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THE LOOP SPACE OF THE Q-CONSTRUCTION[1]

BY

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The higher algebraic K-groups are defined as $K_i\mathcal{M} = \pi_{i+1}|Q\mathcal{M}| = \pi_i\Omega|Q\mathcal{M}|$ for an exact category $\mathcal{M}$. We present a simplicial set $G\mathcal{M}$ with the property that $|G\mathcal{M}|$ is naturally homotopy equivalent to the loop space $\Omega|Q\mathcal{M}|$, and thus $K_i\mathcal{M} = \pi_i|G\mathcal{M}|$. In this way we given an algebraic description of the loop space, which a priori, has no such description.

The case where $\mathcal{M}$ is a category in which all the exact sequences split was done by Quillen with his category $S^{-1}S$. In fact, the definition of our space $G\mathcal{M}$ is a simple generalization of the definition of $S^{-1}S$. Its vertices are all pairs (M, N) of objects of $\mathcal{M}$, and its edges connecting (M, N) to (M', N') are all pairs of exact sequences

$$0 \to M \to M' \to C \to 0, \quad 0 \to N \to N' \to C \to 0.$$

Higher dimensional simplices are defined analogously. There is an isomorphism $\pi_0 G\mathcal{M} \cong K_0\mathcal{M}$, with (M, N) corresponding to $[M] - [N]$.

The simplicial techniques (Section 1) used in the proof that $|G\mathcal{M}| \sim \Omega|Q\mathcal{M}|$ are generalizations of Theorems A and B of Quillen [5]; they apply to simplicial sets rather than just to categories. There is a canonical procedure (subdivision) for converting simplicial sets to categories, but our techniques are not based on this.

The main idea from Quillen's proof of the statement $S^{-1}S = \Omega|Q\mathcal{M}|$ also appears here, but in a more understandable guise. The motto might be "use algebra to add, and topology to subtract". This can be explained briefly by considering supermodules $M \supset N$ of an R-module N (i.e., injections from N into another module). Using algebra, we may "add" them thus: $M_1 + M_2 := M_1 \amalg_N M_2$. We have the equation $M + M = M + (N \oplus M/N)$, and it turns out that by topology (i.e., in the homotopy groups) subtraction is allowed, and yields the equation $M = N \oplus M/N$.

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As an application of the ideas of the proof, we provide a simplification of the original proof of Quillen (as presented by Grayson in [2]) of the homotopy equivalence $S^{-1}S \sim \Omega|Q\mathcal{M}|$.

The first application of the result itself is an explicit representation, in algebraic terms, for an arbitrary element of $K_1\mathcal{M}$.

As our second application, we show how to use $G\mathcal{M}$ to give yet another definition for products in K-theory. We avoid the phony multiplication problem explained by Thomason in [6], in spite of the similarity between $G\mathcal{M}$ and $S^{-1}S$, by iterating G to get $GG\mathcal{M}$, as was done with Q by Waldhausen in [8].

In a future paper we hope to use this construction to provide natural definitions for the λ-operations on algebraic K-theory (coming from exterior powers). This would provide λ-operations on derived K-groups, such as relative K-groups or K-groups with supports, where none exist at the moment. For this purpose a construction such as G is essential, because λ^k is not additive on $K_0\mathcal{M}$, and thus cannot arise as a map $|Q\mathcal{M}| \to |Q\mathcal{M}|$.

1. Simplicial homotopy theory

In [5], Quillen developed fundamental techniques for proving homotopy theoretic facts about the geometric realizations of categories. In this section we show that they work more generally for simplicial sets.

Some notation first. Let Δ^+ be the category of finite ordered sets, let Δ be the category of nonempty finite ordered sets, and use A, B, C for typical objects of these categories. For $p \in \mathbf{N}$ let $[p] := \{0 < 1 < \cdots < p\} \in \Delta$. The p-simplices $X([p])$ of a simplicial set X will also be denoted by X_p.

We have full faithful functors $\Delta \hookrightarrow \Delta^+ \hookrightarrow$ (ordered sets) $\hookrightarrow$ (partially ordered sets) $\hookrightarrow$ (categories) $\overset{N}{\hookrightarrow}$ (simplicial sets), where $N =$ "nerve of". Consequently, we may identify everything in sight with its nerve, and will never write N.

We use $|\ \ |$ to denote "geometric realization of".

By Yoneda's lemma, we may view a p-simplex of a simplicial set X as a map $[p] \to X$.

A 0-simplex will be called a vertex or an object, and a 1-simplex will be called an arrow or an edge.

If $A \in \Delta$, we define $A^{\mathrm{op}} \in \Delta$ as A with the reversed ordering. If X is a simplicial set, we define X^{op} by $X^{\mathrm{op}}(A) := X(A^{\mathrm{op}})$. There is a natural homeomorphism $|X| \cong |X^{\mathrm{op}}|$. If X is a category then X^{op} is the usual opposite category.

Given $A, B \in \Delta^+$ let $AB \in \Delta^+$ denote the disjoint union $A \amalg B$ ordered so $A < B$.

Let Δ^* denote the subcategory of Δ whose arrows are all those maps $A \to B$ in Δ which send the top element of A to the top element of B. We have an embedding $\Delta \hookrightarrow \Delta^*$ defined by adjoining a top element $*$, e.g., $A \mapsto A\{*\}$.

LEMMA 1.1. *If X^*: $\Delta^{*\mathrm{op}} \to$ (sets) is a functor, then the map*

$$\begin{array}{c} X := (A \mapsto X^*(A\{*\})) \\ \downarrow F \\ X_\phi := (A \mapsto X^*(\{*\})) \end{array}$$

of simplicial sets induced by the inclusions $\{*\} \hookrightarrow A\{*\}$ is a homotopy equivalence.

Proof. The retractions $A\{*\} \twoheadrightarrow \{*\}$ provide a map G on the other way so that $FG = 1$.

We define a simplicial homotopy h: $X \times [1] \to X$ from 1_X to GF as follows, the idea being to retract the elements of A onto $*$ one by one. Given a p-simplex (α, β): $[p] \to X \times [1]$ we write $[p] = AB$ where $A = \beta^{-1}\{0\}$ and $B = \beta^{-1}\{1\}$. Let $\tilde{\beta}$: $AB\{*\} \to AB\{*\}$ be the map in Δ^* which is the identity on A and sends B to $*$. We define $h(\alpha, \beta) = X^*(\tilde{\beta})(\alpha)$. When $B = \phi$ we have $\tilde{\beta} = 1$ and $h(\alpha, \beta) = \alpha$; when $A = \phi$ we have $h(\alpha, \beta) = GF(\alpha)$. One checks that h is a simplicial map, and thus is a simplicial homotopy of the type desired. Q.E.D.

Remark. X_ϕ is a discrete simplicial set (i.e., constant), so the lemma provides a homotopy equivalence $X \sim \pi_0 X$. Moreover, the inverse map and the homotopies are natural.

Given $A, B \in \Delta^+$, let $[A[B]]$ denote the inclusion $B \hookrightarrow AB$. Let $\tilde{\Delta}$ denote the category of all such inclusions where B is nonempty; the arrows are all commutative squares

$$\begin{array}{ccc} B & \hookrightarrow & AB \\ \downarrow & & \downarrow \\ B' & \hookrightarrow & A'B'. \end{array}$$

PROPOSITION 1.2. *If $\tilde{X}$: $\tilde{\Delta}^{\mathrm{op}} \to$ (sets) is a functor, then the map*

$$\begin{array}{c} A, B \mapsto \tilde{X}([A[B]]) \\ \downarrow \\ A, B \mapsto \tilde{X}([\phi[B]]) \end{array}$$

of bisimplicial sets induced by the inclusions $[\phi[B]] \hookrightarrow [A[B]]$ is a homotopy equivalence.

Proof. It is enough, by [7, Lemma 5.1], to show that this map is a homotopy equivalence in each degree, i.e., that for each $B \in \Delta$ the map

$$\begin{array}{c} A \mapsto \tilde{X}([A[B]]) \\ \downarrow \\ A \mapsto \tilde{X}([\phi[B]]) \end{array}$$

of simplicial sets is a homotopy equivalence. Define a functor $G\colon \Delta^* \to \tilde{\Delta}$ by setting $G(C) := (B \hookrightarrow C\amalg_* B)$, where $C\amalg_* B$ denotes the quotient of CB obtained by identifying the largest element of C with the smallest element of B. Define $X^* = \tilde{X} \circ G\colon \Delta^{*\mathrm{op}} \to$ (sets). Since $G(A\{*\}) = [A[B]]$, we may apply Lemma 1.1 to get the result. Q.E.D.

Define projections $L, R\colon \Delta \times \Delta \to \Delta$ by $L(A, B) = A$ and $R(A, B) = B$. If X is a simplicial set, then composition gives two bisimplicial sets XL and XR; all three objects have homeomorphic geometric realizations.

If $F\colon X \to Y$ is a map of simplicial sets, then we define a bisimplicial set $Y|F$ by setting

$$(Y|F)(A, B) := \varprojlim \left(\begin{array}{ccc} & & X(B) \\ & & \downarrow F \\ Y(AB) & \to & Y(B) \end{array} \right).$$

The space $|Y|F$ is an analogue of the path space $|Y|^I \times_{|Y|} |X|$.

We write $Y|Y$ for $Y|1_Y$; notice that $(Y|Y)(A, B) = Y(AB)$, so $Y|Y$ enjoys some symmetry.

PROPOSITION 1.3. *The projection map $\pi\colon Y|F \to XR$ is a homotopy equivalence.*

Proof. We define a functor $(Y|F)^\sim\colon \tilde{\Delta}^{\mathrm{op}} \to$ (sets) by

$$(Y|F)^\sim([A[B]]) := \varprojlim \left(\begin{array}{ccc} & & X(B) \\ & & \downarrow F \\ Y(AB) & \to & Y(B) \end{array} \right).$$

Yes, the definition looks the same, but the domain of definition is different; one must check the functoriality again.

Apply 1.2, noticing that

$$(Y|F)^\sim([\phi[B]]) = X(B) = XR(A, B),$$

and

$$(Y|F)^\sim([A[B]]) = (Y|F)(A, B). \qquad \text{Q.E.D.}$$

Suppose $F\colon X \mapsto Y$ is a map of simplicial sets, $A \in \Delta$, and $\rho \in Y(A)$. We define a simplicial set $\rho|F$ (called the "right fiber over ρ") by

$$(\rho|F)(B) := \varprojlim \left(\begin{array}{ccc} & & X(B) \\ & & \downarrow \\ & Y(AB) & \to Y(B) \\ & \downarrow & \\ \{\rho\} & \hookrightarrow Y(A) & \end{array} \right).$$

It is a simplicial analogue of the homotopy fiber of $|F|$ over ρ. We write $\rho|Y$ for $\rho|1_Y$.

LEMMA 1.4. *$\rho|Y$ is contractible, and the contraction is natural.*

Proof. Define $Z^*\colon \Delta^{*\mathrm{op}} \to (\text{sets})$ by

$$Z^*(B) := \varprojlim \begin{pmatrix} & Y(A\amalg_*(B^{\mathrm{op}})) \\ & \downarrow \\ \{\rho\} \hookrightarrow & Y(A) \end{pmatrix}.$$

Since $Z^*(\{*\}) = \{\rho\}$ and $Z^*(C\{*\}) = (\rho|Y)(C^{\mathrm{op}})$, Lemma 1.1 yields contractibility of $(\rho|Y)^{\mathrm{op}}$, so $(\rho|Y)$ is contractible, too. Q.E.D.

THEOREM A$'$. *If $F\colon X \to Y$ is a map of simplicial sets, and for all $A \in \Delta$ and all $\rho \in Y(A)$, $\rho|F$ is contractible, then F is a homotopy equivalence.*

Proof. We mimic the proof of theorem A in [5]. Consider the diagram of projection maps

$$\begin{array}{ccccc} YL & \longleftarrow & Y|F & \xrightarrow{\sim} & XR \\ \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow{\scriptstyle F} \\ YL & \xleftarrow{\sim} & Y|Y & \xrightarrow{\sim} & YR. \end{array}$$

The three arrows marked $\sim$ are homotopy equivalences by Lemma 1.3, or rather, in the case of $Y|Y \to YL$, by the mirror image of 1.3. It is enough to show that $Y|F \to YL$ is a homotopy equivalence, and to do that it suffices to fix an arbitrary $A \in \Delta$ and show that the simplicial map

$$\begin{array}{c} B \mapsto (Y|F)(A, B) \\ \downarrow \\ B \mapsto Y(A) \end{array}$$

is a homotopy equivalence. Since the target of this map is discrete, it is enough to show that for each $\rho \in Y(A)$ the fiber is contractible; but that fiber is $\rho|F$, which was assumed to be contractible. Q.E.D.

LEMMA 1.5. *Suppose Z is a bisimplicial set, Y is a simplicial set, and F: $Z \to YL$ is a map. For $A \in \Delta$ and $\rho \in Y(A)$ define a simplicial set Z_ρ by*

$$Z_\rho(B) = \varprojlim \begin{pmatrix} & Z(A, B) \\ & \downarrow{\scriptstyle F} \\ \{\rho\} \hookrightarrow & Y(A) \end{pmatrix}.$$

Suppose, that for all $A \in \Delta$, *all* $\rho \in Y(A)$, *and all* f: $A' \to A$ *in* Δ, *the natural map* $Z_\rho \to Z_{f^*\rho}$ *is a homotopy equivalence. Then the map*

$$|Z| \xrightarrow{F} |Y|$$

is a quasifibration, and the fiber over any point in the open simplex of $|Y|$ *corresponding to* ρ *is homeomorphic to* $|Z_\rho|$.

Proof. We may consider the simplicial space $A \mapsto |B \mapsto Z(A, B)|$ and write it as

$$A \mapsto \coprod_{\rho \in Y(A)} |Z_\rho|.$$

Now one simply follows the proof of the lemma on p. 90 of [5], replacing the category I by the simplicial set Y to get the result. Q.E.D.

LEMMA 1.6. *Suppose*

$$\begin{array}{ccc} W' & \to & W \\ \downarrow f' & & \downarrow f \\ V' & \underset{g}{\to} & V \end{array}$$

is a commutative square of spaces (each of which is a geometric realization of a simplicial set), f *and* f' *are quasifibrations, and for all* $v \in V'$ *the map* $f'^{-1}(v) \to f^{-1}(g(v))$ *is a homotopy equivalence. Then the square is homotopy cartesian.*

Proof. See [7, p. 167].

If F: $X \to Y$ is a map of simplicial sets, $A \in \Delta$, and $\rho \in Y(A)$, then we define a bisimplicial set $\rho|Y|F$ by

$$(\rho|Y|F)(B,C) := \varprojlim \begin{pmatrix} & & X(C) \\ & & \downarrow \\ & Y(ABC) \to & Y(C) \\ & \downarrow & \\ \{\rho\} \hookrightarrow & Y(A) & \end{pmatrix}.$$

We let $\rho|Y|Y = \rho|Y|1_Y$.

LEMMA 1.7. *If* F: $X \to Y$ *is a map of simplicial sets,* $A \in \Delta$, *and* $\rho \in Y(A)$, *then the map* $\rho|Y|F \to (\rho|F)R$ *(provided by the inclusions* $AC \hookrightarrow ABC$*) is a homotopy equivalence.*

Proof. We define $(\rho|Y|F)^{\sim} : \tilde{\Delta}^{\mathrm{op}} \to (\text{sets})$ by

$$(\rho|Y|F)^{\sim}([B[C]]) := \overset{\lim}{\leftarrow}\left(\begin{array}{ccc} & & X(C) \\ & & \downarrow \\ & Y(ABC) \to & Y(C) \\ & \downarrow & \\ \{\rho\} \hookrightarrow & Y(A) & \end{array}\right).$$

When B is nonempty, then $(\rho|Y|F)^{\sim}([B[C]]) = (\rho|Y|F)(B, C)$, and when B is empty, then $(\rho|Y|F)^{\sim}([\phi, [C]]) = (\rho|F)(C) = (\rho|F)R(B, C)$. Proposition 1.2 gives the result. Q.E.D.

LEMMA 1.7′. *If X is a simplicial set, $A \in \Delta$, and $\rho \in Y(A)$, then the map*

$$\rho|Y|Y \to (\rho|Y)L$$

(provided by the inclusions $AB \hookrightarrow ABC$) is a homotopy equivalence.

Proof. We observe that $\rho|Y|Y = (\rho|Y)|(\rho|Y)$, and apply the mirror image of 1.3. Q.E.D.

THEOREM B′. *Suppose $F: X \to Y$ is a map of simplicial sets. Suppose for any $A \in \Delta$, any $\rho \in Y(A)$, and any $f: A' \to A$ that the map $\rho|F \to f^*\rho|F$ (induced by f) is a homotopy equivalence. Then the square*

$$\begin{array}{ccc} \rho|F & \longrightarrow & X \\ \downarrow & & \downarrow \\ \rho|Y & \longrightarrow & Y \end{array}$$

is homotopy cartesian.

COROLLARY 1.8. *Assuming that the three spaces involved are nonempty, and that basepoints are chosen appropriately, there is a long exact sequence of homotopy groups*

$$\cdots\ \pi_i X \to \pi_i Y \to \pi_{i-1}\rho|F \to \pi_{i-1}X\ \cdots$$

Proof. Consider the following maps of commutative squares of bisimplicial sets:

$$\begin{array}{ccc} \rho|Y|F \to Y|F \\ \downarrow \qquad\quad \downarrow \\ (\rho|Y)L \to YL \end{array} \leftarrow \begin{array}{ccc} \rho|Y|F \to Y|F \\ \downarrow \qquad\quad \downarrow \\ \rho|Y|Y \to Y|Y \end{array} \to \begin{array}{ccc} (\rho|F)R \to XR \\ \downarrow \qquad\quad \downarrow \\ (\rho|Y)R \to YR. \end{array}$$

The horizontal maps are all homotopy equivalences, by Lemma 1.3, 1.7, or 1.7′. By [3, Note 3.13.1] the left square is homotopy cartesian if and only if the right one is. Since the geometric realization of the right hand square is homeomorphic to the one in our statement, it suffices to show the left hand square is homotopy cartesian.

Given $B \in \Delta$ and $\tau \in Y(B)$, we see that $(Y|F)_\tau = \tau|F$, and thus 1.5 holds for the map $Y|F \to YL$.

Given $B \in \Delta$ and $\tau \in (\rho|Y)(B) \subset Y(AB)$, we see that $(\rho|Y|F)_\tau = \tau|F$, so Lemma 1.5 holds for the map $\rho|Y|F \to (\rho|Y)L$.

Finally, given $B \in \Delta$ and $\tau \in (\rho|Y)(B) \subset Y(AB)$, let f denote the inclusion $B \hookrightarrow AB$, so that $f^*\tau$ is the image of τ in $Y(B)$. Then the map

$$(\rho|Y|F)_\tau \to (Y|F)_{f^*\tau}$$

is isomorphic to the base-change map $\tau|F \to f^*\tau|F$, and is thus a homotopy equivalence. Thus Lemma 1.6 applies to the left hand square, and shows it is homotopy cartesian. Q.E.D.

If X is a bisimplicial set, then (thinking of the second variable as the extraneous one) we define the following simplicial sets:

$$\begin{aligned} \operatorname{obj} X&: \quad A \mapsto X(A,[0]) \\ \operatorname{arr} X&: \quad A \mapsto X(A,[1]) \end{aligned}$$

The maps

$$[0] \underset{1}{\overset{0}{\rightrightarrows}} [1]$$

provide maps

$$\operatorname{arr} X \underset{t}{\overset{s}{\rightrightarrows}} \operatorname{obj} X,$$

($s =$ "source", $t =$ "target"). We say that X *supports natural transformations* if there is a simplicial homotopy H from s to t. If Y is a simplicial set, and

$$Y \underset{g}{\overset{f}{\rightrightarrows}} \operatorname{obj} X$$

are maps, then a *natural transformation* from f to g will be a map $h: Y \to \operatorname{arr} X$ such that $s \circ h = f$ and $t \circ h = g$; if X supports natural transformations, then the composite $H \circ (h \times \mathrm{id}_{[1]})$ is a simplicial homotopy from f to g. We say natural transformations on X are *left-stable* if the homotopy H satisfies the following property: given $A, B \in \Delta$ and $\sigma \in (\operatorname{arr} X)(AB)$ with $i^*\sigma$ an identity arrow (where $i\colon A \hookrightarrow AB$ is the inclusion), we have $H(\sigma, f) = t(\sigma)$ if $f\colon AB \to [1]$ is the map sending A to 0 and B to 1.

We will usually apply this terminology in the case where the case where for each $A \in \Delta$, the simplicial set $X_A\colon B \mapsto X(A, B)$ is a groupoid; call such an X a *simplicial groupoid*. If $Z = \operatorname{obj} X$, then we use the notation Z^{Is} to refer to X.

Suppose X and Y are bisimplicial sets which support natural transformations. A map $F\colon X \to Y$ *preserves natural transformations* if the square

$$\begin{array}{ccc} \operatorname{arr} X \times [1] & \to & \operatorname{obj} X \\ \downarrow & & \downarrow \\ \operatorname{arr} Y \times [1] & \to & \operatorname{obj} Y \end{array}$$

commutes.

If $F\colon X \to Y$ is a map of bisimplicial sets, and $\rho \in X(A, [0])$, we define a bisimplicial set $\rho | F$ by

$$(\rho|F)(B,C) = \varprojlim \left(\begin{array}{ccccc} & & & & X(B,C) \\ & & & & \downarrow F \\ & & Y(AB,C) & \to & Y(B,C) \\ & & \downarrow & & \\ \{\rho\} \hookrightarrow Y(A,[0]) & \hookrightarrow & Y(A,C) & & \end{array} \right).$$

LEMMA 1.9. *Suppose $F\colon X \to Y$ is a map of bisimplicial sets. Then*:

(a) $\operatorname{obj}(\rho|F) = \rho|\operatorname{obj} F$,

(b) *The map* $\pi\colon \rho|\operatorname{obj} F \to \operatorname{obj} X$ *of* (1.3) *arises as* obj( ) *of the obvious projection map* $\rho|F \to X$.

(c) *If X and Y are simplicial groupoids, so is $\rho|F$.*

(d) *If X and Y support natural transformations, and F preserves them, and natural transformations on Y are left-stable, then $\rho|F$ supports natural transformations, and $\rho|F \to X$ preserves them. If, moreover, natural transformations on X are left-stable, then they are left-stable on $\rho|F$, too.*

Proof. Parts (a)–(c) are clear. We prove (d).

Let $c\colon A \to [1]$ be the constant map to 0. Since any map $B \to [1]$ extends uniquely to a map $AB \to [1]$ which sends all elements of A to 0, one can check that $\pi\colon c|\mathrm{id}_{[1]} \to [1]$ is an isomorphism. This yields an isomorphism

$$\begin{aligned} (\operatorname{arr} \rho|F) \times [1] &= \operatorname{arr}((\rho|F) \times [1]L) \xleftarrow{\cong} \operatorname{arr}\big((\rho|F) \times (c|\mathrm{id}_{[1]L})\big) \\ &= \operatorname{arr}\big((\rho, c)|F \times \mathrm{id}_{[1]L}\big) \end{aligned}$$

From which a suitable homotopy $(\operatorname{arr} \rho|F) \times [1] \to \operatorname{obj} \rho|F$ can easily be derived. Q.E.D.

2. Loop spaces

Suppose X is any simplicial set, with a base point $0 \in X_0$. A simple type of simplicial loop at 0 in X looks like

$$\begin{tikzcd} 0 \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & x \end{tikzcd}$$

when X is a category, and a simple type of homotopy of such loops looks like

$$\begin{tikzcd} 0 \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[rr, bend left=60, "f'"] \arrow[rr, bend right=60, "g'"'] & x \arrow[r, "h"] & y \end{tikzcd}$$

where the two triangles commute. This suggests defining a simplicial set ΩX by setting

$$\Omega X(A) := \varprojlim \left(\begin{matrix} \{0\} \hookrightarrow X([0]) & \longleftarrow & X([0]A) \\ \uparrow & & \downarrow \\ X([0]A) & \longrightarrow & X(A) \end{matrix} \right)$$

for $A \in \Delta$. The loops described above form the set $\Omega X([0])$, and the homotopies form $\Omega X([1])$. Forgetting one or the other of the components $X([0]A)$ in the definition provides maps

$$\Omega X \rightrightarrows 0|X,$$

and we have a commutative square

$$\begin{matrix} \Omega X & \to & 0|X \\ \downarrow & & \downarrow \\ 0|X & \to & X. \end{matrix}$$

Since $0|X$ is contractible, this square gives a map from $|\Omega X|$ to the homotopy fiber product of $|0|X| \leftarrow |X| \rightarrow |0|X|$, which is a homotopy equivalence if and only if the square is homotopy cartesian. Since $0|X$ is naturally contractible, the homotopy fiber product is naturally homotopy equivalent to the loop space $\Omega|X|$. Thus we have a natural map $|\Omega x| \to \Omega|X|$.

Consider the map P: $0|X \to X$. The fiber $0|P$ is easily seen to be ΩX, so it is natural to try to use Theorem B$'$ in this situation. Accordingly, given $A \in \Delta$

and $\rho \in X(A)$ we define $0,\rho|X$ to be the simplicial set $\rho|P$, so that $\Omega X = 0,0|X$. For any $B \in \Delta$ we have

$$(0,\rho|X)(B) = \lim_{\leftarrow} \begin{pmatrix} & & \{0\} \\ & & \downarrow \\ & X([0]B) \to & X([0]) \\ & \downarrow & \\ X(AB) \to & X(B) & \\ \downarrow & & \\ \{\rho\} \hookrightarrow X(A) & & \end{pmatrix}.$$

LEMMA 2.1. *Suppose X is a simplicial set, and for all maps $f\colon A' \to A$ in Δ and all $\rho \in X(A)$ the natural map $0,\rho|X \to 0,\ f^*\rho|X$ is a homotopy equivalence. Then the map $|\Omega X| \to \Omega|X|$ is a homotopy equivalence.*

Proof. Apply theorem B′. Q.E.D.

Remark The involution of ΩX which interchanges the two $X([0]A)$ components amounts to reversal of loops in $\Omega|X|$. Therefore, when the lemma applies, the involution is multiplication by -1 on the H-space $|\Omega X|$.

3. The loop space of the Q-construction

Let $\mathcal{M}$ be an exact category with a zero object called 0, and recall the simplicial set $S\mathcal{M}$ defined by Waldhausen [8, 1.3]; in his notation, it is $s.\mathcal{M}$—see [8, 1.4]; it is naturally homotopy equivalent to $Q\mathcal{M}$ via edgewise subdivision [8, 1.9], and is defined as follows. Let $\mathrm{Ar}[n]$ denote the category of arrows in $[n]$, and (j/i) denote the arrow from i to j in $[n]$, for $i \le j$. We call a sequence of the form $(j/i) \to (k/i) \to (k/j)$ in $\mathrm{Ar}[n]$ exact. We define $S_n\mathcal{M} = S\mathcal{M}([n])$ as the set of *exact* functors $M\colon \mathrm{Ar}[n] \to \mathcal{M}$, by which we mean that

(a) For all i, $M(i/i) = 0$, and

(b) For all $i \le j \le k$ the complex $0 \to M(j/i) \to M(k/i) \to M(k/j) \to 0$ is an exact sequence of $\mathcal{M}$.

We think of an element of $S_n\mathcal{M}$ as a sequence of admissible monomorphisms $0 = M_0 \rightarrowtail M_1 \rightarrowtail \cdots \rightarrowtail M_n$ together with choices $M(j/i)$ for all quotients M_j/M_i, $j \ge i > 0$. The face and degeneracy maps amount to forgetting or duplicating an M_i, except that if M_0 is forgotten, then we also factor out by M_1.

We define $G\mathcal{M} = \Omega S\mathcal{M}$.

THEOREM 3.1. *There is a homotopy equivalence* $|G\mathcal{M}| \xrightarrow{\sim} \Omega|S\mathcal{M}|$. *It is natural in* $\mathcal{M}$, *i.e., if* $\mathcal{M} \to \mathcal{M}'$ *is an exact functor, then the square*

$$\begin{array}{ccc} |G\mathcal{M}| & \to & \Omega|S\mathcal{M}| \\ \downarrow & & \downarrow \\ |G\mathcal{M}'| & \to & \Omega|S\mathcal{M}'| \end{array}$$

commutes. Direct sum makes $|G\mathcal{M}|$ *into an H-space, and the involution of* $G\mathcal{M}$ *which interchanges the two filtrations is an additive inverse map for this H-space structure.*

Proof. By Lemma 2.1, we must show that for any $f\colon A' \to A$ in Δ and any $M \in S\mathcal{M}(A)$ that the base-change map $f^*\colon 0, M|S\mathcal{M} \to 0, f^*M|S\mathcal{M}$ is a homotopy equivalence.

We will use the following diagram to represent a typical element of $(0, M|S\mathcal{M})(B)$, thinking of A as $[p]$ and B as $[q]$:

$$\left(\begin{array}{ccccccccccccc} & & & & & & 0 & \rightarrowtail & K_0 & \rightarrowtail & \cdots & \rightarrowtail & K_q \\ \hline\hline 0 & \rightarrowtail & M_1 & \rightarrowtail & \cdots & \rightarrowtail & M_p & \rightarrowtail & L_0 & \rightarrowtail & \cdots & \rightarrowtail & L_q \end{array}\right).$$

Row 1 represents a simplex from $S\mathcal{M}([0]B)$, and row 2 represents a simplex from $S\mathcal{M}(AB)$. We don't write the choices for all the quotients. The equality of the faces of these two simplices in $S\mathcal{M}(B)$ is represented by the double line; that equality amounts to giving compatible isomorphisms $K_j/K_i \cong L_j/L_i$ for all $0 \le i \le j \le q$.

Let $S\mathcal{M}^{\mathrm{Is}}$ be the simplicial groupoid whose objects are $S\mathcal{M}$, and which is defined by letting $S\mathcal{M}^{\mathrm{Is}}(A, B)$ be the set of functors $\mathrm{Ar}(A) \times B \to \mathcal{M}$ such that:

(a) For all $i \in A$, and all $b \in B$, $M((i/i), b) = 0$.

(b) For all $i \le j \le k \in A$ and all $b \in B$ the complex $0 \to M((j/i), b) \to M((k/i), b) \to M((k/j), b) \to 0$ is an exact sequence of $\mathcal{M}$.

(c) For all $(j/i) \in \mathrm{Ar}(A)$ and all $b \le b' \in B$ the map $M((j/i), b) \to M((j/i), b')$ is an isomorphism.

Since the family of exact sequences in $\mathcal{M}$ is closed under isomorphism, we deduce:

(b′) For all $i \le j \le k \in A$ and all $b \le c \le d \in B$ the complex $0 \to M((j/i), b) \to M((k/i), c) \to M((k/j), d) \to 0$ is an exact sequence of $\mathcal{M}$.

LEMMA 3.1.1 (a) *$S\mathscr{M}^{\mathrm{Is}}$ supports natural transformations, and on it they are left-stable.*

(b) *Similarly for $(0|S\mathscr{M})^{\mathrm{Is}}$, $(0, M|S\mathscr{M})^{\mathrm{Is}}$, and $\Omega S\mathscr{M}^{\mathrm{Is}}$ (which are all defined by virtue of* 1.9(c)).

Proof. By (1.9) it is enough to prove (a). We follow Waldhausen [8, proof of Lemma 1.4.1] and define

$$(\operatorname{arr} S\mathscr{M}^{\mathrm{Is}} \times [1])(A) \to (\operatorname{obj} S\mathscr{M}^{\mathrm{Is}})(A)$$

to be

$$(M, f) \mapsto ((j/i) \mapsto M((j/i)f(j))).$$

Checking naturality and the desired properties is easy. (Remark: we could not have used $f(i)$ instead of $f(j)$ above, for it would have destroyed the left-stability.) Q.E.D.

Now let $(0, M|S\mathscr{M})'$ be the simplicial set obtained from $(0, M|S\mathscr{M})$ by forgetting the choices of those quotients which involve any $M_i, 0 \le i \le p$, and let

$$F\colon 0, M|S\mathscr{M} \to (0, M|S\mathscr{M})'$$

be the forgetful map. A map F' the other way can be defined by first choosing quotients for all the admissible monomorphisms in $\mathscr{M}$; we remark that these two choices are related by a unique isomorphism. We have $F \circ F' = 1$, and a natural transformation $1 \xrightarrow{\cong} F' \circ F$, showing that F is a homotopy equivalence.

Suppose we consider now the case where the map f above is the inclusion $\{0, p\} \hookrightarrow [p]$, with $p \ge 1$. Then the base change map f^* is the one which forgets $M_1 \ldots M_{p-1}$ and all the choices of quotients involving these. There is a commutative square

$$\begin{array}{ccc} 0, M|S\mathscr{M} & \xrightarrow{f^*} & 0, f^*M|S\mathscr{M} \\ \downarrow F & & \downarrow F \\ (0, M|S\mathscr{M})' & \xrightarrow{f'^*} & (0, f^*M|S\mathscr{M})' \end{array}$$

where f'^* forgets $M_1, \ldots, M_{p-1}$ (the prime indicating that the choice of quotients has already been forgotten). It is clear now that f'^* is a bijection (its inverse simply re-inserts $M_1 \ldots M_{p-1}$), so f^* is a homotopy equivalence.

Now we claim that it is enough to check that the base-change map f^* is homotopy equivalence when f is one of the two maps $[0] \to [1]$. For, if that is done, the case where $\#A \ge 2$ and $\#A' \ge 2$ ($\#$ is cardinality) follows from the diagram

$$\begin{tikzcd}
\{0,3\} \arrow[d, "g"] & \{0\} \arrow[l] \arrow[r] & \{0,1\} \arrow[d] & \{1\} \arrow[l] \arrow[r] & \{1,2\} \arrow[d, "g'"] \\
A & & A \arrow[ll, "1"'] & & A' \arrow[ll, "f"']
\end{tikzcd}.$$

Here g sends $0,3$ to the extreme elements of A, and g' sends $1,2$ to the extreme elements of A'. That g^* and g'^* are homotopy equivalences was proved above, and the maps in the top row are all isomorphic to one of the two maps $[0] \to [1]$. Thus f^* is a homotopy equivalence. The cases where $\#A = 1$ or $\#A' = 1$ follow even more easily.

So now we may assume $p = 1$, and set $N = M_1$. Define f, g: $[0] \to [1]$ by $f(0) = 0$ and $g(0) = 1$. We must show that the base-change maps

$$f^*, g^*: 0, M|S\mathcal{M} \to G\mathcal{M}$$

are homotopy equivalences. The map f^* forgets N and the choices of quotients involving it, and the map g^* factors out by N and forgets all the original objects.

Now, for $\mathcal{M}$, we fix a direct sum operation by choosing, for each pair of objects M and M' in $\mathcal{M}$, a representative $M \oplus M'$ for their direct sum.

We use direct sum with N to define a map H: $G\mathcal{M} \to 0, M|S\mathcal{M}$, namely

$$\left(\frac{0 \rightarrowtail K_0 \rightarrowtail \cdots \rightarrowtail K_q}{0 \rightarrowtail L_0 \rightarrowtail \cdots \rightarrowtail L_q}\right)$$

$$\Big\downarrow H$$

$$\left(\frac{0 \rightarrowtail \qquad K_0 \rightarrowtail \cdots \rightarrowtail \qquad K_q}{0 \rightarrowtail N \rightarrowtail N \oplus L_0 \rightarrowtail \cdots \rightarrowtail N \oplus L_q}\right),$$

where we choose the quotients in the obvious natural way, namely,

$$\frac{N \oplus L_i}{N \oplus L_j} := \frac{L_i}{L_j} \quad \text{and} \quad \frac{N \oplus L_i}{N} := L_i.$$

We see that $g^* \circ H = 1_{G\mathcal{M}}$.

Notice that $f^* \circ H$: $G\mathcal{M} \to G\mathcal{M}$ is the map which adds N to every term in the second filtration; we claim that it is a homotopy equivalence. To see this, let G: $G\mathcal{M} \to G\mathcal{M}$ be the map which adds N to the every term in the first filtration. Then

$$F := G \circ (f^* \circ H) = (f^* \circ H) \circ G$$

is the map which adds N to both filtrations, as pictured here:

$$\begin{array}{c}
\left(\begin{array}{c} 0 \rightarrowtail K_0 \rightarrowtail \cdots \rightarrowtail K_q \\ \hline\hline 0 \rightarrowtail L_0 \rightarrowtail \cdots \rightarrowtail L_q \end{array}\right) \\
\Big\downarrow{\scriptstyle F} \\
\left(\begin{array}{c} 0 \rightarrowtail N \oplus K_0 \rightarrowtail \cdots \rightarrowtail N \oplus K_q \\ \hline\hline 0 \rightarrowtail N \oplus L_0 \rightarrowtail \cdots \rightarrowtail N \oplus L_q \end{array}\right)
\end{array}$$

There is a simplicial homotopy from 1 to F, defined as a map $G\mathcal{M} \times [1] \to G\mathcal{M}$ as follows. Given a simplex $(\alpha, \beta)\colon [q] \to G\mathcal{M} \times [1]$, we choose i, $-1 \le i \le q$, so that $\beta(0) = \cdots = [i) = 0$ and $\beta(i+1) = \cdots = \beta(q) = 1$. If

$$\alpha = \left(\begin{array}{c} 0 \rightarrowtail K_0 \rightarrowtail \cdots \rightarrowtail K_q \\ \hline\hline 0 \rightarrowtail L_0 \rightarrowtail \cdots \rightarrowtail L_q \end{array}\right)$$

then we define

$$h(\alpha, \beta) = \left(\begin{array}{c} 0 \rightarrowtail K_0 \rightarrowtail \cdots \rightarrowtail K_i \rightarrowtail N \oplus K_{i+1} \rightarrowtail \cdots \rightarrowtail N \oplus K_q \\ \hline\hline 0 \rightarrowtail L_0 \rightarrowtail \cdots \rightarrowtail L_i \rightarrowtail N \oplus L_{i+1} \rightarrowtail \cdots \rightarrowtail N \oplus L_q \end{array}\right),$$

using the natural and obvious choices for all the quotients which are needed to make a simplex of $G\mathcal{M}$. One checks that h is a simplicial map, and thus $f^* \circ H$ is a homotopy equivalence.

We claim it is enough now to show that $J := H \circ g^*$ is homotopic to the identity. For, assuming that done, since $g^* \circ H \sim 1$, we see that g^* and H are homctopy equivalences. Since $f^* \circ H$ is a homotopy equivalence, so is f^*, and we are done.

We set $Y = 0, M | S\mathcal{M}$. The map J is the map $Y \to Y$ described by

$$\begin{array}{c}
\left(\begin{array}{r} 0 \rightarrowtail K_0 \rightarrowtail \cdots \rightarrowtail K_q \\ \hline\hline 0 \rightarrowtail N \rightarrowtail L_0 \rightarrowtail \cdots \rightarrowtail L_q \end{array}\right) \\
\Big\downarrow{\scriptstyle J} \\
\left(\begin{array}{ccccc} & 0 \rightarrowtail & K_0 \rightarrowtail & \cdots \rightarrowtail & K_q \\ 0 \rightarrowtail & N \rightarrowtail & N \oplus \dfrac{L_0}{N} \rightarrowtail & \cdots \rightarrowtail & N \oplus \dfrac{L_q}{N} \end{array}\right).
\end{array}$$

For all pairs $L \twoheadleftarrow K \rightarrowtail M$ of admissible monomorphisms in $\mathscr{M}$ we choose a pushout object $L\amalg_K M$. When $K = 0$, we specify that $L\amalg_K M = L \oplus M$.

We make Y into an H-space using pushout over N; i.e., the addition map

$$Y \times Y \xrightarrow{+} Y$$

is described as follows:

$$\begin{pmatrix} & 0 \rightarrowtail K_0 \rightarrowtail \cdots \rightarrowtail K_q \\ & \overline{\overline{\qquad\qquad\qquad}} \\ 0 \rightarrowtail N \rightarrowtail & L_0 \rightarrowtail \cdots \rightarrowtail L_q \end{pmatrix} + \begin{pmatrix} & 0 \rightarrowtail K'_0 \rightarrowtail \cdots \rightarrowtail K'_q \\ & \overline{\overline{\qquad\qquad\qquad}} \\ 0 \rightarrowtail N \rightarrowtail & L'_0 \rightarrowtail \cdots \rightarrowtail L'_q \end{pmatrix}$$
$$= \begin{pmatrix} & 0 \rightarrowtail K_0 \oplus K'_0 \rightarrowtail \cdots \rightarrowtail K_q \oplus K'_q \\ & \overline{\overline{\qquad\qquad\qquad\qquad\qquad}} \\ 0 \rightarrowtail N \rightarrowtail & L_0\amalg_N L'_0 \rightarrowtail \cdots \rightarrowtail L_q\amalg_N L'_q \end{pmatrix}$$

The natural isomorphisms

$$\frac{L_i\amalg_N L'_i}{N} \cong \frac{L_i}{N} \oplus \frac{L'_i}{N} \quad \text{and} \quad \frac{L_i\amalg_N L'_i}{L_j\amalg_N L'_j} \cong \frac{L_i}{L_j} \oplus \frac{L'_i}{L'_j}$$

provide the choices and identifications required for that to make sense. There are natural transformations (coming from the associativity of direct sums and pushouts up to natural isomorphism) which show that Y is a homotopy associative and homotopy commutative H-space. The 0-simplex

$$\begin{pmatrix} & 0 \rightarrowtail 0 \\ & \overline{\overline{\quad}} \\ 0 \rightarrowtail N \rightarrowtail & N \end{pmatrix}$$

serves as the additive identity.

The natural isomorphism

$$L_i\amalg_N L_i \cong L_i\amalg_N\left(N \oplus \frac{L_i}{N}\right)$$

yields an natural isomorphism $1 + 1 \cong 1 + J$ of maps from Y to Y, and thus a homotopy $1 + 1 \sim 1 + J$. We will be done if we can find a map -1: $|Y| \to |Y|$ and a homotopy $-1 + |1| \sim 0$, where 0 denotes the constant map to the additive identity. For then we may use associativity to get

$$|1| \sim -1 + |1| + |1| \sim -1 + |1| + |J| \sim |J|.$$

For this the following lemma is useful.

LEMMA 3.2. *Suppose X is a homotopy associative H-space such that the monoid $\pi_0 X$ is actually a group. Then there is a map $-1\colon X \to X$ and a homotopy* $1 + (-1) \sim 0$.

Proof. This was proved in [2] when X is connected. Let X_0 be the connected component of the additive identity $0 \in X$; it is a connected associative H-space, so we have a map -1 for it. Letting $-x$ denote $(-1)(x)$, we extend this map to all of X as follows.

For each $i \in \pi_0 X$ choose a point x_i in the component X_i. For all $x \in X_i$ we define $-x := x_{-i} + (-(x + x_{-i}))$. Now we have the following homotopies of maps $X_i \to X_{-i}$:

$$\begin{aligned} x + -x &= x + \left(x_{-i} + -(x + x_{-i})\right) \\ &\sim \left(x + x_{-i}\right) + -\left(x + x_{-i}\right) \sim 0. \end{aligned} \qquad \text{Q.E.D.}$$

Continue the proof of the theorem. The preceding lemma tells us it is enough to show that the monoid $\pi_0 Y$ is actually a group. Suppose then that

$$\begin{pmatrix} & 0 \rightarrowtail K_0 \\ & \| \\ 0 \rightarrowtail N \rightarrowtail & L_0 \end{pmatrix}$$

is a vertex of Y. We claim that its connected component is inverse to the connected component of

$$\begin{pmatrix} & 0 \rightarrowtail L_0/N \\ & \| \\ 0 \rightarrowtail N \rightarrowtail & N \oplus K_0 \end{pmatrix}.$$

The sum of these two vertices is isomorphic to

$$\begin{pmatrix} & 0 \rightarrowtail L_0/N \oplus K_0 \\ & \| \\ 0 \rightarrowtail N \rightarrowtail & L_0 \oplus K_0 \end{pmatrix}$$

and therefore is connected to it. The 1-simplex

$$\begin{pmatrix} & 0 \rightarrowtail 0 \rightarrowtail L_0/N \oplus K_0 \\ & \| \\ 0 \rightarrowtail N \rightarrowtail N \rightarrowtail & L_0 \oplus K_0 \end{pmatrix},$$

whose definition uses the isomorphism

$$\frac{L_0 \oplus K_0}{N} \cong \frac{L_0}{N} \oplus K_0,$$

connects that vertex to the identity.

This finishes the proof of Theorem 3.1. Q.E.D.

Remark. If we have an exact sequence $0 \to M \to N \to P \to 0$ in $\mathcal{M}$, then we are tempted to write $P \cong N/M$. Within a single exact category $\mathcal{M}$ one may choose a quotient N/M for every admissible monomorphism $M \rightarrowtail N$, but it is not possible to do this simultaneously for all exact categories in such a way that all exact functors will respect these choices. Nevertheless, the choice of cokernel N/M is unique up to a unique isomorphism. Thus it is possible to specify an isomorphism

$$N/M \xrightarrow{\sim} N'/M'$$

without having previously specified choices for N/M and N'/M'.

If we omit the choices of the quotients from the definition of $G\mathcal{M}$, we get a homotopy equivalent set $G'\mathcal{M}$. A q-simplex of $G'\mathcal{M}$ can be described as a pair

$$K_0 \rightarrowtail \cdots \rightarrowtail K_q, \quad L_0 \rightarrowtail \cdots \rightarrowtail L_q$$

of admissible filtrations in $\mathcal{M}$, together with isomorphisms $K_j/K_i \cong L_j/L_i$ for $i \le j$, such that whenever $i \le i' \le j' \ge j$, the square

$$\begin{array}{ccc} K_j/K_i & \xrightarrow{\sim} & L_j/L_i \\ \downarrow & & \downarrow \\ K_{j'}/K_{i'} & \xrightarrow{\sim} & L_{j'}/L_{i'} \end{array}$$

commutes. Both these concepts make sense without actually choosing the quotients. The relationship of $G'\mathcal{M}$ with $S\mathcal{M}$ can be described by saying that there are natural homotopy equivalences

$$|G'\mathcal{M}| \leftarrow |G\mathcal{M}| \to \Omega|S\mathcal{M}|.$$

A choice of quotients in $\mathcal{M}$ gives a homotopy inverse map $|G'\mathcal{M}| \to |G\mathcal{M}|$ which is unique up to a unique natural isomorphism, but this construction can not be made natural in the variable $\mathcal{M}$.

4. A simplification of another proof

For this section, we assume that $\mathcal{M}$ is an exact category in which all exact sequences split, and S is the category whose arrows are the isomorphisms of $\mathcal{M}$.

In [2] it was proved that $S^{-1}S$ is homotopy equivalent to $\Omega|Q\mathcal{M}|$, and here we sketch a simplification for that proof. It consists of removing the last paragraph on p. 222 of [2], and all of p. 223 and p. 227. On p. 228 we modify the proof of the first theorem as follows. We must show that the map

$$S^{-1}f\colon S^{-1}S \to S^{-1}E_C$$

is a homotopy equivalence. Here $S^{-1}f$ is given by

$$(P, A) \mapsto (P, 0 \to A \to A \oplus C \to C \to 0).$$

We define

$$S^{-1}i^*: S^{-1}E_C \to S^{-1}S$$

by

$$(P, 0 \to A \to B \to C \to 0) \mapsto (P, A).$$

Then $S^{-1}i^* \circ S^{-1}f = 1$, so it is enough to find a homotopy $1 \sim S^{-1}f \circ S^{-1}i^*$.

We make $S^{-1}E_C$ into an H-space using pullback, setting

$$\begin{aligned}(P, 0 \to A \to B \to C \to 0) &+ (P', 0 \to A' \to B' \to C' \to 0)\\ &= (P \oplus P', 0 \to A \oplus A' \to B \times_C B' \to C \to 0).\end{aligned}$$

The natural isomorphism $B \times_C B \cong B \times_C (C \oplus A)$ gives a homotopy

$$1 + 1 \sim 1 + (S^{-1}f \circ S^{-1}i^*).$$

As in Section 3, it is enough to show that an additive inverse map

$$-1: S^{-1}E_C \to S^{-1}E_C$$

exists, and by 3.2 it suffices to show that the monoid $\pi_0 S^{-1}E_C$ is a group. We claim that the connected component of

$$(P, 0 \to A \to B \to C \to 0)$$

is inverse to the connected component of

$$(A, 0 \to P \to P \oplus C \to C \to 0).$$

Adding these two objects gives

$$(P \oplus A, 0 \to A \oplus P \to B \oplus P \to C \to 0),$$

which is isomorphic to

$$(A \oplus P, 0 \to A \oplus P \to A \oplus P \oplus C \to C \to 0),$$

and therefore connected by an arrow to the additive identity

$$(0, 0 \to 0 \to C \to C \to 0).$$

Notice that even though we are switching $A \oplus P$ to $P \oplus A$, we don't run into Thomason's phony multiplication problem. That problem was a failure of naturality, but here we are just describing objects and arrows for our π_0 computation, and not trying to make a functor.

5. Generators for $K_1\mathcal{M}$

Suppose $z \in K_1\mathcal{M}$ $(=\pi_1|G\mathcal{M}|)$. By means of the simplicial approximation theorem we can prove that z is represented by a loop formed combinatorially from 1-simplices of $G\mathcal{M}$. Drawing 1-simplices as arrows, we may represent z by a loop like

$$(0,0) \to \cdot \leftarrow \cdot \to \cdots \leftarrow \cdot \to \cdot \leftarrow (0,0).$$

Consider one of the configurations

$$\begin{array}{ccc} (K,L) & \to & (K'',L'') \\ \downarrow & & \\ (K',L') & & \end{array}$$

in the loop. Form the pushouts $P = K' \amalg_K K''$ and $Q = L' \amalg_L L''$. Since

$$P/K = K'/K \oplus K''/K \quad \text{and} \quad Q/L = L'/L \oplus L''/L,$$

we have isomorphisms of exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & K'/K & \to & P/K & \to & P/K' & \to & 0 \\ & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\ 0 & \to & L'/L & \to & Q/L & \to & Q/L' & \to & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \to & K''/K & \to & P/K & \to & P/K'' & \to & 0 \\ & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\ 0 & \to & L''/L & \to & Q/L & \to & Q/L'' & \to & 0 \end{array}$$

Thus we have two 2-simplices in $G\mathcal{M}$,

$$\left(\begin{array}{c} 0 \rightarrowtail K \rightarrowtail K' \rightarrowtail P \\ \overline{\overline{}} \\ 0 \rightarrowtail L \rightarrowtail L' \rightarrowtail Q \end{array} \right)$$

and

$$\left(\begin{array}{c} 0 \rightarrowtail K \rightarrowtail K'' \rightarrowtail P \\ \overline{\overline{}} \\ 0 \rightarrowtail L \rightarrowtail L'' \rightarrowtail Q \end{array} \right),$$

which fill in the triangles in the diagram

$$\begin{tikzcd} (K,L) \arrow[r] \arrow[d] \arrow[dr] & (K'',L'') \arrow[d] \\ (K',L') \arrow[r] & (P,Q) \end{tikzcd} .$$

Using many simplices like these, we deform the path so it looks as follows:

$$\begin{tikzcd} (0,0) \arrow[r] \arrow[d] & (K_1,L_1) \arrow[r] & \cdots \arrow[r] & (K_{q-1},L_{q-1}) \arrow[d] \\ (K_1',L_1') \arrow[r] & \cdots \arrow[r] & (K_{q-1}',L_{q-1}') \arrow[r] & (K,L) \end{tikzcd} .$$

Such a loop is described completely by the following data:

(1) objects K and L in $\mathscr{M}$;

(2) admissible filtrations

$$0 = K_0 \rightarrowtail K_1 \rightarrowtail \cdots \rightarrowtail K_q = K,$$
$$0 = K_0' \rightarrowtail K_1' \rightarrowtail \cdots \rightarrowtail K_q' = K,$$
$$0 = L_0 \rightarrowtail L_1 \rightarrowtail \cdots \rightarrowtail L_q = L,$$
$$0 = L_0' \rightarrowtail L_1' \rightarrowtail \cdots \rightarrowtail L_q' = L;$$

(3) isomorphisms $K_i/K_{i-1} \cong L_i/L_{i-1}$ and $K_i'/K_{i-1}' \cong L_i'/L_{i-1}'$.

Our result is that every element of $K_1\mathscr{M}$ can be represented by such data. The case $q = 1$ corresponds to the usual element of $K_1\mathscr{M}$ associated to an automorphism of an object of $\mathscr{M}$.

Notice that such data give two proofs that $[K] = [L]$ in $K_0\mathscr{M}$.

6. Products in K-theory

Let $\mathscr{M}$ be an exact category. For each $A \in \Delta$ define a category

$$\Gamma(A) := \operatorname{Ar}\varinjlim \begin{pmatrix} A \to [0]A \\ \downarrow \\ [0]A \end{pmatrix}.$$

We get thus a functor Γ: $\Delta \to$ (categories). Say that a functor $\Gamma(A) \to \mathscr{M}$ is *exact* if its restriction to each copy of $\operatorname{Ar}([0]A)$ in $\Gamma(A)$ is exact. With this terminology we see that $G\mathscr{M}(A)$ can be identified with the set $\operatorname{Exact}(\Gamma(A), \mathscr{M})$ of exact functors $\Gamma(A) \to \mathscr{M}$, and the identification is natural in A. Define *Exact*$(\Gamma(A), \mathscr{M})$ (or $\mathscr{G}\mathscr{M}(A)$ for short) to be the full subcategory of the category of functors $\Gamma(A) \to \mathscr{M}$ whose objects are the exact functors. Define a sequence $0 \to M' \to M \to M'' \to 0$ in $\mathscr{G}\mathscr{M}(A)$ to be *exact* if evaluation on each object of $\Gamma(A)$ yields an exact sequence of $\mathscr{M}$; this makes $\mathscr{G}\mathscr{M}(A)$ into an exact category (as can be checked). Thus we get a simplicial exact category $\mathscr{G}\mathscr{M}$. The definition $(G\mathscr{G}\mathscr{M})(A, B) = (G(\mathscr{G}\mathscr{M}(B))(A)$ (for $A, B \in \Delta$) yields a bisimplicial set $G\mathscr{G}\mathscr{M}$. We define $\mathscr{G}G\mathscr{M}(A, B) = G\mathscr{G}\mathscr{M}(B, A)$.

LEMMA 6.1. *The bisimplicial sets $\mathscr{G}G\mathscr{M}$ and $G\mathscr{G}\mathscr{M}$ are isomorphic.*

Proof.

$$\begin{aligned} \mathscr{G}G\mathscr{M}(A, B) &= \mathrm{Exact}(\Gamma(B), \mathit{Exact}(\Gamma(A), \mathscr{M})) \\ &= \mathrm{Biexact}(\Gamma(B) \times \Gamma(A), \mathscr{M}) \\ &\cong \mathrm{Biexact}(\Gamma(A) \times \Gamma(B), \mathscr{M}) \\ &= G\mathscr{G}\mathscr{M}(A, B). \end{aligned}$$

Here biexact functors are those which are exact in each variable separately. Q.E.D.

Using the lemma as justification, we define $GG\mathscr{M} = G\mathscr{G}\mathscr{M} \cong \mathscr{G}G\mathscr{M}$.

LEMMA 6.2. *Suppose X is a bisimplicial set, and let $X_{n,*}$ denote the simplicial set $[m] \mapsto X_{n,m}$. Assume that $[n] \mapsto X_{n,*}$ is given a product operation in each degree which makes it into a simplicial H-space such that $\pi_0 X_{n,*}$ is a group for all n. Then there is a spectral sequence*

$$E^2_{pq} = \pi_p([n] \mapsto \pi_q(X_{n,*})) \Rightarrow \pi_{p+q}|X|.$$

(All the terms are groups, all the maps are group homeomorphisms, and convergence is as usual.)

Proof. We use theorem B.5 and its proof in [1]. To see that X satisfies the π_*-Kan condition we use [1, B.3.1] and the fact that

$$\pi^v_t(X)_{\mathrm{free}} = \pi^v_t(X) \times \pi^v_0(X). \qquad \text{Q.E.D.}$$

We have an exact functor $\mathscr{M} \to \mathscr{G}_0\mathscr{M} \cong \mathscr{M} \times \mathscr{M}$ defined by $M \mapsto (M, 0)$. Writing $\mathscr{M}c$ for the constant simplicial exact category $B \mapsto \mathscr{M}$, we get a map $\mathscr{M}c \to \mathscr{G}\mathscr{M}$. Now applying G in each degree gives a map

$$G(\mathscr{M}c) = (G\mathscr{M})L \xrightarrow{f} GG\mathscr{M};$$

there is a similar map

$$(G\mathcal{M})R \xrightarrow{g} GG\mathcal{M}.$$

LEMMA 6.3. *The maps f and g are homotopy equivalences.*

Proof. It is enough to show this for f. We apply the previous lemma to the map $X = G(\mathcal{M}c) \to Y = GG\mathcal{M}$, obtaining

$$E^2_{pq}(X) = \begin{cases} K_q\mathcal{M}, & p = 0, \\ 0, & p \neq 0. \end{cases}$$

The exact functor $\mathcal{G}_n\mathcal{M} \to \mathcal{M}^{n+2}$ defined by

$$M \mapsto (M(0/0'), M(0/0''), M(1/0), \ldots, M(n/n-1))$$

induces an isomorphism on K-groups by the additivity theorem, showing that $\pi_q(Y_{*,n}) = K_q\mathcal{M}^{n+2}$. The simplicial set $[n] \mapsto \pi_q(Y_{*,n})$ is thus the groupoid $K_q\mathcal{M} \times K_q\mathcal{M}//K_q\mathcal{M}$, whose objects are pairs $(\alpha, \beta) \in K_q\mathcal{M} \times K_q\mathcal{M}$, and whose arrows are triples (α, β, δ): $(\alpha, \beta) \to (\alpha + \delta, \beta + \delta)$, with $\delta \in K_q\mathcal{M}$. Thus

$$E^2_{pq}(Y) = \begin{cases} K_q\mathcal{M} \times K_q\mathcal{M}/K_q\mathcal{M}, & p = 0, \\ 0, & p \neq 0, \end{cases}$$

and the map $E^2_{pq}(X) \to E^2_{pq}(Y)$ is the evident isomorphism induced by the first inclusion. Thus f induces an isomorphism on the abutments also, whence the result. Q.E.D.

Now using f allows us to identify $\pi_i GG\mathcal{M}$ with $K_i\mathcal{M}$. The proof above shows that the use of g instead would change the sign of the identification. It is necessary to choose f, however, because we want to identify $\pi_0 G\mathcal{M}$ with $K_0\mathcal{M}$ via the map $(M, N) \mapsto [M]$–$[N]$, and we know already how products are to be defined on K_0.

We can now define the product on K-theory using the G-construction. Suppose that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are exact categories and ψ: $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is a biexact functor. The pairing

$$G_p\mathcal{A} \times G_q\mathcal{A} = \mathrm{Exact}(\Gamma_p, \mathcal{A}) \times \mathrm{Exact}(\Gamma_q, \mathcal{B}) \to \mathrm{Exact}(\Gamma_p \times \Gamma_q, \mathcal{C}) = G_pG_q\mathcal{C},$$
$$(M, N) \mapsto \psi \circ (M \times N).$$

gives a map $G\psi$: $|G\mathcal{A}| \wedge |G\mathcal{B}| \to |G\mathcal{C}|$ and thus a pairing $K_i\mathcal{A} \otimes K_j\mathcal{B} \to K_{i+j}\mathcal{C}$.

Note that on 0-skeleta, this pairing may be described as follows. A vertex of $G\mathscr{A}$ is a pair (A_1, A_2) of objects in $\mathscr{A}$, a vertex of $\mathscr{B}$ is a pair (B_1, B_2), and a vertex of $GG\mathscr{C}$ is a quadruple

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

With this notation,

$$G\psi((A_1, A_2),(B_1, B_2)) = \begin{pmatrix} \psi(A_1, B_1) & \psi(A_1, B_2) \\ \psi(A_2, B_1) & \psi(A_2, B_2) \end{pmatrix}.$$

This is analogous to the fact that the tensor product of two complexes (here of length one) is a double complex.

It is an easy exercise to see that this pairing is compatible with the pairing $|Q\mathscr{A}| \wedge |Q\mathscr{B}| \to |QQ\mathscr{C}|$ described by Waldhausen [7].

To check associativity and commutatitivity is a much easier exercise now than it was in [4, appendix].

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