

$$\varphi''\eta(A'')f = \varphi''F(f)\eta(A) = f\varphi\eta(A) = f$$

it follows that $\varphi''\eta(A'') = 1_{A''}$. Similarly since

$$\begin{aligned}\varphi''F(\varphi'')F^2(f) &= \varphi''F(f)F(\varphi) = f\varphi F(\varphi) = f\varphi\mu(A)\\ &= \varphi''F(f)\mu(A) = \varphi''\mu(A'')F^2(f)\end{aligned}$$

and since $F^2(f)$ is an epimorphism, it follows that $\varphi''F(\varphi'') = \varphi''\delta(A'')$. Thus (A', φ'') is in $\mathfrak{A}^F$ and $[f] : (A, \varphi) \to (A'', \varphi'')$. Now let

$$[h] : (A, \varphi) \to (A_1, \varphi_1)$$

be such that $[h]\,[g] = 0$. Then $hg = 0$ and there is a unique morphism $k : A'' \to A_1$ such that $h = kf$. Then

$$k\varphi''F(f) = kf\varphi = h\varphi = \varphi_1 F(h) = \varphi_1 F(k)F(f)$$

implies that $k\varphi'' = \varphi_1 F(k)$. Thus $[k] : (A'', \varphi'') \to (A_1, \varphi_1)$ and $[h] = [k]\,[f]$. Thus $[f]$ is a cokernel of $[g]$ and the proposition is established.

The above two propositions and known facts about abelian categories imply

PROPOSITION 5.3. *If the category* $\mathfrak{A}$ *is abelian and the functor* F *is additive and preserves cokernels, then the category* $\mathfrak{A}^F$ *is abelian and the functor* T^F *preserves and reflects exact sequences.*

Now assume that the category $\mathfrak{A}$ and the functor F are pointed, and let $\mathcal{E}$ be a projective class in the category $\mathfrak{A}$. By the adjoint theorem for projective classes [2, Ch. II, §2], there results in $\mathfrak{A}^F$ a projective class $\mathcal{E}^F = (T^F)^{-1}\mathcal{E}$. Explicitly a sequence

$$(A', \varphi') \xrightarrow{[f]} (A, \varphi) \xrightarrow{[g]} (A'', \varphi'')$$

is in $\mathcal{E}^F$ if and only if

$$A' \xrightarrow{f} A \xrightarrow{g} A''$$

is in $\mathcal{E}$. The $\mathcal{E}^F$ projective objects are the retracts of objects $S^F(A) = (F(A), \mu(A))$ where A is $\mathcal{E}$-projective. Since the functor T^F is faithful, it follows [2, Ch. II, §2] that if the class $\mathcal{E}$ is exact then the class $\mathcal{E}^F$ also is exact.

In particular, if the category $\mathfrak{A}$ is projectively perfect, and $\mathcal{E}_1$ is the class of all exact sequences in $\mathfrak{A}$, then $\mathcal{E}_1^F$ is the class of all exact sequences in the category $\mathfrak{A}^F$, which is therefore projectively perfect.

If in $\mathfrak{A}$ we take $\mathcal{E}_0$ to be the class of all split exact sequences, there results a projective class $\mathcal{E}_0^F$ in $\mathfrak{A}^F$. The $\mathcal{E}_0^F$-projective objects are the retracts of objects $S^F(A) = (F(A), \mu(A))$ where A is any object of $\mathfrak{A}$. This class $\mathcal{E}_0^F$ may also be arrived at in a different way as follows. The relation (5.1) induces in A^F a cotriple

$$\mathbf{G} = \triangle(a) = (S\beta T, \alpha, ST)$$

where the superscript F has been omitted. This cotriple defines a projective class $\mathcal{G}$ in A^F. The $\mathcal{G}$-projective objects of $\mathfrak{A}^F$ are the retracts of objects $ST(A, \varphi) = S(A) = (F(A), \mu(A))$ where $(A, \varphi) \in \mathfrak{A}^F$. Since the composition

$$S \xrightarrow{S\beta} STS \xrightarrow{\alpha S} S$$

is the identity, it follows that for any $A \in \mathfrak{A}$, $S(A)$ is a retract of $STS(A)$. Thus $S(A)$ is $\mathcal{G}$-projective. It follows that the $\mathcal{E}_0^F$-projective and the $\mathcal{G}$-projective objects coincide and thus $\mathcal{E}_0^F = \mathcal{G}$. In particular, the canonical resolution yields an $\mathcal{E}_0^F$-projective resolution for every object of $\mathfrak{A}^F$.

If the category $\mathfrak{A}$ is preadditive and the functor F is additive then the category $\mathfrak{A}^F$ also is preadditive and the functors S, T and $G = ST$ are additive. If further $\mathfrak{A}$ has kernels, then $\mathfrak{A}^F$ has kernels. We shall show that the conditions of 4.1 are satisfied and therefore the standard complex for the cotriple $\mathbf{G}$ yields $\mathcal{E}_0^F$-projective resolutions.

Indeed, the exact sequence

$$0 \to K \xrightarrow{\kappa} G \xrightarrow{\epsilon} 1_{\mathfrak{A}^F} \to 0$$

is the exact sequence

$$0 \to K \xrightarrow{\kappa} ST \xrightarrow{\alpha} 1_{\mathfrak{A}^F} \to 0.$$

Since T preserves exact sequences, it follows that

$$0 \to TK \xrightarrow{T\kappa} TST \xrightarrow{T\alpha} T \to 0 \tag{5.2}$$

is exact. Since $(T\alpha)(\beta T) = 1_T$ it follows that the sequence (5.2) is split exact and therefore since S is additive

$$0 \to STK \xrightarrow{ST\kappa} STST \xrightarrow{ST\alpha} ST \to 0$$

is exact. Thus the sequence

$$0 \to GK \xrightarrow{G\kappa} G^2 \xrightarrow{G\varepsilon} G \to 0$$

is exact, as required.

6. Examples

Let K be a commutative ring and $\mathfrak{A}$ the category of K-modules. Then $\mathfrak{A}$ is an abelian category. $\mathfrak{A}$ is also a K-category, so that $\mathfrak{A}(A, A')$ is again an object of $\mathfrak{A}$. The tensor product $A \otimes B$ over K yields a functor $\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$

and we have the natural isomorphism

$$(6.1)\qquad a : \mathfrak{A}(\Lambda \otimes B, A) \to \mathfrak{A}(B, \mathfrak{A}(\Lambda, A)).$$

We shall also employ the standard identifications

$$(6.2)\qquad K \otimes A = A, \qquad \mathfrak{A}(K, A) = A$$

Let Λ be a K-algebra. Then we have morphisms

$$(6.3)\qquad \bar{\eta} : K \to \Lambda, \qquad \bar{\mu} : \Lambda \otimes \Lambda \to \Lambda$$

satisfying the usual identities. There results a triple $\mathbf{F} = (F, \eta, \mu)$ where $F = \Lambda\otimes$

$$\eta(A) = \bar{\eta} \otimes \Lambda, \qquad \mu(A) = \bar{\mu} \otimes A.$$

We also have a cotriple $\mathbf{G} = (\delta, \varepsilon, G)$ where $G = \mathfrak{A}(\Lambda, \ )$

$$\varepsilon(A) = \mathfrak{A}(\bar{\eta}, A) : \mathfrak{A}(\Lambda, A) \to \mathfrak{A}(K, A) = A$$

and $\delta(A)$ is the composition

$$\mathfrak{A}(\Lambda, A) \xrightarrow{\mathfrak{A}(\bar{\mu}, A)} \mathfrak{A}(\Lambda \otimes \Lambda, A) \xrightarrow{a} \mathfrak{A}(\Lambda, \mathfrak{A}(\Lambda, A)).$$

The relation (6.1) yields $a : F \dashv G$. Further it is easy to verify that $\eta \dashv \varepsilon$ and $\mu \dashv \delta$. Thus we have $a : \mathbf{F} \dashv \mathbf{G}$.

Let ${}_\Lambda M$ be the category of left Λ-modules, $T : {}_\Lambda M \to \mathfrak{A}$ the usual "forgetful" functor and let $S, R : A \to {}_\Lambda M$ be defined by $S = \Lambda\otimes$, $S' = \mathfrak{A}(\Lambda, \ )$. Then the relation (6.1) induces adjointness relations

$$a_1 : S \dashv T : ({}_\Lambda M, \mathfrak{A})$$

$$a_2 : T \dashv R : (\mathfrak{A}, {}_\Lambda M)$$

which are respectively the universal generator for $\mathbf{F}$ and the universal cogenerator for $\mathbf{G}$, in agreement with 3.3.

Using theorems of Watts [5] it is easy to show that (up to isomorphisms) the K-algebras yield all the triples $\mathbf{F}$ and all the cotriples $\mathbf{G}$ in $\mathfrak{A}$ such that F preserves cokernels and (arbitrary) coproducts (i.e. direct sums) while G preserves kernels and (arbitrary) products.

A K-coalgebra Λ is given by morphisms

$$(6.4)\qquad \bar{\varepsilon} : \Lambda \to K, \qquad \bar{\delta} : \Lambda \to \Lambda \otimes \Lambda$$

satisfying the usual identities. There results a cotriple $\mathbf{G} = (\delta, \varepsilon, G)$ in $\mathfrak{A}$ where $G = \Lambda\otimes$

$$\varepsilon(A) = \bar{\varepsilon} \otimes A, \qquad \delta(A) = \bar{\delta} \otimes A.$$

We also have a triple $\mathbf{F} = (F, \eta, \mu)$ where $F = \mathfrak{A}(\Lambda, \ )$

$$\eta(A) = \mathfrak{A}(\bar{\varepsilon}, A) : A = \mathfrak{A}(K, A) \to \mathfrak{A}(\Lambda, A)$$

and $\mu(A)$ is the composition

$$\mathcal{A}(\Lambda, \mathcal{A}(\Lambda, A)) \xrightarrow{a^{-1}} \mathcal{A}(\Lambda \otimes \Lambda, A) \xrightarrow{\mathcal{A}(\bar{\delta}, A)} A(\Lambda, A).$$

We still have the relations $a : G \dashv F$, $\varepsilon \dashv \eta$ and $\delta \dashv \mu$, so that in a sense we have $\mathbf{G} \dashv \mathbf{F}$. However $\mathbf{G}$ being a cotriple and $\mathbf{F}$ being a triple, 3.3 no longer applies. Indeed, the construction of the universal generator for $\mathbf{F}$ yields the category ${}^{\Lambda}M$ of left comodules (i.e. K-modules A with a morphism $A \to \Lambda \otimes A$ satisfying suitable identities) while the construction of the universal cogenerator for $\mathbf{G}$ yields the category ${}^{\Lambda}M^{\#}$ of Λ-contramodules (i.e. K-modules A with a morphism $\mathcal{A}(\Lambda, A) \to A$ satisfying suitable identities) [see 2, Ch. III, §5]. These categories are in general distinct except when Λ is K-projective and finitely generated over K, in which case ${}^{\Lambda}M$ and ${}^{\Lambda}M^{\#}$ are both isomorphic with the category ${}_{\hat{\Lambda}}M$ where $\hat{\Lambda} = \mathcal{A}(\Lambda, K)$ is the K-algebra dual to the coalgebra Λ.

Again it can be shown that (up to isomorphisms) the K-coalgebras yield all the triples $\mathbf{F}$ and all the cotriples $\mathbf{G}$ in which F preserves kernels and (arbitrary) products while G preserves cokernels and (arbitrary) coproducts.

Bibliography

1. A. Dold and D. Puppe, *Homologie nicht-additiver Functoren*, Ann. Inst. Fourier, vol. 11 (1961), pp. 201–312.
2. S. Eilenberg and J. C. Moore, *Foundations of relative homological algebra*, Mem. Amer. Math. Soc., no. 55 (1965).
3. R. Godement, *Theorie de faisceaux*, Paris, Hermann 1958.
4. P. J. Huber, *Homotopy theory in general categories*, Math. Ann., vol. 144 (1961), pp. 361–385.
5. C. E. Watts, *Intrinsic characterizations of some additive functors*, Proc. Amer. Math. Soc., vol. 11 (1960), pp. 5–8.

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