A CLASS OF GENERALIZED TI-GROUPS

BY

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This paper generalizes Suzuki's characterization of TI-groups. Specifically the following theorem is proved.

THEOREM. *Let G be a finite group in which the intersection of any two distinct Sylow 2-subgroups has 2-rank at most one. Let $O'(G)/O(G)$ be the smallest normal subgroup of $G/O(G)$ of odd index. Then $O'(G)/O(G)$ is one of the following:*

(1) *a 2-group,*

(2) $GL_2(3)$, $SL_2(q)$, *or the perfect nontrivial central extension of A_7 by a 2-group, or an extension of rank 1 of such a group,*

(3) *the extension of a 2-group by $L_2(q)$, $Sz(q)$, or $U_3(q)$, q even,*

(4) *the central product of two copies of $SL_2(5)$ with amalgamated centers, or its extension by an automorphism permuting the copies,*

(5) $L_2(q)$, $q \equiv 3, 5 \bmod 8$, *or $J(11)$, the smallest Janko group.*

The proof of the above theorem is a reasonably straightforward application of results of Alperin, Glauberman, and Shult on fusion, plus several classification theorems. The author would like to thank Professor John Walter for pointing out several errors in the original version of this paper.

1. kI-groups

Let G be a finite group. The *2-rank* $r(G)$ of G is the number of generators of an elementary 2-subgroup of G of maximal order if $|G|$ is even; if $|G|$ is odd, $r(G) = 0$. For k a nonnegative integer, we define G to be a *kI-group* if $r(G) > k$ and for any two distinct S_2-groups S and T of G, $r(S \cap T) \leq k$. If $k = 0$, G is a TI or "trivial intersection" group as defined by Suzuki [7].

The following elementary result is essentially Lemma 1 in [7].

LEMMA 1. *Let G be a kI-group. Then*

(1) *if $H \leq G$ with $r(H) > k$ then H is a kI-group,*

(2) *if H is a normal subgroup of odd order in G, then G/H is a kI-group.*

LEMMA 2. *Let G be a kI-group, S an S_2-group of G, $N = N_G S$ and $A \leq S$ such that either $r(A) > k$ or A is elementary of rank $k = 1$. Then*

$$\{A^g : g \in G, A^g \leq N\} = \{A^x : x \in N\}.$$

Proof. Alperin's theorem on fusion [1].

LEMMA 3. *Let G be a 2-nilpotent kI-group. Then either G is 2-closed or $r(G) = k + 1$.*

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Proof. Assume the lemma to be false and let G be a counterexample with $H = O(G)$ of minimal order. Let E be an elementary 2-subgroup of G of rank $k + 2$, and let $A = \{1, a_1, a_2, a_3\}$ be a 4-group in E. If $B \leq E$ with $r(B) = k + 1$ and $[B, H] = 1$, then for all S_2-groups S and T of G, $r(S \cap T) \geq r(O_2(G)) \geq r(B) = k + 1$, and G is 2-closed. So we can choose A such that $H_i = C_H(a_i) \neq H$ for all i. But E acts on each H_i, so by minimality of H, $[E, H_i] = 1$. Thus $H = \prod H_i$ is centralized by E, and G is 2-closed.

LEMMA 4. *Let G be a solvable kI-group. Then either G is 2-closed or $r(G) = k + 1$.*

Proof. Assume the lemma is false and let G be a minimal counterexample. Let $O = O(G)$. If $O \neq 1$, by minimality of G, G/O is 2-closed, and thus by Lemma 3, $O_{2',2}(G)$ and therefore G is 2-closed. So $O = 1$.

Let $K = O_2(G)$, S an S_2-group of G, $H = O_{2,2'}(G)$ and E an elementary subgroup of S with $r(E) = k + 2$. $r(K) \leq k$, so $r(EK/K) \geq 2$. For $X \leq G$ let $\bar{X} = XK/K$. Let e_i, $1 \leq i \leq 3$, be representatives for nontrivial cosets of a 4-group in $\bar{E}$. Let $K \leq H_i$, $\bar{H}_i = C_{\bar{H}}(\bar{e}_i)$. E acts on H_i, so by minimality of G, either $EH_i = G$ or $EK \trianglelefteq EH_i$. If $EH_i = G$, then $\langle e_i, K\rangle \trianglelefteq G$, so $e_i \in K$ contradicting choice of e_i. Thus $EK \trianglelefteq E\prod H_i = EH$. As $r(EK) > k$, $[\bar{S}, \bar{H}] = 1$.

Let $H \leq M$, M/H minimal normal in G/H. Then M/H is a 2-group, so as $[\bar{S}, \bar{H}] = 1$, M is 2-closed. Thus $M \leq H$ and $G = H$ is 2-closed.

2. $1I$-groups

For the remainder of this paper we let G be a minimal counterexample to our main theorem. Let S be an S_2-group of G and $N = N_G S$. We shall refer to the groups described in the statement of the main theorem as "known".

LEMMA 5. *A $1I$-group which is the central product of known groups, or the extension of a known group by a 2-group, is known.*

Proof. Assume H is a minimal counter example. Then $O(H) = 1$ and $O'(H) = H$. Suppose H is the extension of A by a 2-group. Then $|H:A| = 2$ and we can take t in $H - A$ to be a 2-element. Suppose t centralizes A. Then $H/O_2(H)$ is a TI-group and H is thus known. For, if T contains t and is contained in two S_2-groups of H, then $T = \langle t\rangle \leq O_2(H)$. So t induces an outer automorphism of A. If $A \cong L_2(q)$ or $J(11)$, $q \equiv 3, 5 \bmod 8$, then H contains a subgroup isomorphic to $PGL_2(3)$, a contradiction. If $A \cong SL_2(q)$, $q \equiv \varepsilon \bmod 4$, $\varepsilon = \pm 1$, we can choose t to be an involution inducing an automorphism in $PGL_2(q)$. Then $C_A(t)$ has order $2(q + \varepsilon)$ and does not normalize an S_2-group of A unless $q = 3$. So $H = GL_2(3)$. Similarly $A \neq GL_2(3)$ or the perfect extension of A_7. If A is the extension of a 2-group by a TI-group, let S be a t-invariant S_2-group of A. Then $\langle t, N_A S\rangle$ is not

2-closed, while $r(A) > 1$ unless $A = SL_2(5)$, a case handled above. Thus A contains the central product of two copies of $SL_2(5)$ as a subgroup of index at most two. Unless t permutes these two copies we have a contradiction as above.

Next let H be the central product of A and B, with $r(A) \leq r(B)$. Neither A or B is a 2-group by the above. If $r(B) > 1$, then as A centralizes anS_2-group of B, A is a 2-group. So $r(B) = r(A) = 1$. Similarly if $A \cap B = 1$, A and B are TI-groups and thus 2-groups. So A and B have a common center. Now if a and b are elements of order four in A and B respectively, then ab is an involution which must lie in at most one S_2-group of H, so a and b lie in at most one S_2-group of A and B respectively. Therefore $A \cong B \cong SL_2(5)$.

Various classification theorems imply $r(G) > 1$, so G is a $1I$-group. Clearly $O(G) = 1$ and $O'(G) = G$. Further

LEMMA 6. *If $1 \neq H \lhd G$, then $H \leq O_2(G)$.*

Proof. Let $E = O_2(G)$ and $1 \neq H \lhd G$ with $H \nleq E$. By minimality of G, H is known. Let $K = C_G H$. $K \neq G$ since $O(G) = 1$ and $H \nleq E$. Thus K is known. Therefore by Lemma 5, G is known.

LEMMA 7. *Let z be an involution in $Z(S)$.*

(1) *If $C_G(z)$ is 2-closed, $\langle z^G \rangle$ is known.*

(2) *If $z \in Z(G)$, and t is an involution distinct from z with $I = \langle t, z \rangle \trianglelefteq S$, then $\langle t^G \rangle$ is known.*

Proof. (1) Let $W(z) = \langle z^g \colon [z, z^g] = 1 \rangle$. Since $C_G(z)$ is 2-closed and $z \in Z(S)$, Lemma 2 implies W is abelian. Thus a theorem of Schult [6] implies $\langle z^G \rangle$ is a central product of known groups, and thus known.

(2) Let $Z = \langle z \rangle$ and for $X \leq G$ let $\bar{X} = XZ/Z$. Then $\bar{I} \leq Z(\bar{S})$. Further Lemma 2 implies $W(\bar{I})$ is abelian, so Shult's theorem implies $\langle \bar{I} \rangle^{\bar{G}}$ is known and thus also $\langle t^G \rangle$.

LEMMA 8. $E = O_2(G) = 1$.

Proof. Assume $E \neq 1$. As G is not 2-closed, $r(E) = 1$. Let z be the involution in E.

Suppose S contains no normal 4-group. Then S is cyclic, generalized quaternion, dihedral, or semidihedral [5, Proposition 9.5]. As $r(G) > 1$, S is not cyclic or quaternion. Thus G has more than one class of involutions, and transfer implies G has a subgroup of index two, contradicting Lemma 6.

So let $\langle t, z \rangle$ be a 4-group normal in S. $r(E) = 1$, so $t \notin E$. Thus by Lemma 6, $G = \langle t^G \rangle$, and thefore by Lemma 7, G is known.

Let z be an involution in $Z(S)$ and $C = C_G(z)$. C is not 2-closed by Lemmas 6, 7, and 8. By Glauberman's Z^* theorem [2], $z^G \cap S \neq \{z\}$, so by Lemma 2, $r(Z(S)) \geq 2$. Thus since by minimality of G, C is known, C is either the split extension of $K = O_2(C)$ by a TI-group H, or $C/O(C)$ is 2-

closed and by Lemma 4, $r(S) = 2$ so that $Z(S)$ contains all involutions in S. Here we use the fact that the only perfect central extensions of a 2-group of rank one by $L_2(q)$, $Sz(q)$, or $U_3(q)$, q even, are $SL_2(5)$ and $Sz^{(i)}(8)$, $1 \leq i \leq 3$. Also that an S_2-group of $Sz^{(i)}(8)$ has a unique central involution [6].

If $r(S) > 2$, $S = T \times K$, T an S_2-group of H. Now z is conjugate in N to $u \in S - K$, while all involutions in T are conjugate in C. Further all involutions in S lie in $Z(S)$, so a similar result holds for all of them. Therefore G has one class of involutions, and N is transitive on the involutions in S. If T is abelian, it follows from the transitivity of N on its involutions that S is elementary, and Walter's classification of groups with abelian S_2-groups [8] implies G is known. Similarly if T is not abelian then it is of exponent four with elementary center (of order q say) so K is quaternion of order eight. Thus there are exactly $6q$ elements a of order four with $a^2 = z$, where z is the involution in K. It follows that $| S | = 6q(2q - 1) + 2q = 4q(3q - 1) \neq$ power of two, a contradiction.

Therefore S contains exactly three involutions and by a result of G. Higman [3] is isomorphic to an S_2-group of $U_3(4)$. But then a result of Lyons implies $G = U_3(4)$.

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