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ASYMMETRIC TENT MAP EXPANSIONS II. PURELY PERIODIC POINTS

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1. Introduction

The family of asymmetric tent maps T_α: $[0,1] \to [0,1]$ for $\alpha > 1$ is

$$\mathsf{T}_\alpha(x) = \begin{cases} \alpha x & \text{for } 0 \le x \le 1/\alpha. \\ \dfrac{\alpha}{\alpha - 1}(1 - x) & \text{for } 1/\alpha \le x \le 1. \end{cases} \tag{1.1}$$

This family of mappings has been extensively studied as a simple family of one-dimensional dynamical systems, and as a one-dimensional lattice system in statistical mechanics. They give expansions of real numbers $x = [0,1]$, called T_α-*expansions*, which are analogous to the decimal expansion. These are:

$$x = \sum_{j \ge 0} (-1)^j \alpha^{-(n_0 + \cdots + n_j)} \beta^j, \tag{1.2}$$

where $\beta = \alpha/(\alpha - 1)$ and the nonnegative integers n_j are specified by the itinerary $I(x) = L^{n_0}RL^{n_1}RL^{n_2}\ldots$, which encodes the successive iterates $\mathsf{T}_\alpha^{(k)}(x)$ as being in the left interval $[0, 1/\alpha]$ (labelled L) or the half-open right interval $(1/\alpha, 1]$ (labelled R). For certain x the expansion (1.2) contains only finitely many R's, and the corresponding itinerary is then $I(x) = L^{n_0}R \cdots R^{n_j}RL^\infty$; these numbers x are exactly the preperiodic points of 0, denoted $\mathrm{Per}_0(\mathsf{T}_\alpha)$.

Part I studied the set $\mathrm{Per}(\mathsf{T}_\alpha)$ of the eventually periodic points of T_α and proved that for certain values of α, called special Pisot numbers, one has

$$\mathrm{Per}(\mathsf{T}_\alpha) = \mathbb{Q}(\alpha) \cap [0,1].$$

Special Pisot numbers are those real numbers α such that α and $\alpha/(\alpha - 1)$ are both Pisot numbers. (Recall that α is a *Pisot number* if $\alpha > 1$ is a real algebraic integer such that all algebraic conjugates $\sigma(\alpha)$ of α with $\sigma(\alpha) \ne \alpha$ satisfy $|\sigma(\alpha)| < 1$.) Part I showed that there exist only a finite number of

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Table 1. Special Pisot Numbers (α, β)

$\alpha = 2$ $X - 2$	$\beta = 2$ $X - 2$
$\alpha = 1.75487$ $X^3 - 2X^2 + X - 1$	$\beta = 2.32471^+$ $X^3 - 3X^2 + 2X - 1$
$\alpha = 1.61803^+$ $X^2 - X - 1$	$\beta = 2.61803^+$ $X^2 - 3X + 1$
$\alpha = 1.46557^+$ $X^3 - X^2 - 1$	$\beta = 3.14789^+$ $X^3 - 4X^2 + 3X - 1$
$\alpha = 1.38027^+$ $X^4 - X^3 - 1$	$\beta = 3.62965^+$ $X^4 - 5X^3 + 6X^2 - 4X + 1$
$\alpha = 1.32471^+$ $X^3 - X - 1$	$\beta = 4.07959^+$ $X^3 - 5X^2 + 4X - 1$

special Pisot numbers and exhibited eleven such numbers, which are listed in Table 1 below. Since α is a special Pisot number if and only if $\beta = \alpha/(\alpha - 1)$ is also, Table 1 lists only the numbers with $1 < \alpha \leq 2$, the remainder being given by the corresponding β's.

In this paper we study the sets

$$\begin{aligned} \mathrm{Fix}(\mathsf{T}_\alpha) &:= \{x : \mathsf{T}_\alpha^{(m)}(x) = x \text{ for some } m > 0\} \\ &= \{\text{purely periodic points}\}, \\ \mathrm{Per}_0(\mathsf{T}_\alpha) &= \{x : \mathsf{T}_\alpha^{(k)}(x) = 0 \text{ for some } k \geq 0\} \\ &= \{\text{points with terminating } \mathsf{T}_\alpha\text{-expansion}\}, \end{aligned}$$

when α is a special Pisot number.

In §2 we first show that for all special Pisot numbers

$$\begin{aligned} \mathrm{Fix}(\mathsf{T}_\alpha) \subseteq \{\gamma \in \mathbf{Q}(\alpha) : \gamma \in [0,1] \text{ and } \sigma(\gamma) \in \mathsf{A}_\alpha^\sigma \\ \text{for all embeddings } \sigma : \mathbf{Q}(\alpha) \to \mathbf{C} \text{ with } \sigma(\alpha) \neq \alpha\}, \end{aligned} \quad (1.3)$$

where each A_α^σ is a compact subset of $\mathbf{C}$ which is the attractor of a certain hyperbolic iterated function system (Theorem 2.1). Hyperbolic iterated function systems are defined in Barnsley and Demko (1985) and Barnsley (1988); see also §2. We examine when equality holds. We show for $\alpha = 2$ that

$$\mathrm{Fix}(\mathsf{T}_2) = \{p/q : 2 \nmid q \text{ and } 2 | p \text{ and } 0 \leq p < q\};$$

in this case (1.3) is a strict inclusion, since its right side is $\mathbf{Q} \cap [0, 1]$. However when α is a special Pisot number generating either a real quadratic field or a non-totally-real cubic field, we prove that equality holds in (1.3). In particular, for $\alpha = (1+\sqrt{5})/2$, we have

$$\mathrm{Fix}(\mathrm{T}_\alpha) = \left\{\gamma \in \mathbf{Q}(\sqrt{5}) : 0 \le \gamma \le 1 \text{ and } \frac{1-\sqrt{5}}{4} \le \bar{\gamma} \le \frac{1}{2}\right\}, \quad (1.4)$$

where $\bar{\gamma}$ denotes the algebraic conjugate of γ; a similar characterization applies to $\alpha = (3+\sqrt{5})/2$. We are unable to describe precisely the attractors A_α^σ when α generates a non-totally real field; we conjecture however that each such attractor A_α^σ is the closure of a bounded open set in $\mathbf{C}$ having a "fractal" boundary. For the remaining two special Pisot numbers of degree 4 in Table 1 we do not know whether equality holds in (1.3) or not.

In §3 we study $\mathrm{Per}_0(\mathrm{T}_\alpha)$ and related sets. We first show that $\mathrm{Per}_0(\mathrm{T}_2)$ consists of the dyadic rationals in $[0,1]$. For the remaining special Pisot numbers with $\mathbf{Q}(\alpha) \neq \mathbf{Q}$, we consider the set of algebraic integer fixpoints

$$I_\alpha := \{\gamma : \gamma \in \mathrm{Fix}(\mathrm{T}_\alpha) \cap O_K\},$$

where O_K is the ring of integers of $K = \mathbf{Q}(\alpha)$, and prove that I_α is finite. We define

$$\mathrm{Per}^*(\mathrm{T}_\alpha) = \bigcup \{\mathrm{Per}_\gamma(\mathrm{T}_\alpha) : \gamma \in I_\alpha\}$$

where $\mathrm{Per}_\gamma(\mathrm{T}_\alpha)$ denotes the preperiodic points of γ, and prove that

$$\mathrm{Per}^*(\mathrm{T}_\alpha) = O_K \cap [0,1].$$

In particular $\mathrm{Per}^*(\mathrm{T}_\alpha)$ is always closed under multiplication and under addition (mod 1). Now I_α always contains 0 and in some cases $I_\alpha = \{0\}$, and then $\mathrm{Per}_0(\mathrm{T}_\alpha)$ inherits this ring structure. This occurs for $\alpha = (1+\sqrt{5})/2$. Other special Pisot numbers have larger sets, e.g., $\alpha = (3+\sqrt{5})/2$ has $I_\alpha = \{0, (-1+\sqrt{5})/2\}$.

For comparison we mention some related results in the literature. First, the characterization (1.4) for $\mathrm{Fix}(\mathrm{T}_\alpha)$ for $\alpha = (1+\sqrt{5})/2$ is analogous to that for real numbers whose continued fraction expansion is purely periodic. Second, K. Schmidt (1980) studied β-expansion maps $T_\beta^*(x) = \beta x \pmod 1$ and observed that $\mathrm{Per}_0(\mathrm{T}_\beta^*)$ was closed under multiplication and addition (mod 1) for certain special values of β. Solomyak (1991) recently showed that $\mathrm{Per}_0(\mathrm{T}_\beta^*) = \mathbf{Z}[1/\beta] \cap [0,1]$ for a certain class of Pisot numbers. Third, Moussa, Geronimo and Bessis (1984) characterize $\mathrm{Per}(T)$ for monic polynomials $T(X) \in \mathbf{Z}[X]$ acting on $\mathbf{C}$ as being those algebraic integers such that they and all of their algebraic conjugates lie in the Julia set of T. Moussa (1986) extends this result further to polynomials T with algebraic coefficients. Compare this with Theorem 2.1 below.

There remain a number of open questions, including the following.

(1) Obtain the complete list of all special Pisot numbers.

(2) For special Pisot numbers both $\mathrm{Per}(\mathsf{T}_\alpha)$ and $\mathrm{Per}^*(\mathsf{T}_\alpha)$ are closed under multiplication and under addition (mod 1). If we encode $\gamma \in \mathrm{Per}(\mathsf{T}_\alpha)$ (resp. $\mathrm{Per}^*(\mathsf{T}_\alpha)$) using binary sequences for the itinerary in the form (preperiod, period), do these addition and multiplication laws have any interesting structure?

(3) Theorem 2.1 and 2.2 show that for all special Pisot numbers the denominator of points $\gamma \in \mathrm{Fix}(\mathsf{T}_\alpha)$ go to infinity as the period length $p \to \infty$. What can one say about the distribution of period lengths among $\gamma \in \mathrm{Fix}(\mathsf{T}_\alpha)$ of denominator B? Equivalently, for γ having period p, bound the denominator from above and below.

(4) Do there exist any α which are not special Pisot numbers, for which $\mathrm{Fix}(\mathsf{T}_\alpha) = \mathscr{F}_\alpha$, where $\mathscr{F}_\alpha$ is defined in Theorem 2.1 below?

2. Purely periodic points

Associated to the mapping T_α are the two affine maps on $\mathbf{R}$:

$$L_\alpha(x) = \alpha x, \tag{2.1a}$$

$$R_\alpha(x) = \frac{\alpha}{\alpha - 1}(1 - x). \tag{2.1b}$$

Suppose that α is an algebraic number. Then for each embedding σ: $\mathbf{Q}(\alpha) \to \mathbf{C}$ such that $\sigma(\alpha) \neq \alpha$ we have affine maps on $\mathbf{C}$:

$$L_\alpha^\sigma(x) = \sigma(\alpha)x \tag{2.2a}$$

$$R_\alpha^\sigma(x) = \frac{\sigma(\alpha)}{\sigma(\alpha) - 1}(1 - x). \tag{2.2b}$$

Since embeddings preserve field operations, if $\{S_i\}$ denotes any sequence of the operators L_α, R_α and $\{S_i^\sigma\}$ the corresponding sequence of operators L_α^σ, R_α^σ, then for any $x \in \mathbf{Q}(\alpha)$,

$$y = S_1 S_2 \cdots S_k(x)$$

implies that

$$\sigma(y) = S_1^\sigma S_2^\sigma \cdots S_k^\sigma(\sigma(x)). \tag{2.3}$$

For any set of affine maps $\mathscr{S} = \{L_1, \ldots, L_k\}$ on $\mathbf{R}^n$ define the set $\overline{\mathrm{Fix}(\mathscr{S})}$ to be the closure of the set of fixed points of all finite products of the members of $\mathscr{S}$. In general the set $\overline{\mathrm{Fix}(\mathscr{S})}$ is unbounded. However if the mappings L_i

are all strict contractions, i.e., if

$$\| L_i(\mathbf{x}) - L_i(\mathbf{y}) \| < \delta \|\mathbf{x} - \mathbf{y}\| \text{ for some } \delta < 1,$$

then $\overline{\mathrm{Fix}(\mathscr{S})}$ is compact. In that case $\mathscr{S}$ is a *hyperbolic iterated function system* (hyperbolic IFS) in the sense of Barnsley (1988), and

$$\overline{\mathrm{Fix}(\mathscr{S})} = \mathsf{A}(\mathscr{S}),$$

where $\mathsf{A}(\mathscr{S})$ is the *attractor* of the hyperbolic IFS, which is characterized as the unique compact subset of $\mathbf{R}^n$ satisfying the functional equation

$$\mathsf{A} = \bigcup_{i=1}^{k} L_i(\mathsf{A}); \tag{2.4}$$

cf. Hutchinson (1981), Theorem 1. The name *attractor* refers to the property that for any $\varepsilon > 0$, the iterates of any point x_0 of $\mathbf{R}^n$ under any sequence of maps from $\mathscr{S}$ are all within distance ε of A from some point on.

THEOREM 2.1. *For any real algebraic number $\alpha > 1$,*

$$\mathrm{Fix}(\mathsf{T}_\alpha) \subseteq \mathscr{F}_\alpha := \{\gamma : \gamma \in \mathbf{Q}(\alpha) \cap [0,1] \text{ and } \sigma(\gamma) \in \overline{\mathrm{Fix}\{L_\alpha^\sigma, R_\alpha^\sigma\}} \text{ for all embeddings with } \sigma(\alpha) \neq \alpha\} \tag{2.5}$$

Proof. If $\gamma = \mathsf{T}_\alpha^{(k)}(\gamma)$ then γ is a fixed point of some $S_1 S_2 \cdots S_k$ drawn from $\{L_\alpha, R_\alpha\}$, and (2.3) then implies that $\sigma(\gamma)$ is a fixed point of $S_1^\sigma S_2^\sigma \cdots S_k^\sigma$ hence is in $\overline{\mathrm{Fix}\{L_\alpha^\sigma, R_\alpha^\sigma\}}$. ■

Special Pisot numbers are characterized by the properties:

(i) α and $\alpha/(\alpha - 1)$ are algebraic integers which are real and greater than one.

(ii) For each embedding σ with $\sigma(\alpha) \neq \alpha$, both R_α^σ and L_α^σ are contracting maps on $\mathbf{C}$; i.e.,

$$|\sigma(\alpha)| < 1 \text{ and } \left| \frac{\sigma(\alpha)}{\sigma(\alpha) - 1} \right| < 1.$$

A consequence of property (ii) is that $\{R_\alpha^\sigma, L_\alpha^\sigma\}$ forms a hyperbolic IFS for all embeddings σ: $\mathbf{Q}(\alpha) \to \mathbf{C}$ with $\sigma(\alpha) \neq \alpha$. In particular the attractor $\mathsf{A}_\alpha^\sigma := \overline{\mathrm{Fix}(\{L_\alpha^\sigma, R_\alpha^\sigma\})}$ is compact, consequently any purely periodic point γ of T_α has γ and all its conjugates bounded.

Note that while the fixed point sets satisfy

$$\mathrm{Fix}(\{L_\alpha^\sigma, R_\alpha^\sigma\}) = \sigma(\mathrm{Fix}\{L_\alpha, R_\alpha\})$$

holds for all embeddings σ: $\mathbf{Q}(\alpha) \to \mathbf{R}$, this property is not necessarily preserved for the closures of the fixed point sets; i.e.,

$$(\mathbf{Q}(\alpha)) \cap \overline{\mathrm{Fix}(\{L_\alpha^\sigma, R_\alpha^\sigma\}} \neq \sigma\left[\mathbf{Q}(\alpha) \cap \overline{\mathrm{Fix}\{L_\alpha, R_\alpha\}}\right]$$

may occur.

We remark that there exist infinitely many real algebraic numbers $\alpha > 1$ which are not algebraic integers, such that condition (ii) above holds and all the attractors A_α^σ with $\sigma(\alpha) \neq \alpha$ are compact.

Now we study $\mathrm{Fix}(\mathsf{T}_\alpha)$ for special Pisot numbers. The simplest case is $\alpha = 2$ and Theorem 2.1 gives $\mathrm{Fix}(\mathsf{T}_2) \subset \mathbf{Q} \cap [0, 1]$. In fact this inclusion is strict.

THEOREM 2.2. *For* $\alpha = 2$ *one has*

$$\mathrm{Fix}(\mathsf{T}_2) = \left\{\frac{p}{q} : 0 \le p < q \text{ and } 2|p, 2 \nmid q\right\}. \tag{2.6}$$

Proof. Set

$$\mathscr{G} = \left\{\frac{p}{q} : 0 \le p < q \text{ and } 2|P, 2 \nmid q\right\}.$$

Given $x \in \mathrm{Fix}(\mathsf{T}_2)$ write its fixed point equation as

$$x = L_2^{j_1} R_2 L_2^{j_2} R_2 \ldots L_2^{j_m}(x),$$

where all $j_i \ge 0$. Then

$$x = (-1)^{m-1} 2^{m+j_1+j_2+\cdots-j_m} x + \sum_{k=1}^{m-1} (-1)^{k-1} 2^{k+j_1+\cdots+j_k}$$

so that $x = p/q$ where

$$p = \sum_{k=1}^{m-1} (-1)^{m+k-1} 2^{k+j_1+\cdots+j_k},$$

$$q = 2^{m+j_1+\cdots+j_m} + (-1)^m.$$

It is easy to see that $2|p$. Also $2 \nmid q$ and $0 \le p < q$, since the terms in the sum defining p alternate in sign, strictly increase, the first term is ≥ 2, and the last is at most $q+1$. On reducing p/q to lowest terms the conditions $2|p$, $2 \nmid q$ are preserved. Hence $\mathrm{Fix}(\mathsf{T}_2) \subseteq \mathscr{G}$.

To show the other inclusion, let $y = p/q \in \mathscr{G}$ be given in lowest terms. Then $\mathsf{T}_2(y) = 2p/q$ or $2(q-p)/q \in \mathscr{G}$ has the same denominator, so y must be eventually periodic with period $f \le (q-1)/2$. Now we argue by contradiction. If $y \notin \mathrm{Fix}(T_2)$ then there would exist some $T_2^{(j)}(y) \notin \mathrm{Fix}(\mathsf{T}_2)$ with $y' = T_2^{(j+1)}(y) \in \mathrm{Fix}(\mathsf{T}_2)$. Then y' would have two distinct preimages in $\mathscr{G}$, namely $T_2^{(j)}(y)$ and $T_2^{(f-1)}(y') \in \mathrm{Fix}(\mathsf{T}_2) \subseteq \mathscr{G}$. Now we claim that any $z \in \mathscr{G}$ has exactly one preimage in $\mathscr{G}$, which will give a contradiction showing that $y \in \mathrm{Fix}(\mathsf{T}_2)$. To prove the claim, write $z' = 2p'/q' \in \mathscr{G}$, with $0 \le p'/q' < 1/2$ and observe that the preimages of z are p'/q' and $(q'-p')/q'$, and since q' is odd, exactly one of p' and $q'-p'$ is even. ∎

All remaining special Pisot numbers have $\mathbf{Q}(\alpha) \neq \mathbf{Q}$. We show that equality holds in (2.5) for many of them.

THEOREM 2.3. *Let α be a special Pisot number such that $\mathbf{Q}(\alpha)$ is a real quadratic field or a non-totally-real cubic field. Then*

$$\mathrm{Fix}(\mathsf{T}_\alpha) = \mathscr{F}_\alpha := \{\gamma : \gamma \in \mathbf{Q}(\alpha) \cap [0,1] \text{ and } \sigma(\gamma) \in \mathsf{A}_\alpha^\sigma \text{ for all embeddings with } \sigma(\alpha) \neq \alpha.\} \quad (2.7)$$

Proof. Let $\mathscr{F}_\alpha$ denote the right side of (2.7), and suppose that $\gamma \in \mathscr{F}_\alpha$. We show that $\gamma \in \mathrm{Fix}(\mathsf{T}_\alpha)$ by an argument similar to that of Theorem 2.2.

First, $\mathscr{F}_\alpha$ is closed under T_α. To see this, recall by (2.4) that

$$\mathsf{A}_\alpha^\sigma = L_\alpha^\sigma(\mathsf{A}_\alpha^\sigma) \cup R_\alpha^\sigma(\mathsf{A}_\alpha^\sigma)$$

hence if $\sigma(\gamma) \in \mathsf{A}_\alpha^\sigma$ then $\mathsf{T}_\alpha^\sigma(\sigma(\gamma)) \in \mathsf{A}_\alpha^\sigma$.

Second, we claim that each element $\gamma \in \mathscr{F}_\alpha$ has at least one T_α-preimage which is in $\mathscr{F}_\alpha$. Here we use the hypotheses on the field $\mathbf{Q}(\alpha)$. To prove the claim, observe first that for a fixed embedding σ with $\sigma(\alpha) \neq \alpha$, since

$$\mathsf{A}_\alpha^\sigma = L_\alpha^\sigma(\mathsf{A}_\alpha^\sigma) \cup R_\alpha^\sigma(\mathsf{A}_\alpha^\sigma) \quad (2.8)$$

and $\sigma(\gamma) \in \mathsf{A}_\alpha^\sigma$ there is some $\delta' \in \mathsf{A}_\alpha^\sigma$ with either $L_\alpha^\sigma(\delta') = \sigma(\gamma)$ or $R_\alpha^\sigma(\delta') = \sigma(\gamma)$. For definiteness suppose $L_\alpha^\sigma(\delta') = \sigma(\gamma)$. (The proof for $R_\alpha^\sigma(\delta') = \sigma(\gamma)$ is similar.) Now $\delta' \in \sigma(\mathbf{Q}(\alpha))$ so set $\delta' = \sigma(\delta)$ for some $\delta \in \mathbf{Q}(\alpha)$. Applying all automorphisms of $\mathbf{Q}(\alpha)$ to this linear equation gives

$$L_\alpha^\sigma(\tau(\delta)) = \tau(\gamma) \text{ for } \textit{all} \text{ embeddings } \tau. \quad (2.9)$$

To show $\delta \in \mathscr{F}_\alpha$ we must check that $\tau(\delta) \in \mathsf{A}_\alpha^\tau$ for all τ. This always holds if τ is the identity, for *both* R_α^{-1} and L_α^{-1} map $[0,1]$ into $[0,1]$, hence

$\delta = L_\alpha^{-1}(\gamma) \in [0,1]$. Thus, if $\mathbf{Q}(\alpha)$ is a real quadratic field, $\delta \in \mathscr{F}_\alpha$. If $\mathbf{Q}(\alpha)$ is a non-totally real cubic field, then the two embeddings with $\sigma(\alpha) \neq \alpha$ are complex conjugates, call them σ and $\bar{\sigma}$, and applying complex conjugation to (2.8) shows that the attractor $\mathsf{A}_\alpha^{\bar{\sigma}}$ is the complex-conjugate of the attractor A_α^σ. Hence

$$\sigma(\delta) \in \mathsf{A}_\alpha^\sigma \Leftrightarrow \bar{\sigma}(\delta) \in \mathsf{A}_\alpha^{\bar{\sigma}}.$$

This shows that $\delta \in \mathscr{F}_\alpha$, proving the claim.

Third, we consider a sequence of preimages $\gamma, \gamma_{-1}, \gamma_{-2}, \ldots$ with $\mathsf{T}_\alpha(\gamma_{-i}) = \gamma_{-i+1}$ such that all $\gamma_{-i} \in \mathscr{F}_\alpha$. We just showed that such a sequence exists, but is not necessarily unique. We claim that the set $\{\gamma_{-i}: i \geq 0\}$ is finite. If so, it has some $\gamma_{-i} = \gamma_{-j}$, with $i > j$, hence γ is in the cycle $\{\gamma_{-i}, \gamma_{-i+1}, \ldots, \gamma_{-j}\}$, and $\gamma \in \mathrm{Fix}(\mathsf{T}_\alpha)$, proving the theorem. Incidentally this implies that each $\gamma \in \mathscr{F}_\alpha$ has *exactly* one preimage under T_α in $\mathscr{F}_\alpha$, for if it had two, one would not be in a cycle.

To prove this claim, we use the fact, proved in part I, that if α is a special Pisot number with $\mathbf{Q}(\alpha) \neq \mathbf{Q}$ then both α and $\alpha/(\alpha - 1)$ are units in the ring of integers of $\mathbf{Q}(\alpha)$. Consequently if we define the *denominator* of $\gamma \in \mathbf{Q}(\alpha)$ to be the smallest positive $d \in \mathbf{Z}$ such that $d\prod_\sigma(X - \sigma(\gamma)) \in \mathbf{Z}[X]$, then

$$R_\alpha^{-1}(x) = \frac{1}{\alpha}x \quad \text{and} \quad L_\alpha^{-1}(x) = \frac{\alpha - 1}{\alpha}(1 - x)$$

both do not increase denominators. Hence denominator(γ_{-i}) for all $i \geq 0$ is bounded above by $d_0 =$ denominator(γ). Finally since membership in the set $\mathscr{F}_\alpha$ bounds a number $\delta \in \mathbf{Q}(\alpha)$ and all of its conjugates, $\mathscr{F}_\alpha$ only contains finitely many $\delta \in \mathbf{Q}(\alpha)$ having denominators $\leq d_0$. For the lead coefficient of the minimal polynomial in $\mathbf{Z}[X]$ for δ is $\leq d_0$ and the other coefficients are bounded since all roots $\sigma(\delta)$ are bounded. This proves the claim and the theorem. ∎

Next we study the attractors for the numbers covered by Theorem 2.3. The only two real quadratic special Pisot numbers are $(1 + \sqrt{5})/2$ and $(3 + \sqrt{5})/2$, and in this case the attractors A_α^σ have a simple description. For a real quadratic field, let $\bar{\gamma}$ denote the algebraic conjugate of γ.

COROLLARY 2.3a. *For* $\alpha = (1 + \sqrt{5})/2$,

$$\mathrm{Fix}(\mathsf{T}_\alpha) = \left\{\gamma \in \mathbf{Q}(\alpha): \gamma \in [0,1] \textit{ and } \bar{\gamma} \in \left[\frac{1 - \sqrt{5}}{4}, \frac{1}{2}\right]\right\}.$$

For $\alpha = (3 + \sqrt{5})/2$,

$$\mathrm{Fix}(\mathsf{T}_\alpha) = \left\{\gamma \in \mathbf{Q}(\alpha) : \gamma \in [0,1] \text{ and } \bar{\gamma} \in \left[\frac{-1-\sqrt{5}}{2}, 0\right]\right\}.$$

Proof. It suffices to show that the attractors A_α^σ for $\{L_\alpha^\sigma, R_\alpha^\sigma\}$ for the two values of α are exactly the specified intervals. This follows by verifying that (2.4) holds for these intervals. ■

Next we consider the attractors for non-totally real cubic special Pisot numbers.

CONJECTURE 2.4. *For special Pisot numbers such that* $\mathbf{Q}(\alpha)$ *is a non-totally real cubic field, the complex conjugate attractors* A_α^σ *and* $\mathsf{A}_\alpha^{\bar{\sigma}}$ *are the closure of their interiors and* $\partial\mathsf{A}_\alpha^\sigma = \mathsf{A}_\alpha^\sigma - \mathrm{Int}(\mathsf{A}_\alpha^\sigma)$ *has Hausdorff dimension strictly between one and two.*

This conjecture implies that $\partial\mathsf{A}_\alpha^\sigma$ is a "fractal" curve of infinite length. As evidence we exhibit computer plots of some of these attractors A_α^σ. Figure 2.1 shows the attractor A_α^σ for α a root of $X^3 - 2X^2 + X - 1$ with the choice $\sigma(\alpha) \doteq .12256 + .74486i$. Figure 2.2 shows a magnification of part of this attractor near 0 by a factor of ten. Certainly this A_α^σ is not simply connected. It appears to have positive Lebesgue measure. It is not clear whether the set of "holes" is dense in A_α^σ, but we think they are not dense. Figure 2.3 shows the attractor A_β^σ for the corresponding β to the above, which is a root of $X^3 - 3X^2 + 2X - 1$, with the corresponding conjugate $\sigma(\beta) \doteq .33764 - .56228i$. The attractor A_β^σ appears visually different from A_α^σ, but appears consistent with the conjecture.

Under a plausible hypothesis we show that such sets A_α^σ have Hausdorff dimension two. Any finite set of similitudes $\mathscr{S} = \{L_i(\mathbf{x}) = \alpha_i O_i \mathbf{x} + \beta_i : 1 \le i \le m\}$ of $\mathbf{R}^n$ such that each O_i is a rotation and $0 < \alpha_i < 1$ for all i is a hyperbolic IFS with an attractor $\mathsf{A}(\mathscr{S})$. Such a set $\mathscr{S}$ satisfies the *open set condition* if there is an open set U in $\mathbf{R}^n$ such that:

(1) $\bigcup_{i=1}^m L_i(U) \subseteq U$.
(2) For $i \ne j$, $L_i(U) \cap L_j(U) = \emptyset$.

Hutchinson (1981) showed that for any finite set $\mathscr{S}$ of similitudes as above which satisfies the open set condition the Hausdorff dimension d of its

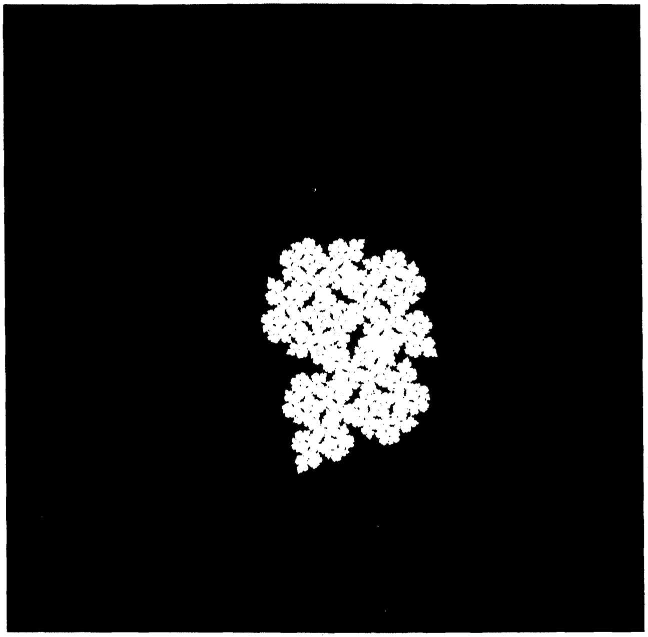

FIG. 2.1 The attractor A_α^σ for $\sigma(\alpha) = .12251 + .74486i$ a root of $X^3 - 2X^2 + X - 1$. (The box is a square of side 4 centered at zero in $\mathbf{C}$.)

attractor $\mathsf{A}(\mathscr{S})$ is the unique positive solution of

$$\sum_{i=1}^{m} \alpha_i^d = 1. \tag{2.10}$$

This dimension d is then also equal to the *box dimension* of $\mathscr{S}$ and to the *Lyapunov dimension* of $\mathscr{S}$; see Geronimo and Hardin (1989). Falconer (1987) indicates that the formula (2.10) determining d is sometimes valid even when the open set condition doesn't hold. Now for a non-totally real cubic special Pisot number α, the let $\mathscr{S}_\alpha = \{L_\alpha^\sigma, R_\alpha^\sigma\}$ is a hyperbolic IFS consisting of similitudes on $\mathbf{C}$ with $\alpha_1 = |\sigma(\alpha)|$ and $\alpha_2 = |\sigma(\beta)|$. Also $\alpha_1 = |\alpha|^{-1/2}$, $\alpha_2 = |\beta|^{-1/2}$ because α and β are units in a non-totally real cubic field and

$$1 = |N_{\mathbf{Q}(\alpha)/\mathbf{Q}}(\alpha)| = |\alpha\sigma(\alpha)\bar{\sigma}(\alpha)| = |\alpha|\,|\sigma(\alpha)|^2.$$

Since $1/\alpha + 1/\beta = 1$ this gives

$$\alpha_1^2 + \alpha_2^2 = 1.$$

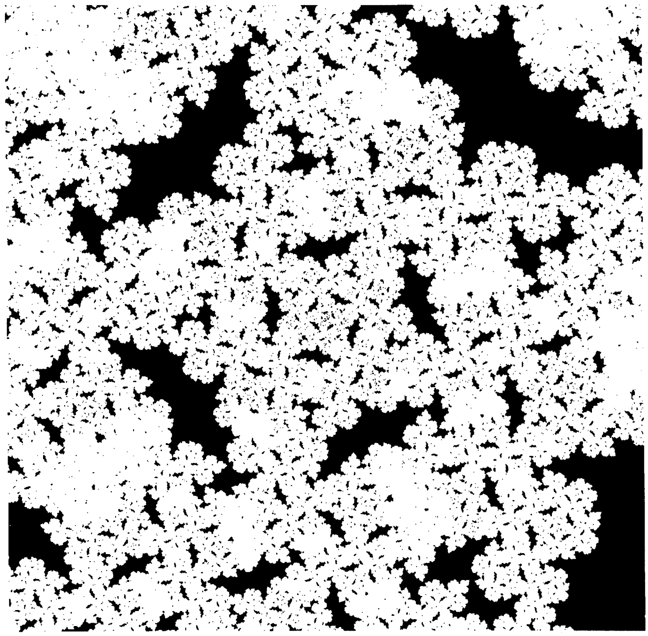

FIG. 2.2 The attractor A_α^σ in Figure 2.1 magnified by ten. (The box is of side 0.4 centered at zero in **C**.)

Comparing this with (2.10), we conclude that if $\mathscr{S}_\alpha$ satisfies the open set condition, then A_α^σ has Huasdorff dimension two.

Theorems 2.2 and 2.3 cover all special Pisot numbers in Table 1 except for two numbers of degree 4. For these two numbers the attractors $\mathsf{A}_\alpha^{\sigma_i}$ consist of one real attractor and a complex conjugate pair of attractors. The hyperbolic IFS associated to these α do not satisfy the open set condition, because the corresponding equations (2.10) have no solution $d \leq 1$ in the real case and no solution $d \leq 2$ in the complex case. We do not know if the equality $\mathrm{Fix}(\mathsf{T}_\alpha) = \mathscr{F}_\alpha$ holds for these α or not.

For comparison with the attractors A_α^σ we mention a set $\mathscr{M}$ constructed by Rauzy (1982) which he calls a "morecellement." The set $\mathscr{M}$ is the attractor of an affine hyperbolic IFS associated to the non-totally-real cubic Pisot number $\alpha = 1.8392^+$ satisfying $X^3 - X^2 - X - 1 = 0$. Rauzy constructs $\mathscr{M}$ in connection with the fixed point of the substitution sequence $1 \to 12$, $2 \to 13$, $3 \to 1$. Other sets that A_α^σ may resemble are the dragon-fractals in **C** constructed in Gilbert (1982).

Finally, we remark that differences between Figures 2.1 and Figure 2.3 might be taken as evidence that the conjugating map taking T_α to T_β is singular, when α is the real root of $X^3 - 2X^2 + X - 1$. Recall that T_α is

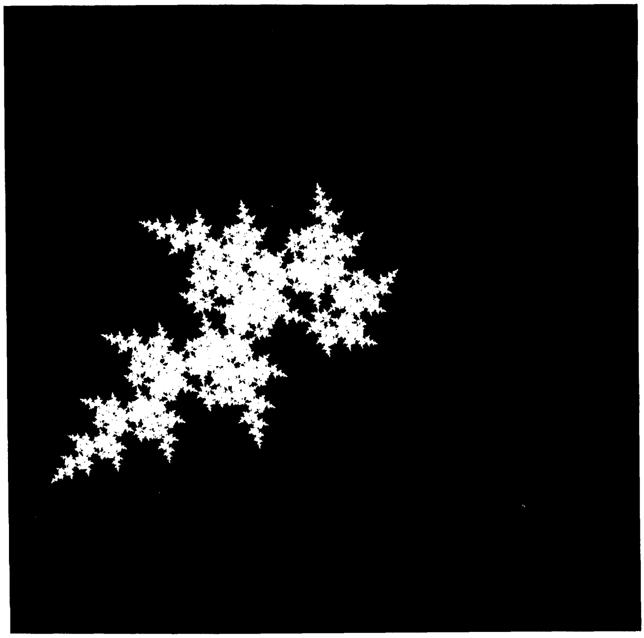

FIG. 2.3 The attractor A_α^σ for $\sigma(\beta) = .33764 - .56628i$ a root of $X^3 - 3X^2 + 2X - 1$. (The box is a square of side 4 centered at zero in **C**.)

topologically conjugate to T_γ for all γ, but Proppe, Byers and Boyarsky (1983) showed that the conjugating map is singular whenever γ does not equal α or β.

3. Preperiodic points

There is a nice characterization of certain sets of preperiodic points of T_α for special Pisot numbers α. We begin with the case $\alpha = 2$.

THEOREM 3.1. *For $\alpha = 2$, one has*

$$\mathrm{Per}_0(\mathsf{T}_2) = \left\{ \frac{m}{2^n} : n \geq 0 \text{ and } 0 \leq m \leq 2^n \text{ with } m \in \mathbf{Z} \right\}. \tag{3.1}$$

Proof. Let $\mathscr{P}$ denote the right side of (3.1). We examine all numbers in $\mathrm{Per}(\mathsf{T}_2) = \mathbf{Q} \cap [0, 1]$. The map T_2 applied to p/q with $(p, q) = 1$ gives a rational with denominator q if $2 \nmid q$ and $q/2$ if $2|q$. Theorem 2.2 shows that the only purely periodic point having denominator dividing a power of 2 is $\{0\}$. Since all rationals are eventually periodic, $\mathscr{P} \subseteq \mathrm{Per}_0(\mathsf{T}_2)$.

The same observation shows that any odd $q > 3$ dividing a denominator is preserved under iteration by T_2, hence $\mathrm{Per}_0(\mathsf{T}_2) \subseteq \mathscr{P}$. ■

COROLLARY 3.1a. $\mathrm{Per}_0(\mathsf{T}_2)$ *is closed under multiplication and under addition* (mod 1).

Now we turn to the remaining special Pisot numbers α having $\mathbf{Q}(\alpha) \neq \mathbf{Q}$. Let O_K denote the ring of integers of $K = \mathbf{Q}(\alpha)$ and $\mathrm{Per}_\gamma(\mathsf{T}_\alpha)$ denote the set of preperiodic points of a purely periodic point $\gamma \in \mathrm{Fix}(\mathsf{T}_\alpha)$. We define

$$\mathrm{Per}^*(\mathsf{T}_\alpha) := \bigcup \left\{\mathrm{Per}_\gamma(\mathsf{T}_\alpha) \colon \gamma \in \mathrm{Fix}(\mathsf{T}_\alpha) \cap O_K\right\}$$

and have the following result.

THEOREM 3.2. *Let α be a special Pisot number with $K = \mathbf{Q}(\alpha) \neq \mathbf{Q}$. Then*

$$I_\alpha := \left\{\gamma \colon \gamma \in \mathrm{Fix}(\mathsf{T}_\alpha) \cap O_K\right\}$$

is a finite set which always includes 0 *and*

$$\mathrm{Per}^*(\mathsf{T}_\alpha) = O_K \cap [0,1]. \tag{3.2}$$

Proof. We showed in part I that for special Pisot numbers α with $\mathbf{Q}(\alpha) \neq \mathbf{Q}$ both α and β are units in O_K. Hence denominator$(\mathsf{T}_\alpha(\gamma))$ = denominator(γ) for all $\gamma \in \mathbf{Q}(\alpha)$. Since $\mathrm{Per}(\mathsf{T}_\alpha) = \mathbf{Q}(\alpha) \cap [0,1]$ and $O_K = \{\gamma \in \mathbf{Q}(\alpha)$: denominator$(\gamma) = 1\}$ this proves (3.2).

To see that the set I_α is finite, observe that any $\gamma \in \mathrm{Fix}(\mathsf{T}_\alpha)$ has γ and all its conjugates bounded by Theorem 2.1, since $\mathsf{A}_\alpha^\sigma = \overline{\mathrm{Fix}(\{L_\alpha^\sigma, R_\alpha^\sigma\})}$ is compact for special Pisot numbers. Since O_K contains finitely many elements having all conjugates in any bounded set, (cf. the end of the proof of Theorem 2.3), I_α is finite. ■

COROLLARY 3.2a. *For all special Pisot numbers α,* $\mathrm{Per}^*(\mathsf{T}_\alpha)$ *is closed under multiplication and addition* (mod 1).

Proof. For $\alpha \neq 2$ this holds by (3.2) and for $\alpha = 2$ it holds by Corollary 3.1a, since $I_2 = \{0\}$. ■

One can determine each I_α by a finite computation. The quadratic special Pisot numbers reveal an asymmetry.

COROLLARY 3.2b. *For real quadratic special Pisot numbers, if* $\alpha = (1+\sqrt{5})/2$ *then* $I_\alpha = \{0\}$, *while if* $\alpha = (3+\sqrt{5})/2$, *then* $I_\alpha = \{0, (-1+\sqrt{5})/2\}$.

Proof. Set $\gamma = (a + b\sqrt{5})/2$ with $a \equiv b \pmod 2$, and use Corollary 2.3a. For $\alpha = (1 + \sqrt{5})/2$ one must have

$$0 \le a + b\sqrt{5} \le 2$$
$$\frac{1-\sqrt{5}}{2} \le a - b\sqrt{5} \le 1.$$

These imply that

$$\frac{1-\sqrt{5}}{2} \le 2a \le 3$$
$$-1 \le 2b \le \frac{3+\sqrt{5}}{2}. \tag{3.3}$$

Now (3.3) with $a \equiv b \pmod 2$ has solutions $(a, b) = (0,0), (1,1)$ and the second is extraneous, so $I_\alpha = \{0\}$.

For $\alpha = (3 + \sqrt{5})/2$ one must have

$$0 \le a + b\sqrt{5} \le 2$$
$$-1 - \sqrt{5} \le a - b\sqrt{5} \le 0,$$

and one easily deduces $I_\alpha = \{0, (-1 + \sqrt{5})/2\}$. ■

Similar asymmetries occur for some other special Pisot numbers.

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