A GENERATOR FOR A SET OF FUNCTIONS

BY

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1. Introduction

Suppose that $H = [S, +, \|\cdot\|]$ is a normed complete abelian group and that D_0 is a subset of S. Suppose furthermore that for each x in D_0, T_x is a function from $[0, 1)$ to S. The main problem considered is that of finding a Stieltjes-Volterra integral equation

$$(*) \qquad T_x(t) = T_x(s) + \int_s^t dF \cdot T_x, \qquad 0 \leq s \leq t < 1,$$

which is satisfied for all x in D_0. The integral used is similar to one used in [6].

Some ways in which such function collections arise are now described.

(i) If H is a linear space and for each $t \geq 0$, $M(t)$ is a bounded linear transformation on H such that $M(s)M(t) = M(s + t)$ if $s, t \geq 0$, then one has a semi-group of bounded linear transformations of the kind considered so extensively in [1]. Here one may define $T_x(s) = M(s)x$ for $0 \leq s < 1$. In some cases an examination of the function F in $(*)$ yields a generator for M (see Section 3 of this paper). Results of this paper seem to apply only to the "uniform" case of [1].

(ii) If f is a continuous function from $S \times R$ (R is the real line) to S so that (I) $f(p, 0) = p$ for all p in S and (II) $f(f(p, t_1), t_2) = f(p, t_1 + t_2)$ for all p in S and t_1, t_2 in R, then f is a dynamical system (see for example [5]). One may define $T_x(t) = f(x, t)$ for all x in S (or perhaps some subset D_0 of S) and $0 \leq t < 1$. In some cases in which f is generated by a system of differential equations, $(*)$ is equivalent to this system (see Example 3, Section 5 of [6]).

(iii) Suppose M is a continuous harmonic operator (see [7] or [3] for a discussion and references), that is, M is a function from $R \times R$ to the set of all bounded linear transformations on S such that M is continuous and of bounded variation with respect to its first place, continuous with respect to its second place and for each number triple r, s, t, $M(r, s)M(s, t) = M(r, t)$ and $M(r, r) = I$. Then, one may define $T_x(s) = M(s, 0)x$ for $0 \leq s < 1$. Then F in $(*)$ generates the restriction of M to $[0, 1) \times [0, 1)$. Results of this paper applied to the harmonic operator case duplicate some results of [7] and [2].[1]

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[1] Some recent results of Mac Nerney [4] extend the linear theory of [3] to nonlinear problems. Some overlap can be seen in both the results and the methods of the present

In [6], the problem of obtaining families of functions like the set of T_x, x in D_0, was considered. The main process used there can be described as an exponential process. In this paper, the opposite, i.e., logarithmic, process is considered.

2. The main result

DEFINITION. If Q is a number set, then the statement that V is a variation function for Q means that V is a function from $Q \times Q$ to a non-negative number set such that if each of s, p and t is in Q and p is in $[s, t]$, then

$$V(s, p) = V(p, s) \quad \text{and} \quad V(s, p) + V(p, t) = V(s, t).$$

A definition for integral and a sufficient condition for existence are given.

DEFINITION. Suppose that $[a, b]$ is a number interval, X is a function from $[a, b]$ to S and F is a function on $[a, b]$ such that if t is in $[a, b]$, $F(t)$ is a transformation from a subset of S to a subset of S. The statement that X is F-integrable from a to b means that there is a point w in S such that if $\varepsilon > 0$, there there is a $\delta > 0$ such that if $t_0, \cdots, t_{n+1}$ is a chain from a to b with mesh $< \delta$ and $s_0, \cdots, s_n$ is an interpolation sequence for $t_0, \cdots, t_{n+1}$, then

$$\left\| w - \textstyle\sum_{i=0}^{n} [F(t_{i+1}) - F(t_i)]X(s_i) \right\| < \varepsilon.$$

Such a point w is of course unique and is denoted by $\int_a^b dF \cdot X$.

LEMMA 0. *Suppose that $[a, b]$ is a number interval, U is a variation function for $[a, b]$ and each of X and F is a function as in the first sentence of the above definition. Suppose in addition that (1) X is continuous and (2) there is a $\delta > 0$ such that if each of s and t is in $[a, b]$, each of u and v is in $[s, t]$ and $|s - t| < \delta$, then each of $X(u)$ and $X(v)$ is in the domain of $F(t) - F(s)$ and*

$$\| [F(t) - F(s)]X(u) - [F(t) - F(s)]X(v) \| \le U(t, s) \| X(u) - X(v) \|.$$

Then X is F-integrable from a to b.

A proof which follows closely an existence proof for ordinary integrals is omitted. This lemma is similar to Theorem E of [6].

With $H = [S, +, \|\cdot\|]$ a normed complete abelian group and D_0 a subset of S, suppose that if x is in D_0, then T_x is a function from $[0, 1)$ to S such that $T_x(0) = x$. If t is in $[0, 1)$, denote by D_t the set of all $T_x(t)$ for all x in D_0. Denote by G the set of all $(t, T_y(t))$ for all y in D_0 and all t in $[0, 1)$. Denote by I the identity transformation on S.

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study and [4] if the underlying linear system of the latter is assumed to be a linear continuum. In making comparisons it should be noted that the functions Δ and M in this paper correspond to V and W respectively in [4]. In the notation of [4], the point y in Lemma 5 is denoted by ${}_t\sum^s[M - I]w$ and the point $T_y(t)$ in part (B) of the theorem is denoted by ${}_t\prod^s[1 + \Delta]w$.

THEOREM. *Suppose that each of U and V is a continuous variation function for* $[0, 1)$ *so that*

(1) $\| T_x(t) - T_x(s) \| \leq V(t, s)$ *if x is in D_0 and* $0 \leq s \leq t < 1$,

(2) $\| [T_x(t) - T_x(s)] - [T_y(t) - T_y(s)] \| \leq U(t, s) \| T_x(s) - T_y(s) \|$ *if each of x and y is in D_0 and* $0 \leq s \leq t < 1$, *and*

(3) *the set G is open with respect to* $[0, 1) \times S$.

There is a function F on $[0, 1)$ *such that*

(1) *if t is in* $[0, 1)$ *then $F(t)$ is a transformation from D_t to S and*

(2) *the following hold:*

(A) $T_y(t) = T_y(s) + \int_s^t dF \cdot T_y$ *if* $0 \leq s \leq t < 1$ *and*

(B) *if* $\varepsilon > 0$, *and* $0 \leq s \leq t < 1$ *there is a* $\delta > 0$ *such that if* $t_0, \cdots, t_{n+1}$ *is a chain from s to t of mesh* $< \delta$, *then*

$$\left\| T_y(t) - \left\{ \prod_{i=0}^{n} [I + F(t_{i+1}) - F(t_i)] \right\} T_y(s) \right\| < \varepsilon.$$

A proof is developed by means of a sequence of lemmas, all of which are under the hypothesis of the theorem.

LEMMA. 1. *If each of x and y is in D_0, s is in* $[0, 1)$ *and* $T_x(s) = T_y(s)$, *then* $T_x(t) = T_y(t)$ *if* $s < t < 1$.

Proof of Lemma 1.

$$\| T_x(t) - T_y(t) \| = \| [T_x(t) - T_x(s)] - [T_y(t) - T_y(s)] \| \leq U(s, t) \| T_x(s) - T_y(s) \| = 0$$

so that $T_x(t) = T_y(t)$.

Notation. If s is in $[0, 1)$ and w is in D_s, then $I(w, s)$ denotes the set of all numbers t such that if u is in $[s, t]$, then w is in D_u. Note that such a set $I(w, s)$ is open with respect to $[0, 1)$. If $0 \leq s \leq t < 1$, then $M(t, s)$ denotes the function from D_s to D_t such that if w is in D_s, $M(t, s)w = T_x(t)$ where x is such that $w = T_x(s)$.

Note that (2) in the hypothesis of the theorem is equivalent to the following: $\| [M(t, s) - I]w - [M(t, s) - I]z \| \leq U(t, s) \| w - z \|$ if $0 \leq s \leq t < 1$ and each of w and z is in D_s. Also note that (1) in the hypothesis of the theorem is equivalent to $\| [M(t, s) - I]w \| \leq V(t, s)$ under the same conditions.

LEMMA 2. *Suppose that* $0 \leq s < 1$, *w is in D_s and t is in $I(w, s)$. If* $s \leq a \leq b \leq t$ *and* $t_0, \cdots, t_{n+1}$ *is a chain from a to b, then*

$$\left\| [M(b, a) - I]w - \left\{ \sum_{i=0}^{n} [M(t_{i+1}, t_i) - I] \right\} w \right\| \leq U(b, a) V(b, a).$$

Proof of Lemma 2.

$$\left\| [M(b, a) - I]w - \left\{ \sum_{i=0}^{n} [M(t_{i+1}, t_i) - I] \right\} w \right\|$$

$$= \left\| \sum_{i=0}^{n} \{ [M(t_{i+1}, a) - M(t_i, a)]w - [M(t_{i+1}, t_i) - I]w \} \right\|$$

$$\leq \sum_{i=0}^{n} \| [M(t_{i+1}, t_i) - I]M(t_i, a)w - [M(t_{i+1}, t_i) - I]w \|$$
$$\leq \sum_{i=0}^{n} U(t_{i+1}, t_i) \| M(t_i, a)w - w \|$$
$$\leq \sum_{i=0}^{n} U(t_{i+1}, t_i)V(t_i, a) \leq U(b, a)V(b, a).$$

LEMMA 3. *Suppose that* $0 \leq s < 1$, w *is in* D_s, t *is in* $I(w, s)$, $a < b$, *each of* a *and* b *is in* $[s, t]$ *and* $r = r_0, \cdots, r_{n+1}$ *is a chain from* a *to* b. *If* $q_0, \cdots, q_{m+1}$ *is a refinement of* r, *then*

$$\| \{\sum_{i=0}^{n} [M(r_{i+1}, r_i) - I]\}w - \{\sum_{i=0}^{m} [M(q_{i+1}, q_i) - I]\}w \| \leq \sum_{i=0}^{n} V(r_{i+1}, r_i)U(r_{i+1}, r_i).$$

This follows easily from Lemma 2 and a proof is omitted.

LEMMA 4. *Suppose that* $0 \leq s \leq t < 1$, w *is in* D_s *and* t *is in* $I(w, s)$. *If* $\varepsilon > 0$, *there is a number* $\delta > 0$ *such that if each of* $r = r_0, \cdots, r_{n+1}$ *and* $q = q_0, \cdots, q_{m+1}$ *is a chain from* s *to* t *of mesh* $< \delta$, *then*

$$\| \{\sum_{i=0}^{n} [M(r_{i+1}, r_i) - I]\}w - \{\sum_{i=0}^{m} [M(q_{i+1}, q_i) - I]\}w \| < \varepsilon.$$

Proof of Lemma 4. Suppose $\varepsilon > 0$. Denote by δ a positive number so that if each of a and b is in $[s, t]$ and $| a - b | < \delta$, then

$$V(b, a) < \varepsilon/[2 + 2U(b, a)].$$

Denote by each of $r = r_0, \cdots, r_{n+1}$ and $q = q_0, \cdots, q_{m+1}$ a chain from s to t of mesh $< \delta$ and by $v = v_0, \cdots, v_{u+1}$ a common refinement of r and q. By Lemma 3,

$$\| \{\sum_{i=0}^{n} [M(r_{i+1}, r_i) - I]\}w - \{\sum_{i=0}^{m} [M(q_{i+1}, q_i) - I]\}w \|$$
$$\leq \| \{\sum_{i=0}^{n} [M(r_{i+1}, r_i) - I]\}w - \{\sum_{i=0}^{u} [M(v_{i+1}, v_i) - I]\}w \|$$
$$+ \| \{\sum_{i=0}^{u} [M(v_{i+1}, v_i) - I]\}w - \{\sum_{i=0}^{m} [M(q_{i+1}, q_i) - I]\}w \|$$
$$\leq \sum_{i=1}^{n} V(r_{i+1}, r_i)U(r_{i+1}, r_i) + \sum_{i=0}^{m} V(q_{i+1}, q_i)U(q_{i+1}, q_i) < \varepsilon$$

since $| r_{i+1} - r_i | < \delta$, $i = 0, \cdots, n$ and $| q_{i+1} - q_i | < \delta$, $i = 0, \cdots, m$.

LEMMA 5. *Suppose that* $0 \leq s \leq t < 1$, w *is in* D_s *and* t *is in* $I(w, s)$. *There is a unique point* y *of* S *with the following property*: *If* $\varepsilon > 0$, *there is a* $\delta > 0$ *so that if* $t_0, \cdots, t_{n+1}$ *is a chain from* s *to* t *with mesh* $< \delta$, *then*

$$\| y - \{\sum_{i=0}^{n} [M(t_{i+1}, t_i) - I]\}w \| < \varepsilon.$$

Indication of proof of Lemma 5. Lemma 4 yields the fact that

$$\{\{\sum_{i=0}^{n} [M(s + (i/n)(t - s), s + [(i - 1)/n](t - s)] - I\}w\}_{n=1}^{\infty}$$

is a Cauchy sequence. Denote its limit by y. A simple argument (which is omitted) gives that y satisfies the conclusion of Lemma 5.

It is remarked that it follows from Lemma 3 that if $t_0, \cdots, t_{n+1}$ is a chain from s to t, then

$$\| y - \{\textstyle\sum_{i=0}^{n} [M(t_{i+1}, t_i) - I]\}w \| \leq \textstyle\sum_{i=0}^{n} V(t_{i+1}, t_i)U(t_{i+1}, t_i)$$

and in particular, $\| y - [M(t, s) - I]w \| \leq V(t, s)U(t, s)$.

A point y satisfying the conclusion to Lemma 5 is denoted by $\Delta(t, s)w$. Note that if $0 \leq s \leq t < 1$, w is in D_s and t is in $I(w, s)$, then $\Delta(t, s)w$ is defined.

LEMMA 6. *If $0 \leq s \leq p \leq t < 1$, w is in D_s and each of p and t is in $I(w, s)$, then $\Delta(t, p)w + \Delta(p, s)w = \Delta(t, s)w$.*

A simple argument is omitted.

LEMMA 7. *If y is in D_0 and $0 \leq s \leq t < 1$, there is a $\delta > 0$ such that if $s \leq u \leq t$, $\| T_y(u) - x \| < \delta$ and $u \leq v \leq u + \delta$, then x is in D_v.*

Proof of Lemma 7. Suppose the lemma is false. Denote by y an element of D_0, by each of s and t a number in $[0, 1)$, by each of $\{u_i\}_{i=1}^{\infty}$ and $\{v_i\}_{i=1}^{\infty}$ a number sequence and by $\{x_i\}_{i=1}^{\infty}$ a point sequence in S so that

$$s \leq u_i \leq t, u_i \leq v_i \leq u_i + 1/i, \| T_y(u_i) - x_i \| < 1/i$$

and x_i is not in D_{v_i}, $i = 1, 2, \cdots$.

Denote by $\{n_i\}_{i=1}^{\infty}$ an increasing sequence of positive integers so that $\{u_{n_i}\}_{i=1}^{\infty}$ converges and denote by u the limit of this sequence. Then, u is also the limit of $\{v_{n_i}\}_{i=1}^{\infty}$. Since u is in $[0, 1)$ and G is open in $[0, 1) \times S$, there is a $\delta > 0$ so that if x is in S, $\| T_y(u) - x \| < \delta$ and q is in both $[0, 1)$ and $[u - \delta, u + \delta]$, then (q, x) is in G. Denote by δ_1 a positive number $\leq \delta$ so that if $| v - u | \leq \delta_1$ then $\| T_y(u) - T_y(v) \| < \delta/2$. Denote by i an integer so that $1/i < \delta_1/2$ and $| u_{n_i} - u | < \delta_1/2$. Then,

$$\| T_y(u_{n_i}) - T_y(u) \| < \delta/2, \quad \| T_y(u_{n_i}) - x_{n_i} \| < 1/n_i < \delta/2,$$

and hence $\| T_y(u) - x_{n_i} \| < \delta$. But $| u - v_{n_i} | < \delta$ so that (v_{n_i}, x_{n_i}) is in G, a contradiction.

It is remarked that since T_y is uniformly continuous on closed subsets of $[0, 1)$, it follows from Lemma 7 that if $0 \leq s \leq t < 1$ and y is in D_0, then there is a $\delta > 0$ so that if each of a and b is in $[s, t]$, $0 \leq b - a < \delta$ and u is in $[a, b]$, then $T_y(u)$ is in the domain of $\Delta(b, a)$.

LEMMA 8. If $0 \leq s \leq t < 1$, each of w and x is in D_s and t is in $I(w, s)$ and $I(x, s)$, then

$$\| \Delta(t, s)w \| \leq V(t, s) \quad \textit{and} \quad \| \Delta(t, s)w - \Delta(t, s)x \| \leq U(t, s) \| w - x \|.$$

Proof of Lemma 8. Suppose that $\varepsilon > 0$. Denote by $t_0, \cdots, t_{n+1}$ a chain from s to t so that

$$\| \Delta(t, s)w - \{\textstyle\sum_{i=0}^{n} [M(t_{i+1}, t_i) - I]\}w \| < \varepsilon$$

and

$$\| \Delta(t, s)x - \{ \textstyle\sum_{i=0}^{n} [M(t_{i+1}, t_i) - I] \} x \| < \varepsilon.$$

Since

$$\begin{aligned} \| \{ \textstyle\sum_{i=0}^{n} [M(t_{i+1}, t_i) - I] \} x \| &\le \textstyle\sum_{i=0}^{n} \| [M(t_{i+1}, t_i) - I]x \| \\ &\le \textstyle\sum_{i=0}^{n} V(t_{i+1}, t_i) \le V(t, s) \end{aligned}$$

and

$$\begin{aligned} &\| \{ \textstyle\sum_{i=0}^{n} [M(t_{i+1}, t_i) - I] \} w - \{ \textstyle\sum_{i=0}^{n} [M(t_{i+1}, t_i) - I] \} x \| \\ &\qquad \le \textstyle\sum_{i=0}^{n} \| [M(t_{i+1}, t_i) - I]w - [M(t_{i+1}, t_i) - I]x \| \\ &\qquad \le \textstyle\sum_{i=0}^{n} U(t_{i+1}, t_i) \| w - x \| \le U(t, s) \| w - x \| \end{aligned}$$

it follows that

$$\| \Delta(t, s)w \| < \varepsilon + V(t, s)$$

and

$$\| \Delta(t, s)w - \Delta(t, s)x \| \le 2\varepsilon + U(t, s) \| w - x \|,$$

from which the lemma follows.

Notation. Denote by F the function from $[0, 1)$ such that if t is in $[0, 1)$, then $F(t)$ is the transformation K from D_t to S with the following property: If w is in D_t and q is the mid-point of $I(w, t)$, then

$$Kw = \Delta(t, q)w \quad \text{if} \quad t \ge q \quad \text{and} \quad Kw = -\Delta(q, t)w \quad \text{if} \quad t < q.$$

Note that if $0 \le s \le t < 1$, w is in D_s and t is in $I(w, s)$, then $I(w, t) = I(w, s)$, $\Delta(t, s)w$ is defined and $[F(t) - F(s)]\, w = \Delta(t, s)w$.

It is remarked that it follows from the second part of the conclusion to Lemma 8 that if $0 \le s < 1$ and w is in D_s, then there is a $\delta > 0$ such that if $s \le t \le s + \delta$, then $F(t) - F(s)$ is continuous at w.

LEMMA 9. *If $0 \le s \le t < 1$ and y is in D_0, then $\int_s^t dF \cdot T_y$ exists.*

Proof of Lemma 9. It follows from the remark following Lemma 7 that there is a number $\delta > 0$ so that if each of a and b is in $[s, t]$ and $| b - a | < \delta$, then $T_y(u)$ is in the domain of $F(b) - F(a)$ for all u in $[a, b]$. Hence by Lemma 8, if each of b and a is in $[s, t]$, $| b - a | < \delta$ and each of u and v is in $[a, b]$, then

$$\begin{aligned} &\| [F(b) - F(a)]T_y(u) - [F(b) - F(a)]T_y(v) \| \\ &\qquad\qquad \le U(b, a) \| T_y(u) - T_y(v) \|. \end{aligned}$$

Since T_y is continuous, this lemma follows from Lemma 0.

Proof of part (A) *of the theorem.* Suppose $\varepsilon > 0$. Denote by δ_1 a positive number so that if $t_0, \cdots, t_{n+1}$ is a chain from s to t with mesh $< \delta_1$ and $s_0, \cdots, s_n$ is an interpolation sequence for $t_0, \cdots, t_{n+1}$ then

$$\| \textstyle\int_s^t dF \cdot T_y - \sum_{i=0}^{n} [F(t_{i+1}) - F(t_i)]T_y(s_i) \| < \varepsilon/2.$$

Denote by δ_2 a positive number so that if each of u and v is in $[s, t]$ and $| u - v | < \delta_2$, then

$$V(u, v) < \varepsilon/[2 + 2U(t, s)].$$

Denote $\min(\delta_1, \delta_2)$ by δ. Denote by $t_0, \cdots, t_{n+1}$ a chain from s to t with mesh $< \delta$. Then,

$$\| \textstyle\int_s^t dF \cdot T_y - \sum_{i=0}^n [F(t_{i+1}) - F(t_i)]T_y(t_i) \| < \varepsilon/2$$

and

$$\begin{aligned}
&\| [T_y(t) - T_y(s)] - \textstyle\sum_{i=0}^n [F(t_{i+1}) - F(t_i)]T_y(t_i) \| \\
&\quad = \| \textstyle\sum_{i=0}^n \{[T_y(t_{i+1}) - T_y(t_i)] - [F(t_{i+1}) - F(t_i)]T_y(t_i)\} \| \\
&\quad \le \textstyle\sum_{i=0}^n \| T_y(t_{i+1}) - T_y(t_i) - [F(t_{i+1}) - F(t_i)]T_y(t_i) \| \\
&\quad = \textstyle\sum_{i=0}^n \| [M(t_{i+1}, t_i) - I]T_y(t_i) - \Delta(t_{i+1}, t_i)T_y(t_i) \| \\
&\quad \le \textstyle\sum_{i=0}^n V(t_{i+1}, t_i)U(t_{i+1}, t_i) \\
&\quad \le \max_{i=0,\cdots,n} V(t_{i+1}, t_i)U(t, s) < \varepsilon/2.
\end{aligned}$$

Hence, $\| [T_y(t) - T_y(s)] - \int_s^t dF \cdot T_y \| < \varepsilon$ for every $\varepsilon > 0$, that is,

$$T_y(t) = T_y(s) + \textstyle\int_s^t dF \cdot T_y .$$

This completes a proof to part (A) of the theorem.

Proof of part (B) *of the theorem.* Suppose that y is in D_0, $0 \le s \le t < 1$ and $\varepsilon > 0$. Denote by δ_1 a positive number such that if $s \le u \le t$, $\| T_y(u) - x \| < \delta_1$ and $u \le v \le u + \delta_1$, then x is in D_v. Denote by δ a positive number $< \delta_1$ so that if each of a and b is in $[s, t]$ and $| b - a | < \delta$, then $V(b, a) < \min(\varepsilon, \delta_1) \exp(-U(t, s))$.

Suppose that $t_0, \cdots, t_{n+1}$ is a chain from s to t with mesh $< \delta$. Denote $I + F(t_i) - F(t_{i-1})$ by J_i, $i = 1, \cdots, n + 1$. Denote

$$\min(\varepsilon, \delta_1) \exp(-U(t, s))$$

by R. It will now be shown that

$$\| T_y(t_i) - J_i \cdots J_1 T_y(s) \| \le R[\exp U(t_i, s) - 1], \quad i = 1, \cdots, n + 1.$$

$$\begin{aligned}
\| T_y(t_1) - J_1 T_y(s) \| &= \| [T_y(t_1) - T_y(s)] - [F(t_1) - F(t_0)]T_y(s) \| \\
&\le V(t_1, t_0)U(t_1, t_0) \le R[\exp U(t_1, s) - 1].
\end{aligned}$$

Suppose that i is a positive integer $< n + 1$ and that

$$\| T_y(t_i) - J_i \cdots J_1 T_y(s) \| < R[\exp U(t_i, s) - 1].$$

Since

$$R[\exp U(t_i, s) - 1] \le \min(\varepsilon, \delta_1) \exp(-U(t, s)) \exp U(t_i, s) \le \delta_1,$$

it follows that $J_{i+1} T_y(t_i)$ and $J_{i+1} J_i \cdots J_1 T_y(s)$ are defined and that

$$\begin{aligned}
&\| T_y(t_{i+1}) - J_{i+1} J_i \cdots J_1 T_y(s) \| \\
&\quad \leq \| M(t_{i+1}, t_i) T_y(t_i) - J_{i+1} T_y(t_i) \| + \| J_{i+1} T_y(t_i) - J_{i+1} \cdots J_1 T_y(s) \| \\
&\quad \leq \| [M(t_{i+1}, t_i) - I] T_y(t_i) - [F(t_{i+1}) - F(t_i)] T_y(t_i) \| \\
&\qquad + \| T_y(t_i) - J_i \cdots J_1 T_y(s) \| \\
&\qquad + \| [F(t_{i+1}) - F(t_i)] T_y(t_i) - [F(t_{i+1}) - F(t_i)] J_i \cdots J_1 T_y(s) \| \\
&\quad \leq V(t_{i+1}, t_i) U(t_{i+1}, t_i) \\
&\qquad + R[\exp U(t_i, s) - 1] + U(t_{i+1}, t_i) R[\exp U(t_i, s) - 1] \\
&\quad \leq R\{ U(t_{i+1}, t_i) + [\exp U(t_i, s) - 1][1 + U(t_{i+1}, t_i)] \} \\
&\qquad = R\{ [1 + U(t_{i+1}, t_i)] \exp U(t_i, s) - 1 \} \leq R[\exp U(t_{i+1}, s) - 1].
\end{aligned}$$

Hence,

$$\begin{aligned}
&\| T_y(t) - \{ \textstyle\prod_{i=0}^{n} [I + F(t_{i+1}) - F(t_i)] \} T_y(s) \| \\
&\qquad = \| T_y(t_{n+1}) - J_{n+1} \cdots J_1 T_y(s) \| \leq R[\exp U(t_{n+1}, s) - 1] < \epsilon.
\end{aligned}$$

This completes a proof to part (B) of the theorem.

3. An application to semi-groups

In this section a connection with semi-groups of transformations is established.

COROLLARY. *If in addition to the hypothesis of the theorem it is true that H is a linear space and $M(t, s) = M(t - s, 0)$ for $0 \leq s \leq t < 1$, then there is a continuous function A from D_0 to S such that if $F(t) = tA$ for $0 \leq t < 1$, then A and B of the theorem hold.*

Proof of the corollary. If $0 \leq \delta < 1$, denote $M(\delta, 0)$ by $Q(\delta)$. Note that $D_s = D_0$ and hence $I(w, s) = [0, 1)$ for all s in $[0, 1)$ and w in D_0. Also note that $Q(s)Q(t) = Q(s + t)$ provided that each of s, t and $s + t$ is in $[0, 1)$. As in the proof of Lemma 5, if w is in D_0,

$$\{ \{ \textstyle\sum_{i=1}^{n} [Q(1/2n) - I] \} w \}_{n=1}^{\infty} = \{ n[Q(1/2n) - I] w \}_{n=1}^{\infty}$$

converges. Denote by A a transformation from D_0 to S such that the limit of this sequence is $\frac{1}{2} Aw$. If $0 \leq s < 1$ and n is a positive integer, denote by s_n the largest integer p so that $p/2n \leq s$. Again as in the proof of Lemma 5,

$$\{ [Q(s - s_n/2n) - I] w + \{ \textstyle\sum_{i=1}^{s_n} [Q(1/2n) - I] \} w \}_{n=1}^{\infty}$$

converges for each point w of D_0. Since $\| [Q(s - s_n/2n) - I] w \| \to 0$ as $n \to \infty$, it follows that

$$\{ \textstyle\sum_{i=0}^{s_n} [Q(1/2n) - I] \} w = s_n/n \{ \textstyle\sum_{i=1}^{n} [Q(1/2n) - I] \} w \to sAw \quad \text{as} \quad n \to \infty.$$

Denote sA by $F_1(s)$ and $(s - \frac{1}{2})A$ by $F(s)$ for $0 \leq s < 1$. Then F is as defined above since $\frac{1}{2}$ is the mid-point of $I(w, s)$ for all w in D_s. Hence (A) and (B) follow. They also are true for F replaced by F_1 since $F_1(t) - F_1(s) = F(t) - F(s)$ for $0 \leq s \leq t < 1$. That A is continuous follows from the remark preceding the statement of Lemma 9. This completes the argument for the corollary.

Since $D_s = D_0$ for $0 \leq s < 1$, the definition of Q may be extended to the nonnegative real axis in the following way: if $t \geq 1$ denote by n a positive integer so that $t/n < 1$. Define $Q(t)$ to be $[Q(t/n)]^n$. Then, Q forms a semi-group of transformations. The transformation A defined above may be said to generate Q.

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