A CHARACTERIZATION OF $S_{p_6}(2)$

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Yamaki [6], [7] has characterized the simple groups having the centralizer of an involution isomorphic to the centralizer of a transvection in $S_{p_6}(2)$. His result is that such a simple group must be isomorphic to $S_{p_6}(2)$, A_{12}, or A_{13}. But a Sylow 2-subgroup of $S_{p_6}(2)$ contans three central involutions whose centralizers are nonisomorphic. The purpose of this paper is to prove the following result.

THEOREM. *Let t_0 be an involution in the center of a Sylow 2-subroup of $S_{p_6}(2)$ such that t_0 is not a transvection. Let H_0 be the centralizer of t_0 in $S_{p_6}(2)$. Let G be a finite simple group containing an involution t such that $C_G(t) \simeq H_0$. Then $G \simeq S_{p_6}(2)$.*

The notation we use is standard. For example:

$\{x, y, \cdots\}$	The set of elements $x, y, \cdots$
$\langle x, y, \cdots \rangle$	The group generated by $x, y, \cdots$
$[x, y]$	$x^{-1}y^{-1}xy$
x^y	$y^{-1}xy$
$x \sim_H y$	x is conjugate to y in H
$cl_H(x)$	The set of elements of H which are conjugate to x in H.
$O_{2'}(G)$	The largest normal odd order subgroup of G.
$\mathsf{M}_G(X, 2')$	The set of odd order subgroups normalized by X which intersect X trivially.

1. Preliminary lemmas

Let G_0 be a group generated by the set of elements

$$\{u_i, w_j \mid 1 \leq i \leq 9, 1 \leq j \leq 3\}$$

with the following relations (for brevity we shall write $u_{ij} = u_i u_j$):

$$u_i^2 = 1 \quad \text{for } 1 \leq i \leq 9 \tag{1.1}$$

$$[u_i, u_j] = 1 \quad \text{for } 4 \leq i, j \leq 9$$

$$(u_{13})^2 = u_2$$

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	u_1	u_2	u_3
u_4	u_4	u_4	u_{45}
u_5	u_5	u_5	u_5
u_6	u_6	u_6	u_6
u_7	u_{47}	u_{57}	u_7
u_8	u_{568}	u_8	u_8
u_9	u_9	u_{469}	u_{789}

$$w_j^2 = 1 \quad \text{for } 1 \leq j \leq 3$$
$$(w_1 w_2)^3 = (w_2 w_3)^4 = (w_1 w_3)^2 = 1$$
$$(w_1 u_1)^3 = (w_2 u_3)^3 = (w_3 u_9)^3 = 1$$

	w_1	w_2	w_3
u_1	—	u_2	u_1
u_2	u_3	u_1	u_4
u_3	u_2	—	u_7
u_4	u_7	u_5	u_2
u_5	u_5	u_4	u_5
u_6	u_8	u_6	u_6
u_7	u_4	u_7	u_3
u_8	u_6	u_9	u_8
u_9	u_9	u_8	—.

The tables indicate the result of conjugation of the element on the left by the element at the top.

We then have the following [6], [7]:

$$G_0 \simeq S_{p_6}(2),$$
$$T_0 = \langle u_i \mid 1 \leq i \leq 9\rangle \text{ is a Sylow 2-subgroup of } G_0,$$
$$Z(T_0) = \langle u_5 \quad u_6\rangle,$$
$$C_{G_0}(u_6) = \langle T_0, w_2, w_3\rangle, \quad C_{G_0}(u_5) = \langle T_0, w_1, w_3\rangle, \quad C_{G_0}(u_{56}) = \langle T_0, w_3\rangle$$

Our theorem may be restated as follows:

THEOREM. *Let G be a finite simple group containing an involution t such that $H = C_G(t)$ is isomorphic to one of*
(a) $C_{G_0}(u_5)$
(b) $C_{G_0}(u_{56})$.
Then $G \simeq S_{p_6}(2)$.

In the proof we will identify the elements of H with the elements of $C_{G_0}(u_5)$ or $C_{G_0}(u_{56})$, and the relations (1.1) between elements of $C_{G_0}(u_5)$ or $C_{G_0}(u_{56})$ are assumed to hold in H. In particular, we have $t = u_5$ or $t = u_{56}$, and

$T = \langle u_i \mid 1 \leq i \leq 9\rangle$ is a Sylow 2-subgroup of H. We begin with a detailed study of important subgroups of T.

LEMMA 1.1. (i) $Z(T) = \langle u_5, u_6\rangle$,

$$T' = \langle u_2, u_4, u_5, u_6, u_{78}\rangle, \quad T'' = \langle u_5\rangle.$$

(ii) $S = \langle u_4, u_5, u_6, u_7, u_8, u_9\rangle$ *is the unique elementary abelian subgroup of T of order* 2^6.

(iii) *There are eight subgroups lying between S and T*:

X	$\lvert X\rvert$	$Z(X)$	X'
$K_1 = \langle S, u_1\rangle$	2^7	$\langle u_4, u_5, u_6, u_9\rangle$	$\langle u_4, u_{56}\rangle$
$K_2 = \langle S, u_{12}\rangle$	2^7	$\langle u_4, u_5, u_6, u_{789}\rangle$	$\langle u_{45}, u_{56}\rangle$
$K_3 = \langle S, u_2\rangle$	2^7	$\langle u_4, u_5, u_6, u_8\rangle$	$\langle u_5, u_{46}\rangle$
$K_4 = \langle S, u_{23}\rangle$	2^7	$\langle u_5, u_6, u_{47}, u_8\rangle$	$\langle u_5, u_{4678}\rangle$
$K_5 = \langle S, u_3\rangle$	2^7	$\langle u_5, u_6, u_7, u_8\rangle$	$\langle u_5, u_{78}\rangle$
$L_1 = \langle S, u_1, u_2\rangle$	2^8	$\langle u_4, u_5, u_6\rangle$	$\langle u_4, u_5, u_6\rangle$
$L_2 = \langle S, u_2, u_3\rangle$	2^8	$\langle u_5, u_6, u_8\rangle$	$\langle u_5, u_{46}, u_{78}\rangle$
$L_3 = \langle S, u_{13}\rangle$	2^8	$\langle u_5, u_6\rangle$	$\langle u_4, u_5, u_6, u_{78}\rangle$

(iv) *T contains exactly eight self-centralizing elementary abelian subgroups of order* 2^5:

X	$N_T(X)$
$M_1 = \langle u_1, u_2, u_4, u_5, u_6\rangle$	T
$M_2 = \langle u_2, u_4, u_5, u_6, u_8\rangle$	T
$M_3 = \langle u_1, u_4, u_5, u_6, u_9\rangle$	L_1
$M_4 = \langle u_{12}, u_4, u_5, u_6, u_{789}\rangle$	L_1
$M_5 = \langle u_3, u_5, u_6, u_7, u_8\rangle$	L_2
$M_6 = \langle u_{23}, u_5, u_6, u_{47}, u_8\rangle$	L_2
$M_7 = \langle u_2, u_3, u_5, u_6, u_8\rangle$	$\langle u_i \mid 1 \leq i \leq 8\rangle$
$M_8 = \langle u_{24}, u_{37}, u_5, u_6, u_8\rangle$	$\langle u_i \mid 1 \leq i \leq 8\rangle$

Proof. (i), (ii), (iii), are in [6]. (iv) is easily computed from the relations between the u_i's. We observe in addition that S, L_1, L_2, L_3 are weakly closed in T, since each is isomorphic to no other subgroup of T.

Lemmas 1.2 through 1.5 correspond to Yamaki's Lemmas 2 through 5 in [6].

LEMMA 1.2. *If two elements of S are conjugate in G then they are conjugate in $N_G(S)$.*

Proof. This will follow from Lemma 2 in [6] once we determine that T is a Sylow 2-subgroup of G.

LEMMA 1.3. $N_G(T) = T$. *In particular T is a Sylow 2-subgroup of G.*

Proof. $N_G(T)$ normalizes $Z(T)$ and permutes the subgroups $\{K_i \mid 1 \leq i \leq 5\}$, and hence the subgroups $\{K_i' \mid 1 \leq i \leq 5\}$. Since u_{56} appears in exactly two of the subgroups K_i', u_5 in exactly three, and u_6 in none, it follows that $N_G(T)$ centralizes $Z(T)$. Hence $N_G(T) = N_H(T) = T$.

LEMMA 1.4. *No two of u_5, u_{56}, u_6 are conjugate in G.*

LEMMA 1.5. *Let $\mathfrak{N} = N_G(S)/S$. Then*
(i) $|\mathfrak{N}/O_{2'}(\mathfrak{N})| = 2^3, 2^3\cdot 3, 2^3\cdot 3\cdot 5, 2^3\cdot 3\cdot 7, 2^3\cdot 3^2\cdot 5$, or $2^3\cdot 3^2\cdot 5\cdot 7$.
(ii) $|O_{2'}(\mathfrak{N})| = 3^k, 0 \leq k \leq 4$.

Proof. With respect to the basis

$$\{u_{4789}, u_{56789}, u_8, u_{568}, u_{459}, u_{569}\}$$

the action of u_2 and u_1 on S is given by

$$\left|\begin{array}{cc|cc|cc} & 1 & & & & \\ 1 & & & & & \\ \hline & & & 1 & & \\ & & 1 & & & \\ \hline & & & & 1 & \\ & & & & & 1 \end{array}\right| \quad \text{and} \quad \left|\begin{array}{cc|cc|cc} & 1 & & & & \\ 1 & & & & & \\ \hline & & 1 & & & \\ & & & 1 & & \\ \hline & & & & & 1 \\ & & & & 1 & \end{array}\right|$$

Hence the proof of Yamaki's Lemma 5 gives $\mathfrak{N}/O_{2'}(\mathfrak{N}) \simeq D_8$, $PGL_2(q)$, $L_2(q)$, or A_7, and since $|\mathfrak{N}| \mid |GL_6(2)|$ we have (i). Now $u_1 \sim u_{12}$ in $N_G(S)$ so we get

$$|O_{2'}(\mathfrak{N})| = |C(u_1) \cap O_{2'}(\mathfrak{N})|^2 |C(u_2) \cap O_{2'}(\mathfrak{N})|.$$

Since $|C(u_i) \cap O_{2'}(\mathfrak{N})| = 1, 3$, or 3^2 for $i = 1, 2$, and since $|\mathfrak{N}| \mid |GL_6(2)|$ we get $|O_{2'}(\mathfrak{N})| = 3^k$, $0 \leq k \leq 4$. Note that if $u_i \sim u_2$ in $N_G(S)$ then $|O_{2'}(\mathfrak{N})| = 1$ or 3^3.

We now prove a few more miscellaneous lemmas which will apply to both cases (a) and (b)

LEMMA 1.6. (i) $u_{46} \not\sim u_4$ *and* $u_{46} \not\sim u_5$.

(ii) $u_4 \not\sim u_{56}$ and $u_4 \not\sim u_6$.

Proof. (i) Assume $u_{46}^x = u$ where $x \in N_G(S)$ and $u = u_4$ or u_5. Then $L_1^x \subseteq C_G(u)$. Let T_0 be a Sylow 2-subgroup of $C_G(u)$ containing L_1. Then there exists $y \in C_G(u)$ such that $L_1^{xy} \subseteq T_0$. By weak closure, $S^{xy} = S$ and $L_1^{xy} = L_1$. Thus xy permutes the groups K_1, K_2, K_3. Now $u_{46} \in K_2'$ and K_3' but $u_4 \in K_1'$ and $u_5 \in K_3'$ only. Since $u_{46}^{xy} = u$ this is impossible.

(ii) This may be proved in the same way as (i).

LEMMA 1.7. *The following mappings are automorphisms*:

(i) $\alpha_1 : C_{G_0}(u_5) \to C_{G_0}(u_5)$ *given by*

$$u_i \to u_i, \quad 1 \le i \le 8, \quad u_9 \to u_{59}, \quad w_1 \to w_1, \quad w_3 \to w_3 u_5;$$

(ii) $\alpha_2 : C_{G_0}(u_5) \to C_{G_0}(u_5)$ *given by*

$$u_1 \to u_{15}, \quad u_i \to u_i, \quad 2 \le i \le 9, \quad w_1 \to w_1 u_5, \quad w_3 \to w_3;$$

(iii) $\alpha_3 : C_{G_0}(u_{56}) \to C_{G_0}(u_{56})$ *given by*

$$u_1 \to u_{15}, \quad u_i \to u_i, \ 2 \le i \le 9, \quad w_3 \to w_3;$$

(iv) $\alpha_4 : C_{G_0}(u_6) \to C_{G_0}(u_6)$ *given by*

$$u_i \to u_i, \quad 1 \le i \le 7, \quad u_8 \to u_{68}, \quad u_9 \to u_{69}, \quad w_2 \to w_2, \quad w_3 \to w_3 u_6;$$

(v) $\alpha_5 : C_{G_0}(u_6) \to C_{G_0}(u_6)$ *given by*

$$u_i \to u_i, \quad 1 \le i \le 7, \quad u_8 \to u_{58}, \quad u_9 \to u_{49}, \quad w_2 \to w_2, \quad w_3 \to w_3 u_6.$$

Proof. We check that each of these mappings is consistent with the defining relations (1.1).

LEMMA 1.8. *Assume that H is contained in a subgroup G_1 of G such that $G_1 \simeq S_{p_6}(2)$ and such that $u_4 \sim_{G_1} u_5$ and $u_{46} \sim_{G_1} u_{78} \sim_{G_1} u_{56}$.*
Then $\mathcal{U}_G(S, 2')$ is trivial.

Proof. Let K be an odd order subgroup of G normalized by S. Then by the theorem of Brauer and Wielandt [5] we have

$$K = C_K(u_4)C_K(u_5)C_K(u_{45}) = C_K(u_{46})C_K(u_{78})C_k(u_{4678}).$$

Since either $H = C_G(u_5)$ or $H = C_G(u_{56})$ it follows that $K \subseteq G_1$. By the structure of $S_{p_6}(2)$ (see [7, Lemma 13]) we must have $K = \{1\}$. Thus $\mathcal{U}_G(S, 2')$ is trivial.

2. The case $H \simeq C_{G_0}(u_5)$

In this section we let $H = \langle u_1, w_1, w_3 \mid 1 \le i \le 9\rangle$ with the relations (1.1) and assume that H is the centralizer of u_5 in a simple group G.
H has 13 classes of involutions:

Table 1

x	u_5	u_6	u_{56}	u_4	u_{46}	u_9	u_{59}	u_{49}	u_{549}	u_{48}	u_1	u_{15}	u_{19}
$\lvert \mathrm{cl}_H(x) \cap S \rvert$	1	3	3	6	6	4	4	12	12	12	0	0	0

Now $\lvert N_H(S) \rvert = 2^9 \cdot 3$. For $s \in S$ we write

$$n(s) = \lvert N_G(S) : N_G(S) \cap C_G(s) \rvert = \lvert \mathrm{cl}_G(s) \cap S \rvert.$$

The following table gives the possible values of $n(u_5)$ corresponding to the possibilities given in Lemma 1.5 and subject to the restriction $1 \leq n(s) < 63$:

Table II

$\lvert \mathfrak{N}/O_{2'}(\mathfrak{N}) \rvert$	$\lvert O_{2'}(\mathfrak{N}) \rvert$ 1	3	3^2	3^3	3^4
2^3	—	1	3	9	27
$2^3 \cdot 3$	1	3	9	27	—
$2^3 \cdot 3 \cdot 5$	5	15	45	—	—
$2^3 \cdot 3 \cdot 7$	7	21	—	—	—
$2^3 \cdot 3^2 \cdot 5$	15	45	—	—	—
$2^3 \cdot 3^2 \cdot 5 \cdot 7$	—	—	—	—	—

LEMMA 2.1. *It is false that* $u_5 \sim u_9 \sim u_{59}$.

Proof. Assume $u_5 \sim u_9 \sim u_{59}$ and assume $u_9^x = u_5$ where $x \in N_G(S)$. Then K_1^x is a 2-subgroup of H, so there exists $y \in H$ such that $K_1^{xy} \subseteq T$. Thus $K_1^{xy} = K_i$ for $1 \leq i \leq 5$, and since $u_9 \notin K_1'$ we must have $u_5 \notin K_i'$. Hence $i = 1$ or 2. If $i = 1$ then xy carries the coset $\{u_9, u_{49}, u_{569}, u_{4569}\}$ of K_1' in $Z(K_1)$ onto $\{u_5, u_{45}, u_6, u_{46}\}$. By assumption $u_9 \sim u_{59} \sim_H u_{4569}$. By Lemmas 1.4 and 1.6 we must have $u_{4569}^{xy} = u_{45}$, and then $u_{456}^{xy} = u_4$. This contradicts Lemma 1.6 since $u_{456} \sim_H u_{46}$. Similarly if $i = 2$ then

$$\{u_9, u_{49}, u_{569}, u_{4569}\}^{xy}$$

is the coset

$$\{u_5, u_4, u_6, u_{456}\}$$

of K_2' in $Z(K_2)$. This time u_{4569}^{xy} must be u_4 so that $u_{456}^{xy} = u_{45}$. Since $u_{45} \sim_H u_4$ this contradicts Lemma 1.6, and the lemma is proven.

LEMMA 2.2. $n(u_5) \not\equiv 0(3)$.

Proof. By Table I, if $n(u_5) \equiv 0(3)$ then $u_5 \sim u_9 \sim u_{59}$ contradicting Lemma 2.1.

It follows from Table II that $n(u_5) = 1, 5,$ or 7.

LEMMA 2.3. $n(u_5) \neq 1$.

Proof. Assume $n(u_5) = 1$. Then $N_G(S) = N_H(S)$ and so there is no fusion in G between H-classes of S. Now by a transfer theorem of Thompson [3, Lemma 5.38], u_{19} is conjugate in G to an element of L_2. But every involution of L_2 is conjugate in H to an element of S, and so u_{19} is conjugate in G to an element of S. In particular, M_3, a Sylow 2-subgroup of $C_H(u_{19})$, is not a Sylow subgroup of $C_G(u_{19})$. Hence there exists an element $x \in G \backslash H$ normalizing M_3.

Table III

u	u_5	u_6	u_{56}	u_4	u_{46}
$\mathrm{cl}_H(u) \cap M_3$	$\{u_5\}$	$\{u_6\}$	$\{u_{56}\}$	$\{u_4, u_{45}\}$	$\{u_{46}, u_{456}\}$

Now if x normalizes any one of the sets $\{u_5\}$, $\{u_4, u_{45}\}$, $\{u_{46}, u_{456}\}$ then $x \in H$. So x must fuse these sets with the three sets

$$\mathrm{cl}_H(u_1) \cap M_3, \quad \mathrm{cl}_H(u_{15}) \cap M_3, \quad \mathrm{cl}_H(u_{19}) \cap M_3.$$

But then x normalizes both $\{u_6\}$ and $\{u_{56}\}$ so that $x \in H$, a contradiction.

LEMMA 2.4. $n(u_5) \neq 5$.

Proof. Let $n(u_5) = 5$. By Table I, either $u_5 \sim u_9$ or $u_5 \sim u_{59}$. Assume $u_5 \sim u_9$. Then there exists a 2-element $x \in G \backslash H$ normalizing K_1 and inducing an automorphism of order 2 on $Z(K_1)$ and centralizing u_9. Since

$$\mathrm{cl}_G(u_5) \cap Z(K_1) = \{u_5, u_9, u_{469}\}$$

we have $u_5^x = u_{469}$. Then x centralizes $u_{456} = u_5 \cdot u_{469} \cdot u_9$. Lemma 1.6 shows that $u_4 \not\sim u_{56}$ and so, since x normalizes K_1', $u_4^x = u_4$. Then we get $u_{45}^x = u_{69}$, $u_6^x = u_{459}$, and $u_{46}^x = u_{59}$. Since $u_4 \sim_H u_{45}$ and $u_{69} \sim_H u_{49}$ we have $u_5 \sim u_9$, $u_6 \sim u_{459}$, $u_4 \sim u_{49}$, $u_{46} \sim u_{59}$. By Lemma 1.6 u_4 is not conjugate to u_6, u_{46}, or u_{56}, and by hypothesis is not conjugate to u_5. Now

$$|(\mathrm{cl}_H(u_4) \cup \mathrm{cl}_H(u_{49})) \cap S| = 18,$$

and $|\mathrm{cl}_G(u_4) \cap S| = n(u_4)$ divides $|N_G(S)/S| = 2^3 \cdot 3 \cdot 5$. We must have, therefore, that $u_4 \sim u_{48}$. By Lemmas 1.4 and 1.6, u_{56} is not conjugate to u_5, u_6, or u_4. If $u_{56} \sim u_{46}$ then by Table I,

$$n(u_{56}) = 3 + 6 + 4 = 13 \nmid 2^3 \cdot 3 \cdot 5,$$

a contradiction. Hence

$$|N_G(S) : N_G(S) \cap C_G(u_{56})| = n(u_{56}) = 3.$$

But $N_G(S)/S \simeq PGL_2(5)$ by Lemma 1.5 and thus has no subgroup of index 3. Therefore $u_5 \not\sim u_9$. Similarly we can show $u_5 \not\sim u_{59}$, and therefore $n(u_5) \neq 5$.

LEMMA 2.5. *There exists an element $w \in G$ whose action on L_1 is the same as that of w_2.*

Proof. By Lemmas 2.4 and 1.6 and Table I we have $u_5 \sim u_4$. There exists $w \in G$ such that $u_4^w = u_5$ and $L_1^w = L_1$. Then $K_1^w = K_3$ and $K_2^w = K_2$. It follows that $u_5^w = u_4$ and $u_{56}^w = u_{46}$, and hence $u_6^w = u_6$. Now since

$$|\mathrm{cl}_H(u_6) \cap S| = 3$$

and since $N_G(S)/S \simeq L_2(7)$ has no subgroup of index 3, u_6 must be conjugate to one or more of the elements u_9, u_{59}, u_{49}, u_{459}, u_{48}. But $n(u_6) = |\mathrm{cl}_G(u_6) \cap S|$ must be a divisor of $|N_G(S)/S|$, so by Table I the only possibility is $n(u_6) = 7$; that is, $u_6 \sim u_9$ or $u_6 \sim u_{59}$. Since the automorphism α_1 of H in Lemma 1.7 interchanges $\mathrm{cl}_H(u_9)$ and $\mathrm{cl}_H(u_{59})$, we can assume, without loss of generality, that $u_6 \sim u_9$. Since $Z(K_1)^w = Z(K_3)$ we have

$$u_9^w \in Z(K_3) \cap \mathrm{cl}_G(u_6) = \{u_6, u_8, u_{568}\}.$$

We have from above that $u_9^w \neq u_6$. If $u_9^w = u_{568}$ we can replace w by wu_1 Hence we may assume $u_9^w = u_8$. Similarly

$$u_8^w \in Z(K_1) \cap \mathrm{cl}_G(u_6) = \{u_6, u_9, u_{469}\}.$$

If $u_8^w = u_{469}$ we can replace w by wu_2; and hence we may assume $u_8^w = u_9$. Finally,

$$u_{789}^w \in Z(K_2) \cap \mathrm{cl}_G(u_6) = \{u_6, u_{789}, u_{456789}\}.$$

If $u_{789}^w = u_{456789}$ then $u_7^w = u_{4567} \sim_H u_{48}$. Since $u_7 \sim_H u_4 \sim u_5$, this is impossible. Hence $u_{789}^w = u_{789}$, and thus $u_7^w = u_7$. Now the action of w on S is completely determined. In particular we have

$$u_5 \sim u_4, \quad u_6 \sim u_9, \quad u_{56} \sim u_{46} \sim u_{49}, \quad u_{59} \sim u_{459} \sim u_{48}.$$

The H-classes of involutions of $K_1 \backslash S$ and $K_3 \backslash S$ are

$$\{u_1, u_{14}, u_{156}, u_{1456}\}, \quad \{u_{15}, u_{145}, u_{16}, u_{146}\},$$

$$\{u_{19}, u_{1469}, u_{149}, u_{1569}, u_{169}, u_{159}, u_{14569}, u_{1459}\}$$

and

$$\{u_2, u_{25}, u_{246}, u_{2456}\} \sim u_5, \quad \{u_{24}, u_{245}, u_{26}, u_{256}\} \sim u_{56},$$

$$\{u_{2468}, u_{2458}, u_{24568}, u_{248}, u_{28}, u_{2568}, u_{258}, u_{268}\} \sim u_{59}.$$

Since $u_5 \nsim u_{56} \nsim u_{59} \nsim u_5$ we must have $u_{19} \sim u_{59}$ and hence $u_5 \sim u_1$ or $u_5 \sim u_{15}$. But the automorphism α_2 of H in Lemma 1.7 interchanges $\mathrm{cl}_H(u_1)$ and $\mathrm{cl}_H(u_{15})$ so, without loss of generality, we may assume $u_5 \sim u_1$. Then

$$u_1^w \in \{u_2, u_{25}, u_{246}, u_{2456}\}.$$

Now S is complemented in T by $\langle u_1, u_2, u_3\rangle$, so by a theorem of Gaschütz [1], S is complemented in $N_G(S)$. We may assume that w lies in a complement of S, and since $w^2 \in C_G(S) = S$ we have $w^2 = 1$. Now if $u_1^w = u_{25}$ replace w by wu_7, if $u_1^w = u_{246}$ replace w by wu_{89}, and if $u_1^w = u_{2456}$ replace w by wu_{789}. Then we get $u_1^w = u_2$, and it is still true that $w^2 = 1$. Hence also $u_2^w = u_1$. The lemma is complete.

LEMMA 2.6. *Let w be the element defined in Lemma 2.5. Then we may assume*
(i) $(wu_3)^3 = 1$,
(ii) $(ww_3)^4 = 1$,
(iii) $(w_1w)^3 = 1$.

Proof. (i) $(wu_3)^3 \in C_G(L_1) = \langle u_4, u_5, u_6\rangle$. Hence $|wu_3| = 3$ or 6. If $|wu_3| = 6$ then

$$(wu_3)^3 \in \langle u_4, u_5, u_6\rangle \cap C_G\langle w, u_3\rangle = \langle u_6\rangle.$$

We can replace w by wu_6, and hence $(wu_3)^3 = 1$.

(ii) $(ww_3)^4 \in C_G(M_1) = M_1$. Hence $|ww_3| = 4$ or 8. Assume $|ww_3| = 8$. Then

$$(ww_3)^4 \in M_1 \cap C_G(w, w_3) = \langle u_{1245}, u_6\rangle.$$

Now $|\langle w, w_3\rangle| = 16$ so there exists $y \in G$ such that $\langle w, w_3\rangle^y \subseteq T$. Let $(ww_3)^y = rs$ where $r \in \langle u_1, u_3\rangle$, $s \in S$. Then $(rs)^2 = r^2 s^r s$ where $r^2 \neq 1$, since $(rs)^4 \neq 1$. Hence $r^2 = u_2$, and we have

$$(rs)^4 = (u_2 s^r s)^2 = [[u_1, u_3], [r, s]] \in T'' = \langle u_5\rangle.$$

Thus $(ww_3)^4 \sim u_5$ and so $(ww_3)^4 = u_{1245}$. Now in the dihedral group $\langle w, w_3\rangle$ we have $w_3 \sim u_{1245} w_3$ and so

$$u_{1245} w_3 \sim w_3^{u_9 w_3} = u_9 \sim_G u_6 .$$

However,

$$(u_{1245} w_3)^{u_9 w_3} = u_{14569} \sim_H u_{19} \sim_G u_{59} ,$$

a contradiction. Therefore, $(ww_3)^4 = 1$.

(iii) $(w_1 w) \in C_G(S) = S$. Hence $|w_1 w| = 3$ or $|w_1 w| = 6$. Assume $|w_1 w| = 6$. Then

$$(w_1 w)^3 \in S \cap C_G(w_1, w) = \langle u_{457}, u_{689}\rangle.$$

Now in the dihedral group $\langle w_1, w\rangle$ we have $w \sim (w_1 w)^3 w_1$ and so w is conjugate to one of the elements $u_{457} w_1$, $u_{689} w_1$, or $u_{456789} w_1$. But

$$(u_{457} w_1)^{u_1 w_1} = u_{145} \sim_H u_{15} \sim_G u_{56}, \quad (u_{689} w_1)^{u_1 w_1} = u_{1569} \sim_H u_{19} \sim_G u_{59},$$

and

$$(u_{456789} w_1)^{u_1 w_1} = u_{1469} \sim_H u_{19} \sim_G u_{59} .$$

On the other hand, by (i), $w \sim u_3 \sim_H u_4 \sim_G u_5$, a contradiction. Therefore, $(w_1 w)^3 = 1$, and the lemma is proven.

Since w satisfies the same relations (1.1) as w_2, we have proved:

LEMMA 2.7. *Let $G_1 = \langle H, w\rangle$. Then $G_1 \simeq S_{p_6}(2)$.*

Now since $N_G(S) \subseteq G_1$, all fusion of involutions occurs in G_1. In order to prove that $G_1 = G$, we wish to show that G_1 contains the centralizer of each of its involutions.

LEMMA 2.8. $C_G(u_{56}) = \langle T, w_3\rangle$.

Proof. Let $x \in C_G(u_{56})$ and assume that $u_6^x \in T$. Then

$$u_5^x = u_{56} u_6^x \in \{u_5, u_{568}, u_8, u_{569}, u_{459}, u_{56789}, u_{4789}\}.$$

The only possibility is $u_5^x = u_5$ and hence $u_6^x = u_6$. By Glauberman's Theorem [2], we must have

$$u_6 \in C_G(C_G(u_{56})/O_{2'}(C_G(u_{56}))).$$

But $S \subseteq C_G(u_{56})$ so by Lemma 1.8, we have $u_6 \in Z(C_G(u_{56}))$. Hence

$$C_G(u_{56}) \subseteq C_G(u_{56}, u_6) \subseteq H,$$

and the lemma is proven.

LEMMA 2.9. $C_G(u_{59}) = \langle S, u_1, w_1\rangle$.

Proof. K_1 is a Sylow 2-subgroup of $C_G(u_{59})$; if not, then $g \in C_G(u_{59})\backslash K_1$ normalizes K_1 and hence K_1'. But then $g \in C_G(u_4) \subseteq G_1$, a contradiction. Conjugating by ww_1 we get K_5 is a Sylow 2-subgroup of $C_G(u_{67})$. Assume that for some $x \in C_G(u_{67})$ we have $u_6^x \in K_5$. Then

$$u_7^x = u_6^x u_{67} \in \{u_7, u_{678}, u_{679}, u_{479}, u_{689}, u_{4589}\},$$

and the only possibility is $u_7^x = u_7$. Hence $u_6^x = u_6$, so by [2],

$$u_6 \in C_G(C_G(u_{67})/O_{2'}(C_G(u_{67}))),$$

and so as in Lemma 2.8 we get $C_G(u_{67}) \subseteq C_G(u_7)$. Conjugating by $w_1 w$, we get $C_G(u_{59}) \subseteq C_G(u_5)$ and the lemma is proven.

LEMMA 2.10. $C_G(u_6) = \langle T, w, w_3 \rangle$.

Proof. Since $\langle T, w, w_3\rangle = C_{G_1}(u_6)$ it is sufficient to prove that $C_G(u_6) \subseteq G_1$. Assume that there exists $g \in C_G(u_6)\backslash G_1$. Then T^g is a Sylow 2-subgroup of $C_G(u_6)$ and does not lie in G_1. Let T_1 be a Sylow 2-subgroup of $C_G(u_6)$ such that $T_1 \not\subseteq G_1$ and $|T_1 \cap G_1|$ is maximal. Let $x \in C_{G_1}(u_6)$ such that $(T_1 \cap G_1)^x \subseteq T$, and let $y \in N_{T_1^x}((T_1 \cap G_1)^x)\backslash G_1$. Let $\hat{T} = T^y$. Then $\hat{T} \not\subseteq G_1$ and $|\hat{T} \cap G_1| = |T_1 \cap G_1|$ is maximal. Let $I = \hat{T} \cap G_1$. It is clear that $u_6 \in I$. We prove that $u_5 \notin I$: if $u_5 \in I$ then $u_5^y \in I$ and u_5^y centralizes $\hat{T}$; since the centralizer of every conjugate of u_5 in G_1 lies in G_1, we get $\hat{T} \subseteq G_1$, a contradiction. But now it follows that every involution of I is conjugate to u_6: if $i \in I$ such that

$$i \in \mathrm{cl}_G(u_5) \cup \mathrm{cl}_G(u_{56}) \cup \mathrm{cl}_G(u_{59})$$

then $u_5^y \in C_G(i) \subseteq G_1$, so that $u_5^y \in \hat{T} \cap G_1 = I$, a contradiction. Since the conjugates of u_6 in T are

$$\{u_6, u_8, u_{568}, u_9, u_{469}, u_{789}, u_{456789}\},$$

we get that u_6 is the only involution in I. Assume that I contains an element rs of order 4 where $r \in \langle u_1, u_3\rangle$ and $s \in S$. Then $u_6 = (rs)^2 = r^2[r, s]$. Since $[r, s] \in S$ we have that r is an involution in $\langle u_1, u_3\rangle$ and so $u_6 = [r, s] \in K_i'$ for some i. Since this is not the case, I must be elementary abelian, and thus $I = \langle u_6\rangle$. Now let z be the central involution in the dihedral group $\langle u_5, u_{59}^y\rangle$. Since z centralizes u_5, $z \in G_1$. But the 2-group $\langle I, z, u_{59}^y\rangle$ does not lie in G_1 and intersects G_1 in $\langle I, z\rangle$. By maximality of $|\hat{T} \cap G_1|$ we get $z \in I$ and hence $z = u_6$. But in the group $\langle u_5, u_{59}^y\rangle$ we have either $u_5 \sim u_5 z$ or $u_{59}^y \sim u_5 z$, and so either $u_5 \sim u_{56}$ or $u_{59} \sim u_{56}$. In either case we have a contradiction, and the lemma is proven.

Now we can prove part (a) of the theorem. Since

$$C_G(u_6) = C_{G_1}(u_6) \simeq C_{G_0}(u_6)$$

It follows from Yamaki's Theorem [7] that $G \simeq A_{12}$, A_{13}, or $S_{p_6}(2)$. Since G contains four classes of involutions, we must have $G \simeq S_{p_6}(2)$.

Or, more directly, since we have proven that G_1 contains the centralizer of each of its involutions, it follows (see [7, Lemma 20]) that $G = G_1 \simeq S_{p_6}(2)$.

3. The case $H \simeq C_{G_0}(u_{56})$

In this section $H = \langle u_i, w_3 \mid 1 \leq i \leq 9 \rangle$ with the relations (1.1) and H is assumed to be the centralizer of u_{56} in a simple group G.

H has 21 classes of involutions:

Table IV

x	u_{56}	u_5	u_6	u_8	u_{58}	u_4	u_{46}	u_7	u_{78}
$\lvert \mathrm{cl}_H(x) \cap S \rvert$	1	1	1	2	2	2	2	4	4

u_9	u_{59}	u_{49}	u_{459}	u_{48}	u_{79}	u_{679}	u_{67}	u_{678}	u_1	u_{15}	u_{19}
4	4	4	4	4	8	8	4	4	0	0	0

This time $\lvert N_H(S) \rvert = 2^9$. Again writing

$$n(s) = \lvert N_G(S) : N_G(S) \cap C_G(s) \rvert = \lvert \mathrm{cl}_G(s) \cap S \rvert$$

we get the possible values of $n(u_{56})$:

Table V

$\lvert \mathfrak{N}/O_{2'}(\mathfrak{N}) \rvert$	$\lvert O_{2'}(\mathfrak{N}) \rvert$: 1	3^3
2^3	1	27
$2^3 \cdot 3$	3	—
$2^3 \cdot 3 \cdot 5$	15	—
$2^3 \cdot 3 \cdot 7$	21	—
$2^3 \cdot 3^2 \cdot 5$	45	—
$2^3 \cdot 3^2 \cdot 5 \cdot 7$	—	—

Here we are assuming that $u_1 \sim u_2$ in $N_G(S)$ so that by Lemma 1.5, $\lvert O_{2'}(\mathfrak{N}) \rvert = 1$ or 3^3. We will not refer to Table V until after this fact is established in Lemma 3.4.

LEMMA 3.1. *Either* $u_{56} \sim u_{46}$ *or* $u_5 \sim u_4$.

Proof. By a Transfer Theorem of Thompson [3], u_{19} is conjugate in G to element of L_2, and hence to an element of S. Therefore, M_3 is not a Sylow 2-subgroup of $C_G(u_{19})$. Let $x \in G \backslash H$, $x \in N_G(M_3)$. We have:

Table VI

v	$\mathrm{cl}_H(v) \cap M_3$
u_5	$\{u_5\}$
u_6	$\{u_6\}$
u_{56}	$\{u_{56}\}$
u_4	$\{u_4, u_{45}\}$
u_{46}	$\{u_{46}, u_{456}\}$
u_9	$\{u_9, u_{469}\}$
u_{49}	$\{u_{49}, u_{69}\}$
u_{59}	$\{u_{59}, u_{4569}\}$
u_{459}	$\{u_{459}, u_{569}\}$
u_1	$\{u_1, u_{14}, u_{156}, u_{1456}\}$
u_{15}	$\{u_{15}, u_{145}, u_{16}, u_{146}\}$
u_{19}	$\{u_{19}, u_{159}, u_{149}, u_{1459}, u_{1569}, u_{169}, u_{14569}, u_{1469}\}$

We observe that $\prod_{y \in \mathrm{cl}_H(v) \cap M_3} y = 1$ for $v = u_1, u_{15}$, and u_{19}. Hence if

$$\mathrm{cl}_G(u_{56}) \cap M_3 \subseteq \mathrm{cl}_H(u_{56}) \cup \mathrm{cl}_H(u_1) \cup \mathrm{cl}_H(u_{15}) \cup \mathrm{cl}_H(u_{19}),$$

then $\prod_{y \in \mathrm{cl}_G(u_{56}) \cap M_3} y = u_{56}$ and so $u_{56}^x = u_{56}$, a contradiction. Thus u_{56} is conjugate in G to one of $u_4, u_{46}, u_9, u_{49}, u_{59}, u_{459}$. By Lemma 1.6, $u_{56} \not\sim u_4$. If $u_{56} \sim u_{46}$ the lemma is proven. Hence we assume u_{56} is conjugate to u for

$$u \in \{u_9, u_{49}, u_{59}, u_{459}\}.$$

Now $C_T(u) = K_1$, and there exists $y \in G$ such that $u^y = u_{56}$ and $K_1^y \subseteq T$. Since $u \notin K_1'$, $u_{56} \notin (K_1^y)'$ so K_1^y is K_3, K_4, or K_5. Since by Lemmas 1.4 and 1.6, $u_{56} \not\sim u_5$ and $u_{46} \sim_H u_{456} \not\sim u_5$, we must have $u_4^y = u_5$. The lemma is proved.

Lemma 3.2. (i) *$u_{56} \sim u_{46}$ and $u_5 \sim u_4$.*

(ii) $K_1 \sim K_3$.

Proof. By Lemma 3.1, either $u_{56} \sim u_{46}$ or $u_5 \sim u_4$. Suppose $u_{56} \sim u_{46}$. Then there exists $x \in G$ such that $u_{46}^x = u_{56}$ and $C_G(u_{46})^x = L_1^x \subseteq T$. Then $L_1^x = L_1$ by weak closure and since $u_{56} \in (K_3')^x$ we have $K_3^x = K_1$ or K_2. Since $K_1 \sim_H K_2$ we have $K_1 \sim K_3$ and thus $u_5 \sim u_4$. Similarly, if $u_5 \sim u_4$ there exists $x \in G$ such that $u_4^x = u_5$, $L_1^x = L_1$, and hence $K_1^x = K_3$. It follows that $u_{56} \sim u_{46}$.

Lemma 3.3. *There exists an element $w \in G$ whose action in M_1 is the same as that of w_2.*

Proof. By Lemma 3.2, there exists $w \in G$ such that $L_1^w = L_1$, $u_4^w = u_5$, and $K_1^w = K_3$. If $K_3^w = K_2$ we can replace w by wu_3, so that without loss of generality we may assume $K_3^w = K_1$, and hence $u_5^w = u_4$. Now w normalizes $Z(L_1) = \langle u_4, u_5, u_6\rangle$ and by Lemma 3.2 it follows that $u_6^w = u_6$. Since w permutes the self-centralizing elementary abelian subgroups of L_1 of order

2^5, namely, M_1, M_2, M_3, M_4, and since $M_3 \subseteq K_1$, $M_4 \subseteq K_2$, $M_2 \subseteq K_3$, we must have $M_1^w = M_1$, $M_3^w = M_2$, and $M_2^w = M_3$. Thus

$$u_2^w \in M_1 \cap M_3 = \langle u_1, u_4, u_5, u_6\rangle$$

and, since $u_5 \sim u_4 \sim_H u_2$, we have $u_5 \sim u_1$ or u_{15}. The automorphism α_3 of H in Lemma 1.7 interchanges $\text{cl}_H(u_1)$ and $\text{cl}_H(u_{15})$. Hence without loss of generality we may take $u_5 \sim u_1$. Then

$$u_1^w \in ((M_1 \cap M_2)\backslash Z(L_1)) \cap \text{cl}_G(u_5) = \{u_2, u_{25}, u_{246}, u_{2456}\}.$$

If u_1^w is u_{25}, u_{246}, or u_{2456} then we may replace w by wu_7, wu_9, or wu_{79} respectively. Thus we may assume that $u_1^w = w_2$. Similarly

$$u_2^w \in \{u_1, u_{14}, u_{156}, u_{1456}\}.$$

If u_2^w is u_{156} or u_{1456} we may replace w by wu_8 so we may assume that u_2^w is u_1 or u_{14}. If $u_2^w = u_{14}$ then $u_{12}^w = u_{124}$ and $u_{24}^w = u_{145}$. Since $u_{124} \sim_H u_{145}$, we get $u_5 \sim u_1 \sim_H u_{12} \sim u_{24} \sim_H u_{46}$, which is impossible by Lemma 1.6. Hence $u_2^w = u_1$, and the lemma is proven.

LEMMA 3.4. *There exists a subgroup N of G containing H such that $N \simeq C_{G_0}(u_6)$.*

Proof. Let w be the element of Lemma 3.3 and let $N = \langle H, w\rangle$. Then N normalizes M_1 and by Lemma 3.3, the action of N on M_1 is uniquely determined. Since $C_N(M_1) = C_H(M_1) = M_1$, the structure of N/M_1 is uniquely determined. Now M_1 is complemented in T by the subgroup $\langle u_3, u_7, u_8, u_9\rangle$ and so, by the theorem of Gaschütz [1], M_1 has a complement $C \simeq N/M_1$ in N. Since M_1 is abelian, the action of C on M_1 is uniquely determined, and thus the multiplication table of N is uniquely determined. But in G_0 we have $\langle C_{G_0}(u_{56}), w_2\rangle \simeq C_{G_0}(u_6)$, and so by uniqueness $N \simeq C_{G_0}(u_6)$.

Henceforth, we let $N = \langle u_i, w_2, w_3 \mid 1 \leq i \leq 9\rangle$ with the relations (1.1) and take $H = \langle u_i, w_3 \mid 1 \leq i \leq 9\rangle$. We now have:

Table VII

x	$\mid \text{cl}_N(x) \cap S \mid$
$u_{56} \sim u_{46} \sim u_{15}$	3
$u_5 \sim u_4 \sim u_1$	3
u_6	1
$u_8 \sim u_9$	6
$u_{58} \sim u_{49}$	6
u_7	4
$u_{78} \sim u_{79}$	12
$u_{678} \sim u_{679}$	12
$u_{59} \sim u_{459} \sim u_{48} \sim u_{19}$	12
u_{67}	4

We now prove a series of lemmas concerning the fusion of N-classes of involutions in G. Since we now have $u_1 \sim u_2$ in $N_G(S)$ we may apply the information in Table V. Since G is assumed to be simple, Glauberman's Theorem [2] implies that u_6 much fuse with some other element of T, and so by Lemma 1.2, $N_G(S) \neq N_N(S)$. It follows that $n(u_{56}) = 15, 21, 27$, or 45

LEMMA 3.5. *It is false that* $u_{56} \sim u_8 \sim u_{58}$.

Proof. Assume $u_{56} \sim u_8 \sim u_{58}$. Then there exists $x \in G$ such that $u_8^x = u_{56}$ and $C_T(u_8)^x = L_2^x \subseteq T$. Then $L_2^x = L_2$ and so x normalizes

$$Z(L_2) = \langle u_5, u_6, u_8 \rangle.$$

By assumption

$$\mathrm{cl}_G(u_{56}) \cap Z(L_2) = \{u_{56}, u_8, u_{568}, u_{58}, u_{68}\},$$

and so $u_5^x = u_5$ and $u_6^x = u_6$. But then $u_{56}^x = u_{56}$, a contradiction.

LEMMA 3.6. *We may assume* $u_{56} \sim u_{79}$.

Proof. Since $n(u_{56}) = 15, 21, 27$, or 45 it follows from Table VII and Lemma 3.5 that u_{56} is conjugate to one of the elements u_{79}, u_{48}, or u_{679}. Assume $u_{56} \sim u_{48}$. Then there exists $x \in G$ such that $u_{48}^x = u_{56}$ and $C_T(u_{48})^x = K_3^x \subseteq T$. Then $K_3^x = K_3, K_4$, or K_5 since $u_6 \notin (K_3^x)'$, and hence $u_5^x = u_5$. Then $u_{458}^x = u_6$, and, since $u_{48} \sim_H u_{458}$, we have a contradiction. Therefore, $u_{56} \sim u_{79}$ or u_{679}. Since the automorphism α_4 in Lemma 1.7 interchanges $\mathrm{cl}_N(u_{79})$ and $\mathrm{cl}_N(u_{679})$, we may assume $u_{56} \sim u_{79}$.

LEMMA 3.7. *We may assume* $u_6 \sim u_8$ *and* $u_{56} \sim u_{58}$.

Proof. We first show that u_6 is not conjugate to any of the elements u_{48}, u_7, u_{67}, or u_{678}. Assume $u^x = u_6$ and $C_T(u)^x \subseteq T$ for some $x \in G$ and some

$$u \in \{u_{48}, u_7, u_{67}, u_{678}\}.$$

Then $C_T(u) = K_3$ or K_5 and since $K_1 \sim K_2 \sim K_3$ and $K_4 \sim K_5$ in N we may assume $C_T(u)^x = K_3$ or K_5. Since $u_{78} \sim u_{578} \sim u_{46} \sim u_{456}$ by Lemma 3.6, we must have $u_5^x = u_5$. Then $(uu_5)^x = u_{56}$. But since $u \sim_H uu_5$ for each $u \in \{u_{48}, u_7, u_{67}, u_{678}\}$, this is a contradiction. Now, by Table VII, the only possibilities for the fusion of u_6 are $u_6 \sim u_8$ or $u_6 \sim u_{58}$. Since the automorphism α_5 in Lemma 1.7 interchanges $\mathrm{cl}_N(u_8)$ and $\mathrm{cl}_N(u_{58})$, but fixes $\mathrm{cl}_N(u_{79})$ and $\mathrm{cl}_N(u_{679})$, we may assume $u_6 \sim u_8$. Finally, let $x \in G$ such that $u_8^x = u_6$ and $C_T(u_8)^x = L_2^x \subseteq T$. Then by weak closure $L_2^x = L_2$ and so x normalizes $Z(L_2) \cap L_2' = \langle u_5 \rangle$. Hence $u_{58}^x = u_{56}$.

LEMMA 3.8. *Now* $n(u_{56}) = 21$, $n(u_6) = 7$, *and* $n(u_5) = 7$.

Proof. By Lemmas 3.6 and 3.7, we have $u_{56} \sim u_{79} \sim u_{58}$, so by Table VII, $n(u_{56}) \geq 21$. Now $n(u_{56}) = 27$ only if $u_{56} \sim u_8$ and $n(u_{56}) = 45$ only if $u_{56} \sim u_{48} \sim u_{679}$. But the proof of Lemma 3.6 yields $u_{56} \not\sim u_{48}$ and by Lemma

3.7, $u_{56} \not\sim u_8$, and therefore $n(u_{56}) = 21$. Thus we have

$$n(u_6) \leq |N_G(S) : N_G(S) \cap C_N(u_6)| = 7$$

$$\text{and} \quad n(u_5) \leq |N_G(S) : N_G(S) \cap C_N(u_5)| = 7.$$

Since $u_6 \sim u_8$ we have by Table VII that $n(u_6) = 7$, and we have $n(u_5) = 3$ or 7. But by Lemma 1.5 we get $N_G(S)/S \simeq L_2(7)$, which has no subgroup of index 3. Thus since $S \subseteq N_G(S) \cap C_G(u_5)$ we must have $n(u_5) = 7$. The lemma is proven.

LEMMA 3.9. *The involutions of G are fused as follows:*

(i) $u_{56} \sim u_{46} \sim u_{15} \sim u_{58} \sim u_{49} \sim u_{78} \sim u_{79}$,
(ii) $u_6 \sim u_8 \sim u_9$,
(iii) $u_5 \sim u_4 \sim u_1 \sim u_7$,
(iv) $u_{678} \sim u_{679} \sim u_{59} \sim u_{459} \sim u_{48} \sim u_{19} \sim u_{67}$.

Proof. Statements (i) and (ii) follow from Table VII and Lemma 3.6, 3.7, and 3.8. Now by Lemma 3.7 there exists $x \in G$ such that $u_8^x = u_6$ and $C_T(u_8)^x = L_2^x \subseteq T$. Thus $L_2^x = L_2$. If x normalizes K_3 then $u_{46}^x \in \{u_{46}, u_{456}\}$ and so $u_{468}^x \in \{u_4, u_{45}\}$. Since $u_{468} \sim_H u_{48}$ this is impossible. Hence x does not normalize K_3 and we may assume $K_5^x = K_3$. Then $u_{78}^x \in \{u_{46}, u_{456}\}$ and so $u_7^x \in \{u_4, u_{45}\}$. Thus $u_7 \sim u_5$ and, because of Lemma 3.8, (iii) holds. Finally, we have from Table VII that $|\mathrm{cl}_N(u_{67}) \cap S| = 4$. Since $L_2(7)$ has no subgroup of index 4 or 16, we must have $u_{67} \sim u_{59} \sim u_{678}$, and thus (iv) holds.

Now we proceed as in Lemmas 2.5 and 2.6 to construct a subgroup of G which is isomorphic to $S_{p_6}(2)$.

LEMMA 3.10. *There exists an element w of G whose action on L_2 is the same as that of w_1.*

Proof. By Lemma 3.7 there is an element $w \in G$ normalizing L_2 such that $u_8^w = u_6$ and $u_5^w = u_5$. Now $u_7^w \in \{u_4, u_{45}\}$, so by replacing w by wu_3 if necessary we may assume $u_7^w = u_4$. Similarly $u_4^w \in \{u_7, u_{57}\}$; since w may be replaced by wu_2 we can assume $u_4^w = u_7$. Now $(K_3')^w = K_5'$ so $u_{46}^w \in \{u_{78}, u_{578}\}$. Since $u_{46}^w = u_{578}$ implies $u_6^w = u_{58} \sim u_{56}$, we must have $u_6^w = u_8$. Also

$$u_9^w \in \{u_9, u_{469}, u_{789}, u_{456789}\}.$$

By computing u_{49}^w and u_{79}^w we arrive at a contradiction unless $u_9^w = u_9$. Thus the action of w on S is determined. Since $w^2 \in C_G(S) = S$ and since S is complemented in any 2-group containing it, we may assume $w^2 = 1$. Now we get $u_2^w \in \{u_3, u_{35}, u_{378}, u_{3578}\}$, and by replacing w by wu_{47}, wu_9, or wu_{479} we may assume $u_2^w = u_3$. We still have $w^2 = 1$, so $u_3^w = u_2$ and the lemma is proven.

LEMMA 3.11. (i) $(wu_1)^3 = 1$.

(ii) $(ww_2)^3 = 1$.
(iii) $(ww_3)^2 = 1$.

Proof. (i) $(wu_1)^3 \in C_G(L_2) = \langle u_5, u_6, u_8\rangle$. Hence $|wu_1| = 3$ or 6. Assume $|wu_1| = 6$. Then $(wu_1)^3 \in C_G(w, u_1)$ so $(wu_1)^3 = u_5$. Replace w by wu_5. Then Lemma 3.10 is still satisfied, and hence we may assume $(wu_1)^3 = 1$.

(ii) $(ww_2)^3 \in C_G(S) = S$. Hence $|ww_2| = 3$ or 6. Assume $|ww_2| = 6$. Then $(ww_2)^3 \in C_G(w, w_2)$ so $(ww_2)^3 \in \{u_{457}, u_{689}, u_{456789}\}$. Now in the dihedral group $\langle w, w_2\rangle$ we have $w \sim w_2(ww_2)^3$. But $w_2u_{457} \sim u_{56}$ and $w_2u_{689} \sim w_2\, u_{456789} \sim u_{59}$, whereas by (i) we must have $w \sim u_1 \sim u_5$. This is a contradiction, and so $(ww_2)^3 = 1$.

(iii) $(ww_3)^2 \in C_G(\langle u_2, u_3, \cdots, u_8\rangle) = \langle u_5, u_6, u_8\rangle$. Hence $|ww_3| = 2$ or 4. Assume $|ww_3| = 4$, and thus

$$(ww_3)^2 \in C_G(w, w_3) \cap \langle u_5, u_6, u_8\rangle = \langle u_5, u_{68}\rangle.$$

Then in the dihedral group $\langle w, w_3\rangle$ we have $w_3 \sim w_3(ww_3)^2$, and we get $w_3 \sim u_{56}$ or $w_3 \sim u_{59}$. But in fact $w_3 \sim_H u_9 \sim u_6$; the lemma is proven.

Now since w satisfies the same relations (1.1) as w_1 we have proved:

LEMMA 3.12. *Let* $G_1 = \langle N, w\rangle$. *Then* $G_1 \simeq S_{p_6}(2)$.

LEMMA 3.13. $C_G(u_{59}) \subseteq C_G(u_5)$.

Proof. We will prove that K_1 is a Sylow 2-subgroup of $C_G(u_{59})$; then the lemma follows by a proof exactly like that of Lemma 2.9. Assume K_1 is not a Sylow subgroup of $C_G(u_{59})$ and let $g \in C_G(u_{59})\backslash K_1$ such that $K_1^g = K_1$. Then g normalizes $(K_1)'$ and so $u_{56}^g \in \{u_{56}, u_{456}\}$. Then there exists $g_1 \in G_1$ such that $u_{56}^{gg_1} = u_{56}$, and so $g \in G_1$. Since

$$C_{G_1}(u_{59}) \cap N_G(K_1) = K_1,$$

we have a contradiction.

LEMMA 3.14. $C_G(u_6) \subseteq G_1$.

Proof. Assume that $C_G(u_6) \not\subseteq G_1$ and as in Lemma 2.10 construct a subgroup $\hat{T} = T^y$ with $I = \hat{T} \cap G_1$. Again we have $u_6 \in I$ and $u_5, u_{56} \notin I$. Since the centralizer of every conjugate of u_{56} lies in G_1, I contains no conjugate of u_{56}. Similarly I contains no element of $\mathrm{cl}_N\,(u_{59})$: for if $u \in I$ and $u^n = u_{59}$ with $n \in N$ then $u_5^{yn} \notin G_1$. But u_5^{yn} centralizes u_{59} and u_6, and so by Lemma 3.13 it centralizes u_{56}, which is impossible. Now it follows that all the involutions of I must lie in

$$\mathrm{cl}_H\,(u_6) \cup \mathrm{cl}_H(u_7) \cup \mathrm{cl}_H\,(u_{67}).$$

As in Lemma 2.10, I is elementary abelian. If $I \neq \langle u_6\rangle$ then we may assume that $u_7 \in I$, and it follows that $I = \langle u_6, u_7\rangle$. Again as in Lemma 2.10 we get either $u_5 \sim u_5z$ or $u_{59}^y \sim u_5z$ where $z \in I\backslash\{1\}$. The only possibilities are $u_5 \sim u_{57}$ or $u_{59}^y \sim u_{567}$. The first case implies $u_{56} \sim u_{567}$ and the second $u_{569} \sim u_{57}$, both of which are impossible. Hence the lemma is proved.

Now we may complete the proof of the theorem. We have

$$C_G(u_6) = C_{G_1}(u_6) \simeq C_{G_0}(u_6).$$

Since G has four classes of involutions, it follows from Yamaki's Theorem [7] that $G \simeq S_{p_6}(2)$.

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