ILLINOIS JOURNAL OF MATHEMATICS
Volume 38, Number 4, Winter 1994

MIXED SIEGEL MODULAR FORMS AND KUGA FIBER VARIETIES

MIN HO LEE

1. Introduction

Mixed automorphic forms were first introduced by Hunt and Meyer [1] in connection with holomorphic forms on elliptic surfaces. A generalization to mixed automorphic forms of higher weights was treated in [5] (see also [7], [8]).

Let E be an elliptic surface and let $\pi: E \to X$ be an elliptic fibration in the sense of Kodaira (cf. [2]). Thus E is a compact smooth surface over $\mathbf{C}$, X is a compact Riemann surface, and the generic fiber of π is an elliptic curve. We assume that π has a global section and that there are no exceptional curves of the first kind in the fibers of π. Let E_0 be the union of the regular fibers of π and let $X_0 = \pi(E_0)$. We identify the universal covering space of X_0 with the Poincaré upper half plane $\mathscr{H}$, and the fundamental group $\pi_1(X_0)$ with a subgroup Γ of $PSL(2, \mathbf{R})$. Thus we have $X_0 = \Gamma \backslash \mathscr{H}$, where Γ acts on $\mathscr{H}$ by linear fractional transformations. Given a point $z \in X_0$, we choose a holomorphic 1-form on $E_z = \pi^{-1}(z)$ and a basis $\{\alpha_z, \beta_z\}$ of $H_1(E_z, \mathbf{Z})$ that depends on $z \in X_0$ in a continuous manner. Then the many-valued function

$$\omega(z) = \frac{\int_{\alpha_z} \Phi}{\int_{\beta_z} \Phi}$$

on X_0 can be lifted to a holomorphic function $\omega: \mathscr{H} \to \mathscr{H}$ satisfying $\omega(\gamma z) = \chi(\gamma)\omega(z)$ for all $\gamma \in \Gamma$ and $z \in \mathscr{H}$, where $\chi: \Gamma \to SL(2, \mathbf{R})$ is the monodromy representation of $\Gamma = \pi_1(X_0)$ for the elliptic fibration $\pi: E \to X$. Hunt and Meyer [1] defined the space of mixed cusp forms $S_{2,1}(\Gamma, \omega, \chi)$ using the automorphy factor

$$j(\gamma, z) = (cz + d)^2 (c_\chi \omega(z) + d_\chi),$$

Received August 27, 1992.
1991 Mathematics Subject Classification. Primary 11F46; Secondary 14K10.

where

$$\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma \quad \text{and} \quad \chi(\gamma) = \begin{pmatrix} * & * \\ c_\chi & d_\chi \end{pmatrix} \in SL(2, \mathbf{R}).$$

They proved that $S_{2,1}(\Gamma, \omega, \chi)$ is canonically isomorphic to the space $H^0(E, \Omega^2)$ of holomorphic 2-forms on E. In [5] the space $S_{2,n}(\Gamma, \omega, \chi)$ of mixed automorphic forms of type $(2, n)$ was defined using the automorphy factor

$$j(\gamma, z) = (cz + d)^2 (c_\chi \omega(z) + d_\chi)^n,$$

and it was proved that the space $S_{2,n}(\Gamma, \omega, \chi)$ is canonically isomorphic to the space $H^0(E^n, \Omega^{n+1})$ of holomorphic $(n + 1)$-forms on the elliptic variety E^n, where E^n is obtained by resolving the singularities of the compactification of the n-fold fiber product of E_0 over X_0.

Assuming that $\Gamma \subset SL(2, \mathbf{R})$ with $-1 \notin \Gamma$ and that χ is an inclusion $\Gamma \hookrightarrow SL(2, \mathbf{R})$, the above result of Hunt and Meyer was proved by Shioda [11] and the higher weight case was proved by Sŏkurov [12].

The purpose of this paper is to obtain a result similar to the ones described above in the case of Siegel modular forms. Let G be a semisimple Lie group whose quotient G/K by a maximal compact subgroup K is isomorphic to $\mathscr{H}^m$ and let Γ be a discrete subgroup of G such that the quotient $X = \Gamma \setminus \mathscr{H}^m$ has a structure of a complex manifold, where $\mathscr{H}^m$ is the Siegel upper half space of degree m. Then both G and $Sp(m, \mathbf{R})$ act on $\mathscr{H}^m$. Let ρ: $G \to Sp(m, \mathbf{R})$ be a homomorphism and let τ: $\mathscr{H}^m \to \mathscr{H}^m$ be a holomorphic map such that

$$\omega(gz) = \rho(g)\omega(z)$$

for all $g \in G$ and $z \in \mathscr{H}^m$. Then the equivariant pair (ρ, ω) defines a Kuga fiber variety π: $Y_\omega \to X$ over X whose fibers are complex tori. In this paper, we define mixed Siegel modular forms and show that the space of holomorphic forms of the highest degree on the fiber space Y_ω^n of the n-fold fiber product π^n: $Y_\omega^n \to X$ of the Kuga fiber variety π: $Y_\omega \to X$ is canonically embedded in the space $\mathscr{M}_{m+1,n}(\Gamma, \omega, \rho)$ of mixed Siegel modular forms of type $(m + 1, n)$ associated to Γ, ω and ρ.

I would like to thank the referee for various helpful suggestions.

2. Kuga fiber varieties

Let V be a vector space over $\mathbf{R}$ of dimension $2m$, and let β be an alternating bilinear form on V. We set

$$Sp(\beta) = \{g \in GL(V) | \beta(gx, gy) = \beta(x, y) \text{ for all } x, y \in V\}.$$

and let $\mathscr{H}(\beta)$ be the set of all complex structures J on V such that the bilinear form $\beta(x, Jy)$ on V is symmetric and positive definite. Then $Sp(\beta)$ acts on $\mathscr{H}(\beta)$ transitively by

$$(g, J) \mapsto gJg^{-1} \quad \text{for } g \in Sp(\beta), \quad J \in \mathscr{H}(\beta).$$

Let $\{e_1, \ldots, e_m, f_1, \ldots, f_m\}$ be a basis for V and let $\varphi\colon V \to \mathbf{R}^{2m}$ be the isomorphism such that

$$\varphi(e_i) = \varepsilon_i \quad \text{and} \quad \varphi(f_j) = \varepsilon_{m+j}$$

for $1 \leq i,\ j \leq m$, where $\{\varepsilon_i | 1 \leq i \leq 2m\}$ is the standard basis for $\mathbf{R}^{2m}$. We assume that the choice of the basis for V has been made in such a way that β corresponds to the $(2m \times 2m)$-matrix

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

i.e.,

$$\beta(e_i, e_j) = \beta(f_i, f_j) = 0,$$
$$\beta(e_i, f_j) = -\delta_{ij}$$

for $1 \leq i,\ j \leq m$, where δ_{ij} is the Kronecker delta. Then φ induces an isomorphism $\tilde{\varphi}$ of $\mathscr{H}(\beta)$ to the Siegel upper half space

$$\mathscr{H}^m = \left\{z \in M_m(\mathbf{C}) | z = {}^t z, \operatorname{Im} z \gg 0\right\}$$

(see [10, §II.8]).

Let G be a semisimple Lie group and let K be a maximal compact subgroup of G. We assume that the symmetric space $D = G/K$ has a G-invariant complex structure. Let $\rho\colon G \to Sp(\beta)$ be a homomorphism and let $\tau\colon D \to \mathscr{H}(\beta)$ be a holomorphic map such that

$$\tau(gz) = \rho(g)\tau(z)$$

for all $g \in G$ and $z \in D$. Then ρ determines the semidirect product $G \ltimes_\rho V$ in which the multiplication is given by

$$(g_1, v_1) \cdot (g_2, v_2) = (g_1 g_2, \rho(g_1) v_2 + v_1)$$

for all g_1, $g_2 \in G$ and v_1, $v_2 \in V$. The group $G \ltimes_\rho V$ acts on $D \times V$ by

$$(g, v) \cdot (x, w) = (gx, \rho(g) w + v)$$

for $(g, v) \in G \ltimes_\rho V$ and $(x, w) \in D \times V$.

Let $u(x) = (u_1(x), \ldots, u_k(x))$ be a global complex analytic coordinate system of the bounded symmetric domain D. Define the map $z\colon D \times V \to \mathbf{C}^m$ by

$$z(x,w) = (\tau'(x),1)E\varphi(w),$$

where $\tau' = \tilde{\varphi} \circ \tau$ and

$$E = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \in M_{2m}(\mathbf{R}).$$

This induces the map $\mu\colon D \times V \to \mathbf{C}^{k+m}$ given by

$$\mu(x,w) = (u(x), z(x,w)).$$

Thus μ is a diffeomorphism of $D \times V$ onto $u(D) \times \mathbf{C}^m$. If J is the natural complex structure on $u(D) \times \mathbf{C}^m$, then $\mathscr{T} = \mu^{-1}(J)$ defines a complex structure on $D \times V$ with global coordinates

$$u_1, \ldots, u_k, z_1, \ldots, z_m.$$

PROPOSITION 2.1. *The complex structure $\mathscr{T}$ on $D \times V$ is invariant under the action of $G \ltimes_\rho V$.*

Proof. We shall give the proof of this proposition for later purpose although it is essentially contained in [3]. If $(g,v) \in G \ltimes_\rho V$ and $(x,w) \in D \times V$, then we have

$$(g,v) \cdot (x,w) = (gx, \rho(g)w + v).$$

Since D is assumed to have a G-invariant complex structure, $u_1(gx), \ldots, u_k(gx)$ are holomorphic functions of $u_1, \ldots, u_k$, and similarly $u_1, \ldots, u_k$ are holomorphic functions of $u_1 \circ (g,v), \ldots, u_k \circ (g,v)$. On the other hand, we have

$$\begin{aligned} z((g,v)\cdot(x,w)) &= z(gx, \rho(g)w + v) \\ &= (\tau'(gx),1)E\varphi(\rho(g)w + v) \\ &= (\tau'(gx),1)E(\rho'(g)\varphi(w) + \varphi(v)), \end{aligned}$$

where ρ' is the symplectic representation of G on $\mathbf{R}^{2m}$ determined by ρ and φ. Hence we have

$$z((g,v)\cdot(x,w)) = (\tau'(gx),1)^t\rho'(g)^{-1}E\varphi(w) + (\tau'(gx),1)E\varphi(v).$$

If

$$\rho'(g) = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

then we have

$$\tau'(gx) = \rho'(g)\tau'(x) = (a'\tau'(x) + b')(c'\tau'(x) + d')^{-1}.$$

Since $\tau'(x)$ and $\tau'(gx)$ are symmetric and

$$\begin{pmatrix} \tau'(gx) \\ 1 \end{pmatrix} = \rho'(g)\begin{pmatrix} \tau'(x) \\ 1 \end{pmatrix}(c'\tau'(x) + d')^{-1},$$

we have

$$(\tau'(gx), 1) = {}^t(c'\tau'(x) + d')^{-1}(\tau'(x), 1)^t\rho'(g).$$

Hence we have

$$\begin{aligned}(\tau'(gx), 1)^t\rho'(g)^{-1}E\varphi(w) &= {}^t(c'\tau'(x) + d')^{-1}(\tau'(x), 1)E\varphi(w) \\ &= {}^t(c'\tau'(x) + d')^{-1}z(x, w).\end{aligned}$$

Thus it follows that

$$z((g, v) \cdot (x, w)) = {}^t(c'\tau'(x) + d')^{-1}z(x, w) + (\tau'(gx), 1)E\varphi(v).$$

Since τ' is holomorphic and the complex structure on D is G-invariant, the right hand side of the above relation is a holomorphic function of $(x, w) \in D \times V$. The same relation also indicates that $z_1, \ldots, z_m$ are holomorphic functions of $z_1 \circ (g, v), \ldots, z_m \circ (g, v)$. Thus it follows that

$$u_1 \circ (g, v), \ldots, u_k \circ (g, v), z_1 \circ (g, v), \ldots, z_m \circ (g, v)$$

are again global complex coordinates of $D \times V$. □

Let L be a lattice in V and let Γ be a torsion-free cocompact discrete subgroup of G such that $\rho(\Gamma)L \subset L$. Then the semidirect product $\Gamma \ltimes_\rho L$ operates on $D \times V$ properly discontinuously, and by proposition 2.1 the complex structure $\mathscr{T}$ on $D \times V$ determined by the holomorphic map τ: $D \to \mathscr{H}(\beta)$ induces a complex structure on the manifold $\Gamma \ltimes_\rho L \backslash D \times V$. We denote by Y_τ the complex manifold $\Gamma \ltimes_\rho L \backslash D \times V$ obtained this way. Then the projection map $D \times V \to D$ induces a fiber bundle π: $Y_\tau \to X$ known as a Kuga fiber variety over the complex manifold $X = \Gamma \backslash D$ whose fibers are complex tori of dimension m (see [3] and [10, Chapter 4] for details; see also [4, §1], [6]).

3. Mixed Siegel modular forms

In this section we define mixed Siegel modular forms which generalize usual Siegel modular forms. Let $\mathscr{H}^m$ be the Siegel upper half space of degree $m > 1$ on which the symplectic group $Sp(m, \mathbf{R})$ operates. We define the automorphy factor

$$j: Sp(m, \mathbf{R}) \times \mathscr{H}^m \to \mathbf{C}^\times$$

by

$$j(\sigma, z) = \det(cz + d),$$

where

$$z \in \mathscr{H}^m \quad \text{and} \quad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(m, \mathbf{R}).$$

Let G be a semisimple Lie group such that the quotient G/K of G by a maximal compact subgroup K is isomorphic to $\mathscr{H}^m$. Thus G can be written as a product $Sp(m, \mathbf{R}) \times G_c$ with G_c compact, and the groups Γ, G and $Sp(m, \mathbf{R})$ act on $\mathscr{H}^m$. We denote by p the natural projection of G onto $Sp(m, \mathbf{R})$. Let $\rho: G \to Sp(m, \mathbf{R})$ be a homomorphism and let $\omega: \mathscr{H}^m \to \mathscr{H}^m$ be a holomorphic map such that

$$\omega(\gamma z) = \rho(\gamma)\omega(z)$$

for all $z \in \mathscr{H}^m$ and $\gamma \in \Gamma$.

Definition 3.1. A holomorphic map $\psi: \mathscr{H}^m \to \mathbf{C}$ is a *mixed Siegel modular form* of type (k, l) associated to Γ, ω and ρ if

$$\psi(\gamma z) = j(p(\gamma), z)^k j(\rho(\gamma), \omega(z))^l \psi(z)$$

for all $\gamma \in \Gamma$ and $z \in \mathscr{H}^m$.

We shall denote by $\mathscr{M}_{k,l}(\Gamma, \omega, \rho)$ the space of mixed Siegel modular forms of type (k, l) associated to Γ, ω and ρ.

4. Kuga fiber varieties determined by subgroups of symplectic groups

In this section, as in §3, G is a semisimple Lie group whose quotient G/K by a maximal compact subgroup K is isomorphic to $\mathscr{H}^m$. Thus G can be written as a product $G = Sp(m, \mathbf{R}) \times G_c$ with G_c compact, and both G and $Sp(m, \mathbf{R})$ act on the Siegel upper half space $\mathscr{H}^m$. Let $\rho: G \to Sp(m, \mathbf{R})$ be a homomorphism and let $\omega: \mathscr{H}^m \to \mathscr{H}^m$ be a holomorphic map such that $\omega(gz) = \rho(g)\omega(z)$ for all $z \in \mathscr{H}^m$ and $g \in G$. If L is a lattice in $\mathbf{R}^{2m}$ and if Γ is a torsion-free cocompact discrete subgroup of G with $\rho(\Gamma)L \subset L$, then, as described in §2, ρ and ω determine a Kuga fiber variety $\pi: Y_\omega \to X$ over the complex manifold X whose fibers are complex tori of dimension m.

THEOREM 4.1. *If $k = m(m+1)/2 = \dim \mathscr{H}^m$, then the space $H^0(Y_\omega, \Omega^{k+m})$ of holomorphic $(k+m)$-forms on the Kuga fiber variety Y_ω is canonically embedded in the space $\mathscr{M}_{m+1,1}(\Gamma, \omega, \rho)$ of mixed Siegel modular forms of type $(m+1, 1)$ associated to Γ, ω and ρ.*

Proof. Let $\psi \in H^0(Y_\omega, \Omega^{k+m})$ be a holomorphic $(k+m)$-form on Y_ω. Then ψ is a holomorphic $(k+m)$-form on $\mathscr{H}^m \times \mathbf{C}^m$ that is invariant under the action of $\Gamma \ltimes_\rho L$, where L is a lattice in $\mathbf{C}^m$. Thus ψ can be written in the form

$$\psi = f(u,z)\, du_1 \wedge \cdots \wedge du_k \wedge dz_1 \wedge \cdots \wedge dz_m,$$

where $u_1, \dots, u_k, z_1, \dots, z_m$ are the global coordinates of $\mathscr{H}^m \times \mathbf{C}^m$ constructed in §2. Given $x \in X$, ψ descends to a holomorphic m-form on the fiber $Y_{\omega,x}$ over x. The fiber $Y_{\omega,x}$ is a complex torus of dimension m, and hence the dimension of the space of holomorphic m-forms on $Y_{\omega,x}$ is one. Since any holomorphic function on a compact complex manifold is constant, the restriction of $f(u,z)$ to the complex torus $Y_{\omega,x}$ is constant. Thus $f(u,z)$ depends only on u; and hence ψ can be written in the form

$$\psi = f(u)\, du_1 \wedge \cdots \wedge du_k \wedge dz_1 \wedge \cdots \wedge dz_m,$$

where f is a holomorphic function on $\mathscr{H}^m$. To consider the invariance of ψ under the group $\Gamma \ltimes_\rho L$, we set

$$du = du_1 \wedge \cdots \wedge du_k \quad \text{and} \quad dz = dz_1 \wedge \cdots \wedge dz_m,$$

and let $(\gamma, v) \in \Gamma \ltimes_\rho L$. Then we have

$$du \circ (\gamma, v) = j(p(\gamma), u)^{-(m+1)}\, du,$$

where p is the natural projection of G onto $Sp(m, \mathbf{R})$ (see e.g., [9, §1.6]). On the other hand, as in the proof of Proposition 2.1, we have

$$dz \circ (\gamma, v) = d\left[{}^t(c_\rho \omega(u) + d_\rho)^{-1} z + (\omega(\gamma u), 1) E v\right],$$

where

$$E = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \in M_{2m}(\mathbf{R}) \quad \text{and} \quad \rho(\gamma) = \begin{pmatrix} * & * \\ c_\rho & d_\rho \end{pmatrix};$$

here ω plays the role of τ' in the proof of Proposition 2.1. Hence we obtain

$$\begin{aligned} \psi = \psi \circ (\gamma, v) &= f(\gamma u) j(p(\gamma), v)^{-(m+1)} \det\left[{}^t(c_\rho \omega(u) + d_\rho)^{-1} \right] du \wedge dz \\ &= f(\gamma u) j(p(\gamma), u)^{-(m+1)} j(\rho(\gamma), \omega(u))^{-1} \, du \wedge dz. \end{aligned}$$

Thus we have

$$f(\gamma u) = j(p(\gamma), u)^{m+1} j(\rho(\gamma), \omega(u)) f(u),$$

and it follows that

$$f(u) \in \mathscr{M}_{m+1,1}(\Gamma, \omega, \rho).$$

Therefore the assignment $\psi \mapsto f(u)$ determines a canonical embedding of $H^0(Y_\omega, \Omega^{k+m})$ in $\mathscr{M}_{m+1,1}(\Gamma, \omega, \rho)$. □

Now we consider fiber products of Kuga fiber varieties. Let $\pi: Y_\omega \mapsto X$ be the Kuga fiber variety constructed above. For each positive integer n, we consider the n-fold fiber product

$$Y_\omega \times_\pi Y_\omega \times_\pi \cdots \times_\pi Y_\omega$$

of Y_ω over X. We shall use Y_ω^n to denote this fiber product and $\pi^n: Y_\omega^n \to X$ to denote the fibration induced by π.

THEOREM 4.2. *The space $H^0(Y_\omega^n, \Omega^{k+mn})$ of holomorphic $(k + mn)$-forms on Y_ω^n is canonically embedded in the space $\mathscr{M}_{m+1,n}(\Gamma, \omega, \rho)$ of mixed Siegel modular forms of type $(m + 1, n)$ associated to Γ, ω and ρ.*

Proof. If ψ is a holomorphic $(k + mn)$-form on Y_ω^n, then ψ can be considered as a holomorphic $(k + mn)$-form on $\mathscr{H}^m \times (\mathbf{C}^m)^n$ that is invariant under the action of $\Gamma \ltimes_\rho L$. Thus, as in the proof of Theorem 4.1, there is a holomorphic function $f(u)$ on $\mathscr{H}^m$ such that

$$\psi = f(u) dy \wedge dz^{(1)} \wedge \cdots \wedge dz^{(n)},$$

where $u = (u_1, \dots, u_k)$, $z^{(j)} = (z_1^{(j)}, \dots, z_n^{(j)})$, and $(u, z^{(j)})$ are the canonical coordinates of $\mathscr{H}^m \times \mathbf{C}^m$ for each j with $1 \le j \le n$ considered in §2. Using the results given in the proof of Theorem 4.1, for each $(\gamma, v) \in \Gamma \ltimes_\rho L$, we obtain

$$\psi \circ (\gamma, v) = f(\gamma u) j(p(\gamma), u)^{-(m+1)} j(\rho(\gamma), \omega(u))^{-n} \times du \wedge dz^{(1)} \wedge \cdots \wedge dz^{(n)}.$$

Thus we have

$$f(\gamma u) = j(p(\gamma), u)^{m+1} j(\rho(\gamma), \omega(u))^n f(u),$$

and it follows that

$$f(u) \in \mathscr{M}_{m+1,n}(\Gamma, \omega, \rho).$$

Therefore the assignment $\psi \mapsto f(u)$ determines a canonical embedding of $H^0(Y_\omega, \Omega^{k+mn})$ in $\mathscr{M}_{m+1,n}(\Gamma, \omega, \rho)$. □

References

1. B. Hunt and W. Meyer, *Mixed automorphic forms and invariants of elliptic surfaces*, Math. Ann. **271** (1985), 53–80.
2. K. Kodaira, *On compact analytic surfaces* II–III, Ann. Math. **77** (1963), 563–626; **78** (1963), 1–40.
3. M. Kuga, *Fiber varieties over a symmetric space whose fibers are abelian varieties I, II*, Lect. Notes, Univ. Chicago, 1963/64.
4. ______, *Invariants and Hodge cycles*, Advanced Studies in Pure Math. **15** (1989), 373–413.
5. M.H. Lee, *Mixed cusp forms and holomorphic forms on elliptic varieties*, Pacific. J. Math. **132** (1988), 363–370.
6. ______, *Conjugates of equivariant holomorphic maps of symmetric domains*, Pacific J. Math. **149** (1991), 127–144.
7. ______, *On the Kronecker pairing for mixed cusp forms*, Manuscripta Math. **71** (1991), 35–44.
8. ______, *Periods of mixed cusp forms*, Manuscripta Math. **73** (1991), 163–177.
9. A. Nenashev, *Siegel cusp modular forms and cohomology*, Math. USSR Izv. **29** (1987), 559–586.
10. I. Satake, *Algebraic structures of symmetric domains*, Princeton Univ. Press, Princeton, N.J., 1980.
11. T. Shioda, *On elliptic modular surfaces*, J. Math. Soc. Japan **24** (1972), 20–59.
12. V. Sŏkurov, *Holomorphic differential forms of higher degree on Kuga's modular varieties*, Math USSR Sb. **30** (1976), 119–142.

University of Northern Iowa
Cedar Falls, Iowa