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ORTHOGONAL PROJECTIONS ON MARTINGALE H^1 SPACES OF TWO PARAMETERS

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1. Introduction

Given collections of pairwise disjoint dyadic rectangles $\mathscr{A}_k$, we wish to find conditions which ensure that the natural orthogonal projection

$$Pf = \sum_k (f|\phi_k)\phi_k$$

where

$$\phi_k = \Bigl(\sum_{I\times J\in\mathscr{A}_k} h_{I\times J}\Bigr)\Big/\Bigl\|\sum_{I\times J\in\mathscr{A}_k} h_{I\times j}\Bigr\|_{L^2}$$

defines a bounded operator on $H^1(\delta^2)$. (In the one dimensional case such conditions where found by P. W. Jones in [J]).

More precisely we are interested in the special case where ϕ_k is equivalent in $H^1(\delta^2)$ to the Haar basis $\{h_{I\times J}: I, J \text{ dyadic}\}$. The condition given in Theorem 1 implies that the boundedness of P is determined by its action on *dyadic rectangles*. Adjusting a construction of Michele Capon we then apply this condition to prove that $H^1(\delta^2)$ is primary.

If the collections $\mathscr{A}_k$ are of product structure then the boundedness of the projection P follows simply from the corresponding one-dimensional result. It is therefore natural to ask under which conditions one finds sufficiently rich collections of dyadic rectangles which are of product structure. Here a geometric version of Ramsey's theorem is proved.

The motivation for this study of $H^1(\delta^2)$ and its isomorphic structure comes from the fact that $H^1(\delta^2)$ is not isomorphic to the one-dimensional H^1. This was shown by Jean Bourgain in [B]. More precisely, it was shown there that the vector valued Hardy space $H^1(l^2)$ is not isomorphic to a complemented subspace of H^1. Therefore it may be noteworthy that a sequence of uniformly complemented subspaces of H^1 can be constructed which are uniformly isomorphic to $H^1(l_n^2)$, $n \in \mathbf{N}$. This sequence of examples was constructed during conversation of the present author with Przemyslaw Wojtaszczyk.

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2. Notation and definitions

Given $n \in \mathbf{N}$ and $1 \le i \le 2^n$ we will use (n, i) to denote the dyadic interval

$$\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right[.$$

$\mathscr{D}$ denotes the collection of all dyadic intervals and $\mathscr{D}^n$ contains dyadic intervals of length bigger than 2^{-n}. Accordingly $h_{(n,i)}$ denotes the L^∞ normalized Haar functions, supported on the interval (n, i). On the unit square $[0, 1] \times [0, 1]$ we consider the tensor product of Haar functions

$$h_{(n,i)\times(m,j)}(s, t) := h_{(n,i)}(s)h_{(m,j)}(t).$$

Rectangles of the form $(n, i) \times (m, j)$ are called dyadic rectangles. Given

$$F = \sum{}' c_{(n,i)\times(m,j)} h_{(n,i)\times(m,j)}$$

on the square $[0, 1] \times [0, 1]$, the corresponding square function is

$$S(F) = \left(\sum c^2_{(n,i)\times(m,j)} |h_{(n,i)\times(m,j)}|^2\right)^{1/2}.$$

We use the square functions to define

$$H^1(\delta^2) := \left\{F \in L^1([0, 1]^2) : \int_{[0,1]^2} S(F) < \infty\right\}.$$

See [B], [Ch], [Ch-F], [G], [Ma, Ch V] and the references therein for results which relate this space to analytic functions.

3. The main technical result

For a collection $\mathscr{A}$ of dyadic rectangles we denote by $\mathscr{A}$ the pointset covered by $\mathscr{A}$. We consider $\mathscr{A}_{(n,i)\times(m,j)}$, pairwise disjoint collections of pairwise disjoint dyadic rectangles such that for $m, n \in \mathbf{N}$, $1 \le i \le 2^n$ and $1 \le j \le 2^m$ the following conditions hold.

$$A_{(0,1)\times(0,1)} \neq \emptyset \tag{3.1}$$

$$A_{(n+1,2i-1)\times(m,j)} \cap A_{(n+1,2i)\times(m,j)} = \emptyset$$

$$A_{(n+1,2i-1)\times(m,j)} \cup A_{(n+1,2i)\times(m,j)} \subset A_{(n,i)\times(m,j)}$$

$$A_{(n,i)\times(m+1,2j-1)} \cap A_{(n,i)\times(m-1,2j)} = \emptyset$$

$$A_{(n,i)\times(m+1,2j-1)} \cup A_{(n,i)\times(m+1,2j)} \subset A_{(n,i)\times(m,j)}$$

$$\frac{2^{-n-m}}{C} \le |A_{(n,i)\times(m,j)}| \le 2^{-n-m}C.$$

The block basis over the Haar system induced by $\mathscr{A}_{(n,i)\times(m,j)}$ is

$$\tilde{h}_{(n,j)\times(m,j)} = \sum_{I\times J\in\mathscr{A}_{(n,i)\times(m,j)}} h_{I\times J}.$$

The orthogonal projection P onto

$$\operatorname{span}\left\{\tilde{h}_{(n,i)\times(m,j)}: m, n \in \mathbf{N}, 1 \le i \le 2^n, 1 \le j \le 2^m\right\}$$

is given by

$$Pf = \sum_{(n,i)\times(m,j)} \left(f|\tilde{h}_{(n,i)\times(m,j)}\right) \frac{\tilde{h}_{(n,i)\times(m,j)}}{\|\tilde{h}_{(n,i)\times(m,j)}\|^2_{L^2}}.$$

And by our assumption on $\mathscr{A}_{(n,i)\times(m,j)}$ we have

$$\|Pf\|_{H^1(\delta^2)} = \int_{[0,1]^2}\left(\sum_{(n,i)\times(m,j)} \left(f|\tilde{h}_{(n,i)\times(m,j)}\right)^2 \frac{\tilde{h}^2_{(n,i)\times(m,j)}}{\|\tilde{h}_{(n,i)\times(m,j)}\|^4_2}\right)^{1/2} ds\,dt.$$

Our main theorem gives a criterion for the boundedness of P on $H^1(\delta^2)$.

THEOREM 1. *If there exists $C \in \mathbf{N}^+$ so that for each $n_0 \in \mathbf{N}$, $1 \le i_0, j_0 \le 2^{n_0}$ and for any $I \times J \in \mathscr{A}_{(n,i)\times(m,j)}$, with $(n,i) \times (m,j) \supseteq (n_0, i_0) \times (n_0, j_0)$ we have*

$$\text{(3.2)} \qquad \frac{1}{C}|I \times J| \le |I \times J \cap A_{(n_0,i_0)\times(n_0,j_0)}| 2^{n_0-n} 2^{n_0-m} \le C|I \times J|,$$

then P extends to a bounded linear operator on $H^1(\delta^2)$ and the range of P is isomorphic to $H^1(\delta^2)$.

Theorem 1 implies that the boundedness of P on $H^1(\delta^2)$ is determined by a condition which involves only dyadic rectangles $I \times J$ and does *not* involve arbitrary open sets of $\Omega \subseteq [0,1] \times [0,1]$. This makes our condition quite simple and easy to verify in specific situations (see Section 4).

The price we have to pay is that BMO-techniques—or atoms— are not at our disposal. Instead we will exploit the fact that $H^1(\delta^2)$ is a sequence space and carefully study how P and the embedding of $H^1(\delta^2)$ into $L^1(l^2)$ interact. The example which ultimately led to the proof given below is described at the end of Section 5.

Proof. Let $f \in H^1(\delta^2)$. The product Haar system $\{h_{I\times J}: I, J \in \mathscr{D}\}$ is an unconditional basis in $H^1(\delta^2)$. We therefore assume that f is a finite linear combination of the form

$$f = \sum_{I\times J\in\mathscr{A}} a_{I\times J} h_{I\times J}$$

where $\mathscr{A} := \cup \mathscr{A}_{(Cn,i)\times(Cm,j)}$. Therefore $\mathscr{T} = \{I \times J\colon a_{I\times J} \neq 0\}$ is a finite collection of rectangles. Hence there exists $n_0 \in \mathbf{N}$ so that for any $I \times J \in \mathscr{T}$ any i_0, j_0 and any $I_0 \times J_0 \in \mathscr{A}_{(n_0,i_0)\times(n_0,j_0)}$ we have: $I_0 \times J_0 \cap I \times J \neq \emptyset$ implies $I \times J \supseteq I_0 \times J_0$. $S(f)$ can therefore be minorized pointwise by

$$\sum_{i_0,j_0=1}^{2^{n_0}} \sum_{I_0\times J_0\in\mathscr{A}_{(n_0,i_0)\times(n_0,j_0)}} \left(\sum_{I\times J\supseteq I_0\times J_0} a_{I\times J}^2 h_{I\times J}^2 \right) \mathbf{1}_{I_0\times J_0}.$$

Consequently the norm of f in $H^1(\delta^2)$ is minorized by

$$\sum_{i_0,j_0=1}^{2^{n_0}} \sum_{I_0\times J_0\in\mathscr{A}_{(n_0,i_0)\times(n_0,j_0)}} \left(\sum_{I\times J\supseteq I_0\times J_0} a_{I\times J}^2 \right)^{1/2} |I_0 \times J_0|. \tag{3.3}$$

If $I_0 \times J_0 \in \mathscr{A}_{(n_0,j_0)\times(n_0,j_0)}$ and $I \times J \in \mathscr{A}_{(n,i)\times(m,j)}$ then (by (3.2)) $I \times J \supseteq I_0 \times J_0$ only if $(n,i) \times (m,j) \supseteq (n_0,i_0) \times (n_0,j_0)$. Moreover for fixed $n, m \in \mathbf{N}$ and fixed i_0, j_0, the condition $(n,i) \times (m,j) \supseteq (n_0,i_0) \times (n_0,j_0)$ uniquely determines $1 \leq i \leq 2^n$ and $1 \leq j \leq 2^m$. Therefore (3.3) can be rewritten as

$$\sum_{i_0,j_0=1}^{2^{n_0}} \sum_{I_0\times J_0\in\mathscr{A}_{(n_0,i_0)\times(n_0,j_0)}} \left(\sum_{m,n=1}^{n_0} \sum_{\substack{I\times J\in\mathscr{A}_{(n,i)\times(m,j)} \\ I\times J\supseteq I_0\times J_0}} a_{I\times J}^2 \right)^{1/2} |I_0 \times J_0|. \tag{3.4}$$

we may further rewrite (3.4) as

$$\sum_{i_0,j_0=1}^{2^{n_0}} \sum_{I_0\times J_0\in\mathscr{A}_{(n_0,i_0)\times(n_0,j_0)}} \left(\sum_{(n,i)\times(m,j)\supseteq(n_0,i_0)\times(n_0,j_0)} \sum_{\substack{I\times J\in\mathscr{A}_{(n,i)\times(m,j)} \\ I\times J\supseteq I_0\times J_0}} a_{I\times J}^2 \right)^{1/2} \times |I_0 \times J_0|. \tag{3.5}$$

Let us mention that by hypothesis for fixed $(n,i) \times (m,j)$, $(n_0,i_0) \times (n_0,j_0)$, and $I_0 \times J_0$ satisfying $(n,i) \times (m,j) \supseteq (n_0,i_0) \times (n_0,j_0)$ and $I_0 \times J_0 \in \mathscr{A}_{(n_0,j_0)\times(n_0,j_0)}$, only one rectangle $I \times J \in \mathscr{A}_{(n,i)\times(m,j)}$ can satisfy $I \times J \supseteq I_0 \times J_0$. Therefore the inner sum in (3.5) has at most one summand. To handle this expression we need the next result.

LEMMA 2. *There exists $C > 0$ so that for any $n_0 \in \mathbf{N}$ and $(n,i) \times (m,j)$ satisfying $(n,i) \times (m,j) \supseteq (n_0, i_0) \times (m_0, j_0)$ we have*

$$\sum_{I_0 \times J_0 \in \mathscr{A}_{(n_0,i_0)\times(n_0,j_0)}} |I_0 \times J_0| \sum_{\substack{I\times J \in \mathscr{A}_{(n,i)\times(m,j)} \\ I\times J \supseteq I_0 \times J_0}} |a_{I\times J}| \tag{3.6}$$

$$\geq \frac{1}{C} \left| \sum_{I\times J \in \mathscr{A}_{(n,i)\times(m,j)}} |I \times J| a_{I\times J} \right| 2^{-n_0+n} 2^{-n_0+m}.$$

Proof. Interchanging the order of summation in the LHS of (3.6) gives

$$\sum_{I\times J \in \mathscr{A}_{(n,i)\times(m,j)}} \left(\sum_{\substack{I_0 \times J_0 \in \mathscr{A}_{(n_0,i_0)\times(n_0,j_0)} \\ I\times J \supseteq I_0 \times J_0}} |I_0 \times J_0| \right) |a_{I\times J}|.$$

As $\mathscr{A}_{(n_0,i_0)\times(n_0,j_0)}$ consists of pairwise disjoint dyadic rectangles the above expression coincides with

$$\sum_{I\times J \in \mathscr{A}_{(n,i)\times(m,j)}} |I \times J \cap A_{(n_0,i_0)\times(n_0,j_0)}| |a_{I\times J}|.$$

Condition (3.2) implies that this sum admits a minorization by

$$\frac{1}{C} \sum_{I\times J \in \mathscr{A}_{(n,j)\times(m,j)}} |I \times J| |a_{I\times J}| 2^{-n_0+n} 2^{-n_0+m}$$

which proves the lemma. ∎

We return to the proof of Theorem 1. Applying triangle inequality (for the vector space ℓ^2) to (3.5) gives

$$\|f\|_{H^1(\delta^2)}$$

$$\geq \sum_{i_0, j_0 = 1}^{2^{n_0}} \left(\sum_{(n,i)\times(m,j) \supseteq (n_0,i_0)\times(n_0,j_0)} \left(\sum_{I_0 \times J_0 \in \mathscr{A}_{(n_0,i_0)\times(n_0,j_0)}} |I_0 \times J_0| \times \sum_{\substack{I\times J \in \mathscr{A}_{(n,i)\times(m,j)} \\ I\times J \supset I_0 \times J_0}} |a_{I\times J}| \right)^2 \right)^{1/2}.$$

By Lemma 2, one minorizes the above expression by

(3.7)
$$\frac{1}{C}\sum_{i_0,j_0=1}^{2^{n_0}}\left(\sum_{(n,i)\times(m,j)\supseteq(n_0,i_0)\times(m_0,j_0)}\left(\sum_{I\times J\in\mathscr{A}_{(n,i)\times(m,j)}}|I\times J|a_{I\times J}\right)^2 \times 4^{-n_0+n}4^{-n_0+m}\right)^{1/2}.$$

It remains to relate $\|Pf\|_{H^1(\delta^2)}$ to (3.7). But this is easy: We first have

$$\left(f|\tilde{h}_{(n,i)\times(m,j)}\right) = \sum_{I\times J\in\mathscr{A}_{(n,i)\times(m,j)}}|I\times J|a_{I\times J}. \tag{3.8}$$

Next one observes that

$$\|\tilde{h}_{(n,i)\times(m,j)}\|_2^4 = |A_{(n,i)\times(m,j)}|^2 \tag{3.9}$$

and

$$\mathbf{1}_{A_{(n,i)\times(m,j)}} = \tilde{h}^2_{(n,i)\times(m,j)}. \tag{3.10}$$

Finally, (3.2) implies that there exists $C_0 > 0$ so that

$$2^{-n}2^{-m}\frac{1}{C_0} \le |A_{(n,i)\times(m,j)}| \le C_0 2^{-n}2^{-m}. \tag{3.11}$$

Combining (3.1) with (3.8)–(3.11) we see that the expression (3.7) $\ge$ $C\|Pf\|_{H^1(\delta^2)}$. ■

4. Primarity of $H^1(\delta^2)$

In this section we give some applications to Banach space properties of $H^1(\delta^2)$. Adjusting an idea of Michele Capon we verify that for any collection $\mathscr{C}$ of dyadic rectangles either $\mathscr{C}$ or its complement $\mathscr{D}\times\mathscr{D}\setminus\mathscr{C}$ contains collections $\mathscr{A}_k$ which satisfy the hypothesis of Theorem 1. This dichotomy is, of course, the basis of our proof that $H^1(\delta^2)$ is primary.

Covering lemmas for dyadic rectangles involve the Orlicz norm $\exp(L)$ rather then the L_∞ norm. See [Ch-F], p. 15. Therefore the mere fact that the collections $\mathscr{A}_k$ consist of pairwise disjoint rectangles is not at all obvious.

DEFINITION 3 (Michele Capon). Let

$$\mathscr{C} := \{J \times I : I \in \mathscr{D} \text{ and } J \in \mathscr{E}_I\}$$

where $\mathscr{E}_I \subset \mathscr{D}$. Then $\sigma(\mathscr{E}_I) := \{t \in [0,1]: t$ lies in infinitely many $J \in \mathscr{E}_I\}$

$$\mathscr{M}_t := \{I \in \mathscr{D} : t \in \sigma(\mathscr{E}_I)\} \text{ and } B := \{t \in [0,1] : |\sigma(\mathscr{M}_t)| > \tfrac{1}{2}\}.$$

THEOREM 4. *If $|B| > 0$ then there exist collections $\mathscr{A}_{(n,i)\times(m,j)} \subset \mathscr{C}$ which satisfy the hypothesis of Theorem* 1; *consequently*

$$\operatorname{span}\{h_{I\times J} : I \times J \in \mathscr{C}\}$$

contains a complemented copy of $H^1(\delta^2)$.

Proof. We fix $t \in B$ and sequences $\varepsilon_n > 0$, $\varepsilon_J > 0$ of positive real numbers.

Part 1. *Step* (1). Define

$$\mathscr{K}_{(0,1),t} := \{K \in \mathscr{M}_t : |K \cap \sigma(M_t)| \geq (1 - \varepsilon_1)|K|\}. \tag{4.1}$$
$$\mathscr{F}_{(0,1),t} := \{K \in \mathscr{K}_{(0,1),t} : K \text{ maximal}\}.$$

Then choose $N((0,1),t) \in \mathbf{N}$ so that

$$\sum_{K \in \mathscr{F}_{(0,1),t}} \{|K| : |K| \geq 2^{-N((0,1),t)}\} \geq (1 - \varepsilon_1)\left|\bigcup_{K \in \mathscr{F}_{(0,1),t}} K\right| \tag{4.2}$$

Let

$$\mathscr{T}_{(0,1),t} := \{K \in \mathscr{F}_{(0,1),t} : |K| \geq 2^{-N((0,1),t)}\}$$
$$h_{(0,1),t} := \sum_{K \in \mathscr{T}_{(0,1),t}} h_K.$$

Step $(n+1)$. Having constructed $\mathscr{T}_{(n,i),t}$ and $h_{(n,i),t}$ we let

$$E^+ := \{s : h_{(n,i),t}(s) = 1\} \quad \text{and} \quad E^- := \{s : h_{(n,i),t}(s) = -1\}.$$

Now define

$$\mathscr{K}_{(n+1,2i-1),t} := \{K \in \mathscr{M}_t : K \subset E^+, |K \cap \sigma(\mathscr{M}_t)| \geq (1 - \varepsilon_{n+1})|K|\} \tag{4.3}$$
$$\mathscr{K}_{(n+1,2i),t} := \{K \in \mathscr{M}_t : K \subset E^-, |K \cap \sigma(\mathscr{M}_t)| \geq (1 - \varepsilon_{n+1})|K|\}.$$

Fix $\delta \in \{-1, 0\}$ and define

$$\mathscr{F}_{(n+1,2i+\delta),t} := \{K \in \mathscr{K}_{(n+1,2i+\delta),t} \colon K \text{ maximal}\}.$$

Next we determine a natural number $N = N((n+1, 2i+\delta)t)$ so that for each $J \in \mathscr{T}_{(n,i),t}$

$$\sum_{K \in \mathscr{F}_{(n+1,2i+\delta),t}} \{|K| \colon K \subset J,\ |K| > 2^{-N}\} \geq (1 - \varepsilon_{n+1})|J| \tag{4.4}$$

and we let

$$\mathscr{T}_{(n+1,2i+\delta),t} := \{K \in \mathscr{F}_{(n+1,2i+\delta),t} \colon |K| > 2^{-N}\}$$

and

$$h_{(n+1,2i+\delta),t} := \sum_{k \in \mathscr{T}_{(n+1,2i+\delta),t}} h_K.$$

Having completed the construction of part 1, we now collect six important consequences thereof:

1. Let

$$E_{(n,i),t} := \operatorname{supp}\{h_{(n,i),t}\}$$

then for every $(m, j) \supseteq (n, i)$ and $J \in \mathscr{T}_{(m,j),t}$ we have by (4.1)–(4.4).

$$D_{m,n} 2^{m-n}|J| \leq |J \cap E_{(n,i),t}| \leq C_{m,n} 2^{m-n}|J| \tag{4.5}$$

where

$$D_{m,n} = |E_{(1,1),t}| \prod_{l=m}^{n} (1 + 2\varepsilon_l)^2$$

$$C_{m,n} = |E_{(1,1),t}| \prod_{l=m}^{n} (1 - 2\varepsilon_l)^2.$$

2. $E_{(n,i),t}$ forms a tree of sets in the sense that

$$|E_{(1,1),t}| > \tfrac{1}{4}$$

$$E_{(n+1,2i-1),t} \cup E_{(n+1,2i),t} \subset E_{(n,i),t}$$

$$E_{(n+1,2i-1),t} \cap E_{(n+1,2i),t} = \emptyset$$

$$2(1 - \varepsilon_{n+1}) \leq \frac{|E_{(n,i),t}|}{|E_{(n+1,2i+\delta),t}|} \leq 2(1 + \varepsilon_{n+1}),$$

where $\delta \in \{-1, 0\}$.

3. During this process, for $t \in B$ we inductively selected integers $N((n,i),t)$. Looking back at the construction we observe that the dependence

$$t \to N((n,i),t), \quad t \in B,$$

can be chosen to be measurable. Hence a repeated application of Egorov's theorem implies the existence of $B' \subset B$ so that for any (n,i) there exists $N((n,i))$ satisfying

$$\text{(4.6)} \qquad \sup\{N((n,i),t) : t \in B'\} \leq N((n,i)) \quad \text{and} \quad \frac{|B'|}{|B|} \geq \frac{1}{2}.$$

4. Let $n \in \mathbf{N}$, $1 \leq i \leq 2^n$ and let $\mathscr{K}$ be a finite collection of intervals. Then

$$I_{\mathscr{K}} = \{t \in B' : \mathscr{T}_{(n,i),t} = \mathscr{K}\}$$

is a measurable set and for any $J \in \mathscr{K}$ we have the importanct inclusion

$$\text{(4.7)} \qquad I_{\mathscr{K}} \subset \sigma(\mathscr{E}_J).$$

Let us emphasize that (4.7) is systematically exploited in the construction given below. Moreover by (4.6) the cardinality of $\{\mathscr{K} : I_{\mathscr{K}} \neq \emptyset\}$ is finite.

5. Given $\delta_1, \delta_2, \delta_3 > 0$ and $J \in \mathscr{D}$. By Lebesgue's theorem on differentiation of integrals we may select a finite collection of pairwise disjoint dyadic intervals $\mathscr{G}_J \subset \mathscr{E}_J$ so that for any $K \in \mathscr{G}_J$,

$$\text{(4.8)} \qquad |K \cap \sigma(\mathscr{E}_J)| > (1 - \delta_1)|K$$

$$\text{(4.9)} \qquad |K| \leq \delta_2$$

Moreover for $G_J := \bigcup_{K \in \mathscr{G}_J} K$ we have

$$\text{(4.10)} \qquad |G_J \Delta \sigma(\mathscr{E}_J)| \leq \delta_3.$$

6. Before going on with the proof the reader is advised to have a look at [C], page 91, line 3. There M. Capon states that $|B| > 0$ implies the existence of blockbasis

$$Z_{(n,i)\times(m,j)} \in \operatorname{span}\{h_{I\times J} : I \times J \in \mathscr{C}\}$$

so that

$$|Z_{(n,i)\times(m,j)}(t,y)| = |f_{(n,i)}(t)|\,|h_{(m,j),t}(y)|$$

where$\{|f_{(n,i)}|(t)\}$ is the characteristic functions of a tree $\{B_{(n,i)}\}$ in $[0,1]$. A moments reflection shows that this is already very close to (3.2). And in fact, all we have to do is, to adjust the sets $B_{(n,i)}$ properly to obtain (3.2). ∎

We continue now with our proof of Theorem 4.

Part 2. *Step* (1). Here we shall construct $\mathscr{A}_{(0,1)\times(0,1)}$. The main ingredients of this construction are (4.7)—(4.10) and will reappear several times during the induction argument.

Let $B_{(0,1)} := B'$ and for a finite collection of dyadic intervals $\mathscr{K}$ let

$$I_{\mathscr{K}} := \{t \in B_{(0,1)} : \mathscr{T}_{(0,1),t} = \mathscr{K}\}.$$

Then $\{I_{\mathscr{K}} : \mathscr{K} \subset \mathscr{D} \text{ finite}\}$ is a sequence of pairwise disjoint, measurable subsets of $B_{(0,1)}$ so that

$$B_{(0,1)} = \bigcup_{\mathscr{K} \text{ finite}} I_{\mathscr{K}}.$$

Using Remark 3 we then find $N \in \mathbf{N}$ and $\mathscr{K}_1, \dots, \mathscr{K}_N$, collections of pairwise disjoint dyadic intervals so that

$$B_{(0,1)} = \bigcup_{j=1}^{N} I_{\mathscr{K}_j}.$$

Next fix $J \in \mathscr{K}_j$, $j \le N$. By (4.7)–(4.10) we find finite collections of pairwise disjoint intervals $\mathscr{G}_J \subset \mathscr{E}_J$ so that with $G_J := \bigcup_{\mathscr{K} \in \mathscr{G}_J} K$,

$$K \in \mathscr{G}_J \text{ implies} |K \cap I_{\mathscr{K}_j}| > |K|(1 - \varepsilon_J) \tag{4.11}$$

$$|G_J \,\Delta\, I_{\mathscr{K}_j}| \le \varepsilon_J. \tag{4.12}$$

For $k \ne l$, $k, l \le N$ we may obtain moreover that

$$\bigcup_{J \in \mathscr{K}_l} G_J \cap \bigcup_{J \in \mathscr{K}_k} G_J = \emptyset, \tag{4.13}$$

because for $\mathscr{K}_l \ne \mathscr{K}_k$ we have $I_{\mathscr{K}_l} \cap I_{\mathscr{K}_k} = \emptyset$.

Finally we define

$$\mathscr{A}_{(0,1)\times(0,1)} := \bigcup_{j=1}^{N} \bigcup_{J \in \mathscr{K}_j} \mathscr{G}_J \times J$$

which is a collection of pairwise disjoint dyadic rectangles.

Having constructed $\mathscr{A}_{(m,j)\times(m',j')}$ for $m, m' \le n$, $1 \le j \le 2^m$ and $1 \le j' \le 2^{m'}$ we define now the collections $\mathscr{A}$ of level $n+1$:

Let

$$\mathscr{B}_n := \left\{ L \in \mathscr{D} : \exists J \in \mathscr{D}\ L \times J \in \bigcup_{m,m'=1}^{n} \bigcup_{j=1}^{2^m} \bigcup_{j'=1}^{2^{m'}} \mathscr{A}_{(m,j)\times(m',j')} \right\}.$$

$\mathscr{B}_n$ is then a finite collection of dyadic intervals. The induction step is divided into three steps. In the first step we shall define

$$\mathscr{A}_{(0,1)\times(n+1,2i+\delta)}$$

where $\delta \in \{+1, 0\}$. The second step describes the construction of

$$\mathscr{A}_{(n+1,2i+\delta)\times(0,1)}.$$

We then complete the induction in the third step where

$$\mathscr{A}_{(m,j)\times(n+1,2i+\delta)}$$

and

$$\mathscr{A}_{(n+1,2i+\delta)\times(m,j)}$$

are constructed for $2 \le m \le n+1$ and $1 \le j \le 2^m$.

Step $(n+1, a)$. Fix $\delta \in \{-1, 0\}$ and $\mathscr{K} \subset \mathscr{D}$ finite. Let

$$I_{\mathscr{K},\delta} = \left\{ t \in B_{(1,1)} : \mathscr{T}_{(n+1,2i+\delta),t} = \mathscr{K} \right\}.$$

Then

$$\{ I_{\mathscr{K},\delta} : \mathscr{K} \subseteq \mathscr{D} \text{ finite} \}$$

is a sequence of pairwise disjoint measurable subsets of $B_{(1,1)}$ so that

$$\bigcup_{\mathscr{K} \text{ finite}} I_{\mathscr{K},\delta} = B_{(1,1)}. \tag{4.14}$$

Using Remark 3 we next choose $N \in \mathbf{N}$ and $\mathscr{K}_1, \ldots, \mathscr{K}_N$, finite collections of pairwise disjoint dyadic intervals, so that

$$\bigcup_{j=1}^{N} I_{\mathscr{K}_j,\delta} = B_{(0,1)}. \tag{4.15}$$

Now fix $j \le N$ and $J \in \mathscr{K}_j$. By (4.7)–(4.10) there exists a collection of pairwise disjoint dyadic intervals $\mathscr{G}_{J,\delta} \subseteq \mathscr{E}_J$ so that

$$K \in \mathscr{G}_{J,\delta} \text{ implies } |K \cap I_{\mathscr{K}_{J,\delta}}| \ge |K|(1-\varepsilon_J). \tag{4.16}$$

$$|G_{J,\delta}\,\Delta I_{\mathscr{K}_j,\delta}| \le \varepsilon_J. \tag{4.17}$$

For $k \ne l, k, l \le N$ we may obtain

$$\bigcup_{J\in\mathscr{K}_l} G_{J,\delta} \cap \bigcup_{J\in\mathscr{K}_k} G_{J,\delta} = \emptyset. \tag{4.18}$$

Having done this construction for $\delta = -1$ we repeat the same construction for $\delta = 0$ and using (4.9) we may do this in such a way that

$$K \in \bigcup_{l=1}^{N} \bigcup_{J\in\mathscr{K}_l} \mathscr{G}_{J,-1} \text{ implies } K \notin \bigcup_{l=1}^{N} \bigcup_{J\in\mathscr{K}_l} \mathscr{G}_{J,0}.$$

Finally, for $\delta \in \{-1, 0\}$, we put

$$\mathscr{A}_{(0,1)\times(n+1,2i+\delta)} := \bigcup_{l=1}^{N} \bigcup_{J\in\mathscr{K}_l} \mathscr{G}_{J,\delta} \times J.$$

Step$(n+1, b)$. We consider the following finite set of intervals.

$$\mathscr{C}_{n+1} := \mathscr{B}_n \cup \left\{I \in \mathscr{D}\colon \exists J \in \mathscr{D}\; I \times J \in \mathscr{A}_{(1,1)\times(n+1,2i)} \cup \mathscr{A}_{(1,1)\times(n+1,2i-1)}\right\}$$

Choose now two disjoint measurable subsets $B_{(n+1,2i-1)}, B_{(n+1,2i)}$ of $B_{(n,i)}$ so that for every $I \in \mathscr{C}_{n+1}$, $\delta \in \{-1, 0\}$,

$$\tfrac{1}{2}|B_{(n\,i)} \cap I| = |B_{(n+1,2i+\delta)} \cap I| \tag{4.19}$$

$$B_{(n,i)} = B_{(n+1,2i-1)} \cup B_{(n+1,2i)}. \tag{4.20}$$

Let now $\mathscr{K} \subseteq \mathscr{D}$ be finite and define

$$I_{\mathscr{K},\delta} := \left\{t \in B_{(n+1,2i+\delta)}\colon \mathscr{T}_{(1,1),t} = \mathscr{K}\right\}.$$

Then

$$\left\{I_{\mathscr{K},\delta}\colon \mathscr{K} \subseteq \mathscr{D}, \text{finite}; \delta \in \{-1, 0\}\right\}$$

is a measurable partition of $B_{(n+1,2i-1)} \cup B_{(n+1,2i)}$. We next choose $N \in \mathbf{N}$, $\mathscr{K}_1, \ldots, \mathscr{K}_N$ so that for $\delta \in \{-1, 0\}$,

$$B_{n+1,2i+\delta} = \bigcup_{k=1}^{N} I_{\mathscr{K}_k,\delta} \tag{4.21}$$

and obviously for $I \in \mathscr{C}_{n+1}$ we have, by (4.20),

$$I \cap \bigcup_{k=1}^{N} I_{\mathscr{K}_k,\delta} = I \cap B_{n+1,2k-\delta}. \tag{4.22}$$

Next we fix $k \leq N$, $J \in \mathscr{K}_k$. By (4.7)–(4.10) there exists a collection of pairwise disjoint dyadic intervals $\mathscr{G}_{J,\delta} \subseteq \mathscr{E}_J$ so that

$$K \in \mathscr{G}_{J,\delta} \text{ implies } |K \cap I_{\mathscr{K}_k,\delta}| \geq |K|(1-\varepsilon_J). \tag{4.23}$$

And for $G_{J,\delta} = \bigcup_{K \in \mathscr{G}_{J,\delta}} K$,

$$|G_{J,\delta}\, \Delta I_{\mathscr{K}_j,\delta}| < \varepsilon_J. \tag{4.24}$$

For $k \neq l$, $k, l \leq N$ and $\delta, \delta' \in \{-1, 0\}$ we may obtain

$$\bigcup_{J \in \mathscr{K}_l} G_{J,\delta} \cap \bigcup_{J \in \mathscr{K}_k} G_{J,\delta'} = \emptyset. \tag{4.25}$$

By (4.9) we may achieve that

$$\mathscr{A}_{(n+1,2i+\delta)\times(0,1)} := \bigcup_{j=1}^{N} \bigcup_{J \in \mathscr{K}_j} \mathscr{G}_{J,\delta} \times J$$

satisfies (3.1).

Step $(n+1, c)$. Here we complete the induction step and shall first construct

$$\mathscr{A}_{(m,j)\times(n+1,2i+\delta)} \quad \text{for } m \leq n+1,\ 1 \leq j \leq 2^m.$$

Then we shall define

$$\mathscr{A}_{(n+1,2i+\delta)\times(m,j)} \quad \text{for } m \leq n,\ 1 \leq j \leq 2^m.$$

Fix (m, j) $m \leq n+1, 1 \leq j \leq 2^m$ and $\delta \in \{-1, 0\}$ and consider the following procedure: In step $(n+1, a)$ we defined collections of dyadic intervals $\mathscr{K}_k$ to build $\mathscr{A}_{(0,1)\times(n+1,2i+\delta)}$. We use those $\mathscr{K}_k$'s now to construct $\mathscr{A}_{(m,j)\times(n+1,2i+\delta)}$: Take $J \in \mathscr{K}_k$ and consider the collection $\mathscr{G}_{J,\delta}$ which was defined in step $(n+1, a)$ as well. By (4.7)–(4.10) for $I \in \mathscr{G}_{J,\delta}$ there exists $\mathscr{L}_I \subset \mathscr{E}_J$ so that

$$K \in \mathscr{L}_I \text{ implies } |K \cap B_{(m,j)}| \geq (1 - \varepsilon_{n+1}\varepsilon_I)|K|. \tag{4.26}$$

For $L_I := \bigcup_{K \in \mathscr{L}_I} K$,

(4.27) $$|L_I \,\Delta I \cap B_{(m,j)}| \le \varepsilon_I \varepsilon_{n+1}.$$

Then we define

$$\mathscr{A}_{(m,j)\times(n+1,2i+\delta)} := \bigcup_{k=1}^{N} \bigcup_{J\in\mathscr{K}_k} \bigcup_{I\in\mathscr{G}_{J,\delta}} \mathscr{L}_I \times J.$$

We start this construction at $(m,j) = (1,1)$ and continue until $(m,j) = (n+1, 2^{n+1})$. By (4.9) we can guarantee that the resulting collections $\mathscr{A}_{(m,j)\times(n+1,2i+\delta)}$ are pairwise disjoint and satisfy conditions (3.1). Now fix (m,j), $m \le n, 1 \le j \le 2^m$, $\delta \in \{-1,0\}$ and consider the following procedure. Fix $\mathscr{K} \subseteq \mathscr{D}$ finite and define

$$I_{\mathscr{K}_\delta} := \{t \in B_{(n+1,2i+\delta)} \colon \mathscr{T}_{(m,j),t} = \mathscr{K}\}.$$

$\{I_{\mathscr{K}_\delta} \colon \mathscr{K}$ finite$\}$ is a measurable partition of $B_{(n+1,2i+\delta)}$. We choose $N \in \mathbf{N}$ and $\mathscr{K}_{\delta,1}, \dots, \mathscr{K}_{\delta,N}$, finite collections of pairwise disjoint dyadic intervals, so that

(4.28) $$\bigcup_{k=1}^{N} I_{\mathscr{K}_{\delta,k}} = B_{(n+1,2i+\delta)}.$$

By (4.7)–(4.10) for every $k \le N$ and $J \in \mathscr{K}_{\delta,k}$ we find $\mathscr{G}_J \subset \mathscr{E}_J$ so that

(4.29) $$K \in \mathscr{G}_J \text{ implies} |K \cap B_{(n+1,2i-\delta)}| \ge (1 + \varepsilon_{n+1})|K|.$$

For $G_J := \bigcup_{k \in \mathscr{G}_J} K$ we have

(4.30) $$|G_J \,\Delta I_{\mathscr{K}_k,\delta}| \le \varepsilon_{n+1}\varepsilon_J.$$

Then we define

$$\mathscr{A}_{(n+1,2i+\delta)\times(m,j)} = \bigcup_{k=1}^{N} \bigcup_{J\in\mathscr{K}_k} \mathscr{G}_J \times J.$$

Again we start this construction with $(m,j) = (1,1)$ and stop at $(m,j) = (n, 2^n)$. By (4.9) the resulting collections $\mathscr{A}_{(n+1,2i+\delta)\times(m,j)}$ can be chosen to be disjoint. This completes the induction step and Part 2 of the proof of Theorem 4.

It remains to observe that the resulting families $\mathscr{A}_{(n,i)\times(m,j)}$ satisfy (3.2). To do so we simply have to trace back the construction. Suppose (3.2) holds for $n_0 \in \mathbf{N}$ with constant C_0. We then show that (3.2) holds for $n_0 + 1$ and constant $C_0(1 + \tilde{\varepsilon}_{n_{0+1}})$. Indeed, this follows from (4.29), (4.30), (4.26), (4.27), (4.21), (4.22), (4.19), (4.20) and (4.5) provided $\varepsilon_J > 0$ and $\varepsilon_k > 0$ are chosen small enough. ∎

It is easily observed that for any collection $\mathscr{C}$ of dyadic rectangles, either $\mathscr{C}$ or $\mathscr{D} \times \mathscr{D} \setminus \mathscr{C}$ satisfies the hypothesis of theorem (see [C]). Using this stability property, it is now clear that:

THEOREM 5. *$H^1(\delta^2)$ is primary.*

5. Examples and remarks

In this section we discuss several observations which relate results concerning the 1-dimensional dyadic H^1 to the construction given above.

For a collection $\mathscr{A}$ of dyadic intervals its "Carleson constant" is given by

$$CC\{\mathscr{A}\} := \sup_{I \in \mathscr{A}} \sum_{\{J \in \mathscr{A}:\, J \subset I\}} \frac{|J|}{|I|}.$$

This quantity, which is of great importance to questions of classical function theory (see [Ga]), determines the relation of the subspace span $\{h_I: I \in \mathscr{A}\}$ of H^1 to the spaces l^1 and H_n^1 (see [M]).

The next observation which may be considered as a geometric version of Ramsey's theorem shows that Carleson's condition is also relevant to detect copies of $H_n^1 \otimes H_n^1$ in $\mathscr{C}$ or $\mathscr{D} \times \mathscr{D} \setminus \mathscr{C}$.

LEMMA 6. *For $n_0 \in \mathbf{N}$ there exists $n \in \mathbf{N}$ so that for any collection $\mathscr{C} \subset \mathscr{D}^n \times \mathscr{D}^n$, one finds $\mathscr{A}, \mathscr{B} \subset \mathscr{D}^n$ such that*

(i) *either $\mathscr{A} \times \mathscr{B} \subset \mathscr{C}$ or $\mathscr{A} \times \mathscr{B} \subset \mathscr{D}^n \times \mathscr{D}^n \setminus \mathscr{C}$,*

(ii) $\sup_{I \in \mathscr{A}} \sum_{\{J \in \mathscr{A}:\, J \subset I\}} |I|/|J| \geq 2^{n_0}$ *and* $\sup_{I \in \mathscr{B}} \sum_{\{J \in \mathscr{B}:\, J \subset I\}} |I|/|J| \geq 2^{n_0}$.

Remark. The one dimensional results of [M, Main Lemma 2] imply now that for any $n_0 \in \mathbf{N}$ and any collection $\mathscr{C} \subset \mathscr{D}^n \times \mathscr{D}^n$ (with n big enough) of dyadic rectangles either

$$\operatorname{span}\{h_{I \times J}: I \times J \in \mathscr{C}\}$$

or

$$\operatorname{span}\{h_{I \times J}: I \times J \in \mathscr{C} \setminus \mathscr{D}^n \times \mathscr{D}^n\}$$

contains well complemented copies of $H_{n_0}^1 \otimes H_{n_0}^1$.

Proof. For n_0 given, we choose $k \in \mathbf{N}$ so that $2^{k-1} \geq n_0$ and select $n \in \mathbf{N}$ so that $2^n \geq n_0 2^{2^k}$. Let $I_1 \cdots I_{2^{n+1}}$ be enumeration of the intervals in $\mathscr{D}^n$. Now we define collections

$$\mathscr{E} = \{J \in \mathscr{D}^n : I_1 \times J \notin \mathscr{D}\}$$
$$\mathscr{F} = \{J \in \mathscr{D}^n : I_1 \times J \in \mathscr{C}\}.$$

We use them to define a function

$$f(I_1) = \begin{cases} 0 & \text{if } CC\{\mathscr{E}\} \geq CC\{\mathscr{F}\} \\ 1 & \text{otherwise.} \end{cases}$$

We put

$$\mathscr{G}_1 = \begin{cases} \mathscr{E} & \text{if } CC\{\mathscr{E}\} > CC\{\mathscr{F}\} \\ \mathscr{F} & \text{otherwise.} \end{cases}$$

Having defined

$$f(I_1), \ldots, f(I_{m-1})$$
$$\mathscr{G}_1, \ldots, \mathscr{G}_{m-1}$$

we let

$$\mathscr{E} = \{J \in \mathscr{G}_{m-1} : I_m \times J \notin \mathscr{C}\}$$
$$\mathscr{F} = \{J \in \mathscr{G}_{m-1} : I_m \times J \in \mathscr{C}\}$$
$$f(I_m) = \begin{cases} 0 & \text{if } CC\{\mathscr{E}\} > CC\{\mathscr{F}\} \\ 1 & \text{otherwise.} \end{cases}$$

Finally we let

$$\mathscr{G}_m = \begin{cases} \mathscr{E} & \text{if } CC\{\mathscr{E}\} \geq CC\{\mathscr{F}\} \\ \mathscr{F} & \text{otherwise.} \end{cases}$$

Having completed the construction of f for $I_1, \ldots, I_{2^t}$, we set

$$\mathscr{T}^1 = \{J \in \mathscr{D}^k : f(J) = 1\}$$
$$\mathscr{T}^0 = \{J \in \mathscr{D}^k : f(J) = 0\}$$

Then

$$\mathscr{T}^1 \cup \mathscr{T}^0 = \mathscr{D}^k$$
$$\mathscr{T}^1 \times \mathscr{G}_{2^k} \subset \mathscr{C}$$
$$\mathscr{T}^0 \times \mathscr{G}_{2^k} \subset \mathscr{D}^k \times \mathscr{D}^k \setminus \mathscr{C}$$

and

$$CC\{\mathscr{G}_{2^k}\} \geq \frac{2^n}{2^{2^k}}.$$

Finally we let

$$\mathscr{A} := \begin{cases} \mathscr{T}^1 & \text{if } CC\{J^1\} > CC\{J^2\} \\ \mathscr{T}^2 & \text{otherwise.} \end{cases}$$

$$\mathscr{B} = \mathscr{G}_{2^k}.$$

Our initial choice of k and n gives now the result. ■

The examples constructed below should be compared with a result of J. Bourgain [B] which says that $H^1(l^2)$ is *not* isomorphic to a complemented subspace of H^1.

THEOREM 7. *There exists a sequence of uniformly complemented isometric copies of* $H^1(l_n^2)$ *in* H^1.

Proof. Fix $n \in \mathbf{N}$. We pick a subsequence $\{s_i \in \mathbf{N}\}$ of natural numbers, and a sequence of subsets R_i so that for each $i \in \mathbf{N}$, the cardinality of R_i equals n and

$$s_1 < \inf R_1 < \sup R_1 < s_2 < \cdots < s_{m-1} < \inf R_m < \sup R_m < s_m < \cdots$$

We use the sequence $\{s_i\colon i \in \mathbf{N}\}$ in the usual way to construct "Haar" functions $\tilde{h}_{(0,0)} := r_{s_1}$. Having constructed $\tilde{h}_{(k,j)}$, for $k \leq m$, and $j \leq 2^k$ we let

$$\tilde{h}_{(m+1,2j)} = \mathbf{1}_{\{\tilde{h}_{(m,j)}=1\}} r_{s_{m+1}}$$

$$\tilde{h}_{(m+1,2j+1)} = \mathbf{1}_{\{\tilde{h}_{(m,j)}=-1\}} r_{s_{m+1}}$$

To build the components of l_n^2 we use Rademacher functions associated to R^n: We denote the k-th element of R_m by m_k. The linear extension of the map

$$h_{(m,i)} \otimes e_k \mapsto \tilde{h}_{(m,i)} r_{m_k}$$

gives us an isometric embedding of $H^1(l_n^2)$ into H^1. Indeed given vectors

$\vec{a}_{(m,i)} = (a_{k,(m,i)})_{k=1}^n$ in l_n^2 we obtain

$$\left\| \sum_{m,l,k} \tilde{h}_{(m,i)} r_{m_k} a_{k(m,i)} \right\|_{H^1} = \int \left(\sum_{(mi),k} |h_{(m,i)}| a_{k(m,i)}^2 r_{m_k}^2 \right)^{1/2} dt$$
$$= \int \left(\sum_{(mi)} |h_{(m,i)}| \left(\sum_{k=1}^n a_{k(m,i)}^2 \right) \right)^{1/2} dt$$
$$= \int \left(\sum |h_{m,i}| \| \vec{a}_{(m,i)} \|_2^2 \right)^{1/2}$$
$$= \left\| \sum h_{m\,i} \vec{a}_{(m,i)} \right\|_{H^1(l_n^2)}$$

The span $\{\tilde{h}_{(m,i)r_{m_k}}\colon\ m \in \mathbf{N},\ i \leq 2^m, m_k \in R_m\}$ is complemented in H^1, because the orthogonal projection onto this subspace is bounded. (The best way to see this is to observe that the criterion in [Jo] is satisfied.)

Remark. It is also natural to ask if one can find a sequence of uniformly complemented copies of $H_n^1 \otimes H_n^1$ in H^1. A related problem is to prove that $\mathrm{BM}O(\delta^2)$ is isomorphic to $\mathrm{BM}O$.

The following discussion is included to isolate the idea of the proof of Theorem 1 in a very special, simple and one dimensional setting: Given real numbers $a_1, b_1, \ldots, b_4$ and $c_1, \ldots, c_{16}$ and consider the matrix

$$A = \begin{Bmatrix} a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 \\ b_1 & b_1 & b_1 & b_1 & b_2 & b_2 & b_2 & b_2 & b_3 & b_3 & b_3 & b_3 & b_4 & b_4 & b_4 & b_4 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \end{Bmatrix}$$

Then we form the following sums:

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} a_1 \\ b_1 \\ c_3 \end{pmatrix} + \begin{pmatrix} a_1 \\ b_3 \\ c_9 \end{pmatrix} + \begin{pmatrix} a_1 \\ b_3 \\ c_{11} \end{pmatrix} = \begin{pmatrix} a_1 \\ (b_1 + b_3)/2 \\ (c_1 + c_3 + c_9 + c_{11})/4 \end{pmatrix} 4 = v_1$$

$$\begin{pmatrix} a_1 \\ b_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} a_1 \\ b_1 \\ c_4 \end{pmatrix} + \begin{pmatrix} a_1 \\ b_3 \\ c_{10} \end{pmatrix} + \begin{pmatrix} a_1 \\ b_3 \\ c_{12} \end{pmatrix} = \begin{pmatrix} a_1 \\ (b_1 + b_3)/2 \\ (c_1 + c_4 + c_{10} + c_{12})/4 \end{pmatrix} 4 = v_2$$

$$\begin{pmatrix} a_1 \\ b_2 \\ c_5 \end{pmatrix} + \begin{pmatrix} a_1 \\ b_2 \\ c_7 \end{pmatrix} + \begin{pmatrix} a_1 \\ b_4 \\ c_{13} \end{pmatrix} + \begin{pmatrix} a_1 \\ b_4 \\ c_{15} \end{pmatrix} = \begin{pmatrix} a_1 \\ (b_2 + b_4)/2 \\ (c_5 + c_7 + c_{13} + c_{15})/4 \end{pmatrix} 4 = v_3$$

$$\begin{pmatrix} a_1 \\ b_2 \\ c_6 \end{pmatrix} + \begin{pmatrix} a_1 \\ b_2 \\ c_8 \end{pmatrix} + \begin{pmatrix} a_1 \\ b_4 \\ c_{14} \end{pmatrix} + \begin{pmatrix} a_1 \\ b_4 \\ c_{16} \end{pmatrix} = \begin{pmatrix} a_1 \\ (b_2 + b_4)/2 \\ (c_5 + c_7 + c_{13} + c_{15})/4 \end{pmatrix} 4 = v_4$$

For c_j, $j \leq 16$, $A(c_j)$ denotes the j-th column of the matrix A. l_3^2 denotes the three dimensional Hilbert space. Triangle inequality implies that

$$\sum_{j=1}^{16} \|A(c_j)\|_{\ell_3^2} \geq \sum_{k=1}^{4} \|v_k\|_{\ell_3^2}.$$

To relate the above considerations with orthogonal projections consider

$$\begin{aligned}
\tilde{h}_1 &= h_{(0,0)} \\
\tilde{h}_2 &= h_{(2,1)} + h_{(2,3)} \\
\tilde{h}_3 &= h_{(2,2)} + h_{(2,4)} \\
\tilde{h}_4 &= h_{(3,1)} + h_{(3,3)} + h_{(3,9)} + h_{(3,11)} \\
\tilde{h}_5 &= h_{(3,2)} + h_{(3,4)} + h_{(3,10)} + h_{(3,12)} \\
\tilde{h}_6 &= h_{(3,5)} + h_{(3,7)} + h_{(3,13)} + h_{(3,15)} \\
\tilde{h}_7 &= h_{(3,6)} + h_{(3,8)} + h_{(3,4)} + h_{(3,16)}.
\end{aligned}$$

Now consider

$$Pf = \sum_{k=1}^{7} (f|\tilde{h}_k) \frac{\tilde{h}_k}{\|\tilde{h}_k\|_2^2}.$$

Obviously $\|f\|_{H^1(\delta)}$ can be realized as $\sum_{j=1}^{16} 1/16 \|A(c_j)\|_{\ell_3^2}$, where A is of the form considered above, such that

$$\|Pf\|_{H^1(\delta)} = \sum_{k=1}^{4} \frac{1}{16} \|v_k\|_{\ell_3^2}.$$

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