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CHARACTERIZATIONS OF H^1 AND APPLICATIONS TO SINGULAR INTEGRALS

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ABSTRACT. We give a necessary and sufficient condition for an integrable compactly supported function with mean value zero on the line to be in the Hardy space $H^1(\mathbf{R}^1)$. As a corollary, we obtain a new characterization of $H^1(\mathbf{S}^1)$ and p independence of the spectrum of homogeneous Calderón-Zygmund operators.

1. Introduction and statement of results

It is a well-known result that if a compactly supported function is in $L^p(\mathbf{R}^n)$ and has mean value zero, then it is in the Hardy space $H^1(\mathbf{R}^n)$. In the fundamental paper [2], Calderón and Zygmund proved that L^p can be replaced by $\int_K |f|\text{Log}^+(|f|) < \infty$ for any compact subset K of $\mathbf{R}^n$. Moreover, it is known that $\text{Log}^+(|f|)$ cannot be replaced by $(\text{Log}^+(|f|))^{1-\varepsilon}$ for any $\varepsilon > 0$. Therefore the following question is natural:

Question 1. Find a necessary and sufficient size condition for an integrable compactly supported function on $\mathbf{R}^n$ with mean value zero to be in the Hardy space H^1.

Although, it is unclear whether this question can be answered with a purely size condition, we do have a satisfactory answer in dimension one.

Given a function f on the line, define almost everywhere a function m_f on $\mathbf{R}^1$ by setting

$$m_f(a) = \int_R f(x) \ln \frac{1}{|x-a|}\, dx. \tag{1}$$

The content of Theorem 1 below is that $f \in H^1(\mathbf{R}^1)$ if and only if m_f is a function of finite variation. We provide the following heuristic explanation for this. An integrable function is in $H^1(\mathbf{R}^1)$ if and only if its Hilbert transform is in $L^1(\mathbf{R}^1)$. The result in Theorem 1 states that m_f is of total variation if and only if Hf is integrable. But formally speaking, the derivative of the function m_f is the "Hilbert transform of the function f" which explains the relationship between the variation condition and the space H^1. This heuristic argument makes sense for some functions f but it cannot

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be justified for a general integrable f since the integral giving Hf is never absolutely convergent in this case. Question 1 posed above is related to the following question about Calderón-Zygmund singular operators with homogeneous kernels. Consider an operator

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbf{R}^n} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\, dy, \tag{2}$$

where Ω is an integrable function on the sphere $\mathbf{S}^{n-1}$ with mean value zero.

In 1956 Calderón and Zygmund [2] proved that T_Ω is bounded on $L^p(R^n)$ if $\int_{\mathbf{S}^{n-1}} |\Omega| \mathrm{Log}^+(|\Omega|) < \infty$. The $L\mathrm{Log}^+L$ condition is also sufficient for weak type $(1, 1)$ boundedness of T_Ω, as shown by A. Seeger [10].

Coifman-Weiss [4] and Connett [5] proved that if Ω is in the Hardy space $H^1(\mathbf{S}^{n-1})$, then T_Ω maps $L^p(\mathbf{R}^n)$ into itself. We will refer to this condition as the "H^1 condition". In Theorem 2 we discuss other characterizations of $H^1(\mathbf{S}^{n-1})$, one of which gives a very simple proof of the results of Coifman-Weiss [4] and Connett [5].

The case $n = 2$ is studied in further detail. Define

$$m_\Omega(\xi) = \int_{\mathbf{S}^1} \Omega(\theta) \ln \frac{1}{|\langle \theta, \xi \rangle|}\, d\theta. \tag{3}$$

As a consequence of Theorem 2 below (a $\Leftrightarrow$ e) we have

$$\mathrm{Var}_{\mathbf{S}^1}(m_\Omega) < +\infty \Longleftrightarrow \Omega \in H^1(S^1).$$

which gives a characterization of the multipliers corresponding to $H^1(S^1)$ kernels, since

$$m(\xi) = \mathcal{F}(\Omega(x)/|x|^n)(\xi) = \int_{\mathbf{S}^{n-1}} \Omega(\theta) \left[\frac{\pi i}{2} sgn\langle \theta, \xi \rangle + \ln \frac{1}{|\langle \theta, \xi \rangle|} \right] d\theta$$

and therefore

$$\mathrm{Var}_{\mathbf{S}^1}(m) < +\infty \Longleftrightarrow \Omega \in H^1(\mathbf{R}^1).$$

THEOREM 1. *Let f be a compactly supported function, $f \in L^1(\mathbf{R}^1)$, $\int f(x)\, dx = 0$. Define a function m_f on a full measure subset $A_f \subseteq \mathbf{R}^1$ (i.e., $\mathbf{R}^1 \backslash A_f$ has measure zero) by*

$$m_f(y) = \int_{\mathbf{R}^1} f(x) \ln \frac{1}{|x-y|}\, dx. \tag{4}$$

Then if $\mathrm{Var}_{A_f}(m_f) < \infty$, *then*

$$\|f\|_{H^1} \le \mathrm{Var}_{A_f}(m_f) + C\,\|f\|_{L^1}$$

and

$$\mathrm{Var}_{-\infty}^{+\infty}(m_f) = \mathrm{Var}_{A_f}(m_f) < +\infty.$$

Conversely, let $f \in H^1(\mathbf{R}^1)$ *be a compactly supported function. Then*

$$m_f(a) = \langle f(\cdot), \ln \frac{1}{|\cdot - a|} \rangle$$

is well defined (via the H^1*-BMO duality) and*

$$\mathrm{Var}_{-\infty}^{+\infty}(m_f) \le C\,\|f\|_{H^1} < \infty.$$

Remark. We are uncertain as of now how to formulate a higher dimensional analogue of Theorem 1.

The next theorem gives an equivalence between different definitions of $H^1(\mathbf{S}^{n-1})$.

THEOREM 2. *Let* $\Omega \in L^1(\mathbf{S}^{n-1})$, $\int_{\mathbf{S}^{n-1}} \Omega = 0$. *Let* R_j *be the* j^{th} *Riesz operator. Then the following conditions are equivalent*:

(a) $\Omega \in H^1(\mathbf{S}^{n-1})$.
(b) $\frac{\Omega(x/|x|)}{|x|^n}\chi_{(1\le|x|\le 2)} \in H^1(\mathbf{R}^n)$.
(c) *For every* j, $R_j\left(\frac{\Omega(x/|x|)}{|x|^n}\chi_{(1\le|x|\le 2)}\right) \in L^1(\mathbf{R}^n)$.
(d) *If* $R_j\left(\frac{\Omega(x/|x|)}{|x|^n}\right) = \frac{V_j(x)}{|x|^n}$, *then* $V_j \in L^1(\mathbf{S}^{n-1})$, $j = 1 \dots n$.
(e) *(Only for* $n = 2$.) $\mathrm{Var}_{\mathbf{S}^1}(m_\Omega) < \infty$.
(f) *(Only for* $n = 2$.) $H\left(\Omega(e^{2\pi i x}\chi_{[0,1]}\right) \in L^1(\mathbf{R}^1)$.

As an easy consequence of Theorem 2, we obtain the following

COROLLARY 1 *(Coifman-Weiss 77, Connett 79). Let* T_Ω *be defined as in* (2). *Then if* $\Omega \in H^1(\mathbf{S}^{n-1})$, *then* T_Ω: $L^p \to L^p$, $1 < p < \infty$.

COROLLARY 2. *Using the implication* $a \Rightarrow c$ *in Theorem* 2 *we can easily deduce* L^p *bounds for the maximal operator*

$$T_\Omega^* f(x) = \sup_{\varepsilon>0}\left| \int_{|y|>\varepsilon} \frac{\Omega(y/|y|)}{|y|^n} f(x-y)\,dy \right|.$$

We refer the reader to [7] and [8] for detailed proof of the corollaries. We omit the proofs of both corollaries.

As another application of the results in Theorem 2, we prove p independence of the spectrum of Calderón-Zygmund operators with H^1 kernels. More precisely, we have

PROPOSITION 1. *Let $\Omega \in H^1(\mathbf{S}^1)$ and $1 < p < \infty$. Let $m = \mathcal{F}(\Omega(\cdot/|\cdot|)/|\cdot|^2)$ be the multiplier corresponding to T_Ω. Then*

$$\sigma(T_{\Omega,p}) = \sigma(T_{\Omega,2}) = \text{essrange}\, m,$$

where $T_{\Omega,p} = T_\Omega\colon L^p \to L^p$.

Proof. The implication $\sigma(T_{\Omega,2}) \subseteq \sigma(T_{\Omega,p})$ is trivial, for if $\lambda \in \rho(T_{\Omega,p})$, then $(T_\Omega - \lambda)^{-1}\colon L^p \to L^p$ and therefore by duality and interpolation $(T_\Omega - \lambda)^{-1}\colon L^2 \to L^2$. The converse follows essentially from Theorem 2. Indeed pick $\lambda \notin \text{essrange}\, m$. Since $\text{Var}_{\mathbf{S}^1}(m) < \infty$, we have $\text{Var}_{\mathbf{S}^1}((m(\cdot) - \lambda)^{-1}) < \infty$ too and by Theorem 2, we conclude that the multiplier $\tilde{m}(\cdot) = (m(\cdot) - \lambda)^{-1}$ gives rise to an operator $T_{\tilde{\Omega}}$ with $\tilde{\Omega} \in H^1$. Therefore

$$(T_\Omega - \lambda)^{-1} = T_{\tilde{\Omega}}\colon L^p \to L^p. \quad \square$$

Remark. To the best of our knowledge, the question of the p independence of the spectrum remains open in higher dimensions ($n \geq 3$).

2. Proof of Theorem 1

In this section we give a characterization of compactly supported functions in $H^1(\mathbf{R}^1)$. Let us remark that the compact support condition is necessary, since otherwise the function m_f may not be well defined even *a.e.* By Fubini's theorem, if $f \in L^1(\mathbf{R}^1)$ has compact support then indeed $m_f(a) = \int_{\mathbf{R}^1} f(x) \ln \frac{1}{|x-a|}\, dx$ is defined on a full measure subset $A_f \subset \mathbf{R}^1$. We will pass gradually from simple functions to Lipschitz functions and then we extend our result to arbitrary compactly supported function in L^1.

Let us start off with the simplest possible case, a simple function supported in $[0, 1]$. Set $f(x) = \sum_{i=0}^{2^k-1} c_i \chi_{(a_i, a_{i+1})}$, where $a_i = i/2^k$. For symmetry, set $c_i = 0$ for $i < 0$.

LEMMA 1. *For any f simple function as above we have*

$$2^{-k} \sum_{i_0=-2^k+1}^{2^{k+1}-1} |b_{i_0}| - C\, \|f\|_{L^1} \leq \int_{-1}^{2} |Hf(x)|\, dx \leq 2^{-k} \sum_{i_0=-2^k+1}^{2^{k+1}-1} |b_{i_0}| + C\, \|f\|_{L^1},$$

where

$$b_{i_0} = \sum_{i=0, i\neq i_0}^{2^k-1} \frac{c_i}{i-i_0}.$$

Proof. We have

$$Hf(x) = \int_0^1 \frac{f(y)}{x-y}\,dy = \sum_{i=0}^{2^k-1} c_i \int_{a_i}^{a_{i+1}} \frac{dy}{x-y} = \sum_{i=0}^{2^k-1} c_i \ln\left|\frac{x-a_i}{x-a_{i+1}}\right|.$$

Note that

$$(5)\ [-1+2^{-k-1}, 2-2^{-k-1}] = \bigcup_{i_0=-2^k+1}^{2^{k+1}-1} \bigcup_{n=k+2}^{+\infty} \{x\colon 2^{-n} \le |x-i_0 2^{-k}| \le 2^{-n+1}\}.$$

If $i \neq i_0, i_0-1$ and $2^{-n} \le |x-i_0 2^{-k}| \le 2^{-n+1}$, then

$$\ln\left|\frac{x-a_i}{x-a_{i+1}}\right| = \ln\left(1+\frac{1}{i-i_0} + O\left(\frac{1}{(i-i_0)^2}\right)\right),$$

since $2^k|x-i_0 2^{-k}| \le 2^{k-n+1} \le 1/2$. Therefore

$$\ln\left|\frac{x-a_i}{x-a_{i+1}}\right| = \frac{1}{i-i_0} + O\left(\frac{1}{(i-i_0)^2}\right).$$

Observe also that

$$\left|\ln\left|\frac{x-a_{i_0-1}}{x-a_{i_0}}\right| - (n-k)\right| \le 1,$$

$$\left|\ln\left|\frac{x-a_{i_0}}{x-a_{i_0+1}}\right| - (k-n)\right| \le 1,$$

when $2^{-n} \le |x-a_{i_0}| \le 2^{-n+1}$. Thus

$$(6)\ Hf(x) = \sum_{i\neq i_0, i_0-1} \frac{c_i}{i-i_0} + (c_{i_0}-c_{i_0-1})(k-n+O(1)) + O\left(\sum_{i\neq i_0} \frac{c_i}{(i-i_0)^2}\right),$$

for $2^{-n} \le |x-a_{i_0}| \le 2^{-n+1}$. We integrate (6) to get

$$\int\limits_{2^{-n}\le|x-a_{i_0}|\le 2^{-n+1}} |Hf(x)|\,dx = 2^{-n+1}\left|\sum_{i\neq i_0, i_0-1} \frac{c_i}{i-i_0} + (c_{i_0}-c_{i_0-1})(k-n+O(1))\right|$$

$$+ 2^{-n} O\left(\sum_{i\neq i_0} \frac{c_i}{(i-i_0)^2}\right).$$

We sum over n and use (5) to get

$$\int_{0<|x-a_{i_0}|<2^{-k-1}} |Hf(x)|\,dx$$
$$\geq \sum_{n=k+2}^{\infty} \frac{|b_{i_0}|}{2^{n-1}} - \frac{(|c_{i_0}|+|c_{i_0-1}|)(n-k-C)}{2^{n-1}} - C2^{-n+1}\sum_{i\neq i_0}\frac{|c_i|}{(i-i_0)^2},$$

$$\int_{0<|x-a_{i_0}|<2^{-k-1}} |Hf(x)|\,dx$$
$$\leq \sum_{n=k+2}^{\infty} \frac{|b_{i_0}|}{2^{n-1}} + \frac{(|c_{i_0}|+|c_{i_0-1}|)(n-k+C)}{2^{n-1}} + c2^{-n+1}\sum_{i\neq i_0}\frac{|c_i|}{(i-i_0)^2}.$$

Taking into account that $\sum_{n=k+2}^{\infty}(n-k\pm C)2^{-n+1} \sim 2^{-k}$, we conclude that

$$\sum_{i_0=-2^k+1}^{2^{k+1}-1}\left[\frac{|b_{i_0}|}{2^k} - c\left(\frac{|c_{i_0}|+|c_{i_0-1}|}{2^k} + 2^{-k}\sum_{i\neq i_0}\frac{|c_i|}{(i-i_0)^2}\right)\right] \leq \int_{-1+2^{-k-1}}^{2-2^{-k-1}} |Hf(x)|\,dx,$$

$$\int_{-1+2^{-k-1}}^{2-2^{-k-1}} |Hf(x)|\,dx \leq \sum_{i_0=-2^k+1}^{2^{k+1}-1}\left[\frac{|b_{i_0}|}{2^k} + c\left(\frac{|c_{i_0}|+|c_{i_0-1}|}{2^k} + 2^{-k}\sum_{i\neq i_0}\frac{|c_i|}{(i-i_0)^2}\right)\right]$$

and therefore

$$2^{-k}\sum_{i_0=-2^k+1}^{2^{k+1}-1}|b_{i_0}| - c2^{-k}\left(\sum_{i_0=0}^{2^k-1}|c_{i_0}| + \sum_{i=0}^{2^k-1}|c_i|\right) \leq \int_{-1+2^{-k-1}}^{2-2^{-k-1}} |Hf(x)|\,dx,$$

$$\int_{-1+2^{-k-1}}^{2-2^{-k-1}} |Hf(x)|\,dx \leq 2^{-k}\sum_{i_0=-2^k+1}^{2^{k+1}-1}|b_{i_0}| + C2^{-k}\left(\sum_{i_0=0}^{2^k-1}|c_{i_0}| + \sum_{i=0}^{2^k-1}|c_i|\right).$$

The same calculations applied on the intervals $[-1,-1+2^{-k-1}]$ and $[2-2^{-k-1},2]$ give

$$\int_{-1}^{-1+2^{-k-1}} |Hf(x)|\,dx \leq \|f\|_{L^1},\ \int_{2-2^{-k-1}}^{2} |Hf(x)|\,dx \leq \|f\|_{L^1}.$$

Since $\|f\|_{L^1} = 2^{-k}\sum_{i=0}^{2^k-1}|c_i|$, we finally get the desired inequality. □

LEMMA 2. *With notations as in Lemma* 1 *we have*

$$\frac{b_{i_0}}{2^k} = \left(\int_0^1 f(x)\ln\frac{1}{|x-a_{i_0+1}|}\,dx - \int_0^1 f(x)\ln\frac{1}{|x-a_{i_0}|}\,dx\right) + 2^{-k} O\left(\sum_{i\neq i_0}\frac{|c_i|}{(i-i_0)^2}\right).$$

Proof.

$$\int_0^1 f(x)\ln\frac{1}{|x-a_{i_0+1}|}\,dx - \int_0^1 f(x)\ln\frac{1}{|x-a_{i_0}|}\,dx$$

$$= \int_0^1 f(x)\ln\frac{|x-a_{i_0}|}{|x-a_{i_0+1}|}\,dx$$

$$= \sum_{i=0, i\neq i_0}^{2^k-1} c_i \int_{a_i}^{a_{i+1}} \ln\frac{|x-a_{i_0}|}{|x-a_{i_0+1}|}\,dx + c_{i_0}\int_{a_{i_0}}^{a_{i_0+1}} \ln\frac{|x-a_{i_0}|}{|x-a_{i_0+1}|}\,dx.$$

Case 1. $i < i_0$ and $x \in (a_i, a_{i+1})$.
We have

$$\frac{|x-a_{i_0}|}{|x-a_{i_0+1}|} = 1 - \frac{1}{2^k|x-a_{i_0+1}|} = 1 + \frac{1}{i-i_0} + O\left(\frac{1}{(i-i_0)^2}\right).$$

Thus

$$c_i\int_{a_i}^{a_{i+1}} \ln\frac{|x-a_{i_0}|}{|x-a_{i_0+1}|}\,dx = 2^{-k}\frac{c_i}{i-i_0} + 2^{-k}c_i\, O\left(\frac{1}{(i-i_0)^2}\right).$$

Case 2. $i > i_0$ and $x \in (a_i, a_{i+1})$.
A similar argument shows that

$$c_i\int_{a_i}^{a_{i+1}} \ln\frac{|x-a_{i_0}|}{|x-a_{i_0+1}|}\,dx = 2^{-k}\frac{c_i}{i-i_0} + 2^{-k}c_i\, O\left(\frac{1}{(i-i_0)^2}\right).$$

Finally since

$$\int_{a_{i_0}}^{a_{i_0+1}} \ln\frac{|x-a_{i_0}|}{|x-a_{i_0+1}|}\,dx = 0,$$

we get

$$\int_0^1 f(x)\ln\frac{|x-a_{i_0}|}{|x-a_{i_0+1}|}\,dx = 2^{-k}b_{i_0} + 2^{-k}O\left(\sum_{i\neq i_0}\frac{|c_i|}{(i-i_0)^2}\right). \qquad \square$$

Observe also that

$$2^{-k}\sum_{i_0=-2^k+1}^{2^{k+1}-1}|b_{i_0}| = \sum_{i_0=-2^k+1}^{2^{k+1}-1}\left|\int_0^1 f(x)\ln\frac{1}{|x-a_{i_0+1}|}\,dx - \int_0^1 f(x)\ln\frac{1}{|x-a_{i_0}|}\,dx\right|$$

$$+ 2^{-k}O\left(\sum_{i_0}\sum_{i\neq i_0}\frac{|c_i|}{(i-i_0)^2}\right)$$

$$= \sum_{i_0=-2^k+1}^{2^{k+1}-1}\left|\int_0^1 f(x)\ln\frac{1}{|x-a_{i_0+1}|}\,dx - \int_0^1 f(x)\ln\frac{1}{|x-a_{i_0}|}\,dx\right|$$

$$+ O\left(2^{-k}\sum_{i=1}^{2^k}|c_i|\right).$$

By this observation and Lemma 1, we obtain

$$\sum_{i_0=-2^k+1}^{2^{k+1}-1}\left|\int_0^1 f(x)\ln\frac{|x-a_{i_0}|}{|x-a_{i_0+1}|}\right|dx - C\,\|f\|_1 \le \int_{-1}^2 |Hf(x)|\,dx, \tag{7}$$

$$\int_{-1}^2 |Hf(x)|\,dx \le \sum_{i_0=-2^k+1}^{2^{k+1}-1}\left|\int_0^1 f(x)\ln\frac{|x-a_{i_0}|}{|x-a_{i_0+1}|}\right|dx + C\,\|f\|_1. \tag{8}$$

This is basically the assertion of Theorem 1 for simple functions, since it is easy to control the contribution of $\int_{-\infty}^{-1}|Hf(x)|\,dx$ and $\int_2^{\infty}|Hf(x)|\,dx$.

LEMMA 3. *Let $f\colon \mathbf{R}^1 \to \mathbf{R}^1$ be a Lipschitz function and* $\operatorname{supp} f \subseteq [0, 1]$. *Let* $m_f(a) = \int_0^1 f(x)\ln\frac{1}{|x-a|}\,dx$. *Then*

$$\operatorname{Var}_{-1}^{+2}(m_f) - c\,\|f\|_{L^1} \le \int_{-1}^2 |Hf(x)|\,dx \le \operatorname{Var}_{-1}^{+2}(m_f) + c\,\|f\|_{L^1}, \tag{9}$$

where c is a constant independent of f.

Proof. Note that there exists a sequence of simple functions $\{f_k\}$, such that

(1) $f_k \to f$ pointwise, $|f_k| \le |f|$,

(2) $f_k = \sum_{i=0}^{2^k-1} c_i\chi_{(a_i,a_{i+1})}$,

(3) $|f_k(x) - f(x)| \leq L2^{-k}$ for every $x \in [0, 1]$, where L is the Lipschitz constant.

These can be taken to be the lower Darboux sums for f corresponding to the uniform partition $0 = a_1 \leq \ldots \leq a_{2^k} = 1$.

I. Let us start with the left inequality first. We may apply (7) to each individual f_k (these are simple functions supported on [0, 1]):

$$\sum_{i_0=-2^k+1}^{2^{k+1}-1} \left| \int_0^1 f_k(x) \ln \frac{|x - a_{i_0}|}{|x - a_{i_0+1}|} \right| dx - C \, \|f_k\|_1 \leq \int_{-1}^2 |Hf_k(x)| \, dx. \tag{10}$$

Also

$$\int_{-1}^2 |Hf_k(x)| \, dx \to \int_{-1}^2 |Hf(x)| \, dx. \tag{11}$$

On the other hand, m_f is well defined and continuous (due to the Lipschitz assumption on f), so the total variation of this function can be evaluated as a total variation on a dense subset. In particular,

$$\mathrm{Var}_{-1}^{+2}(m_f) = \sup_{-1 \leq b_0 \leq \ldots \leq b_s \leq 2} \sum_{i=0}^{s-1} |m_f(b_{i+1}) - m_f(b_i)|,$$

where the supremum is taken on a dyadic partition of $[-1, 2]$, i.e., $b_i = q_i/2^{n_i}$ $i = 0, \ldots, s$ for some integers q_i and n_i. Let us fix such a partition. Choose $n = \max\{n_1, \ldots, n_s\}$. Then obviously the b_i's will be part of the dyadic partition with diameter 2^{-n} and therefore part of any dyadic partition with diameter 2^{-k}, $k \geq n$. Consider (10) for $k \geq n$. Since the b_i's are part of the dyadic partition (with diameter 2^{-k}), we have

$$\begin{aligned}
&\sum_{i=0}^{s-1} \left| \int_0^1 f_k(x) \ln \frac{1}{|x - b_{i+1}|} \, dx - \int_0^1 f_k(x) \ln \frac{1}{|x - b_i|} \, dx \right| \\
&\leq \sum_{i_0=-2^k+1}^{2^{k+1}-1} \left| \int_0^1 f_k(x) \ln \frac{1}{|x - a_{i_0+1}|} \, dx - \int_0^1 f_k(x) \ln \frac{1}{|x - a_{i_0}|} \, dx \right| \\
&\leq \int_{-1}^2 |Hf_k(x)| \, dx + C \, \|f_k\|_1
\end{aligned}$$

and therefore taking limit in k yields

$$\sum_{i=0}^{s-1} \left| \int_0^1 f(x) \ln \frac{1}{|x - b_{i+1}|} \, dx - \int_0^1 f(x) \ln \frac{1}{|x - b_i|} \, dx \right| \leq \int_{-1}^2 |Hf(x)| \, dx + C \, \|f\|_1 .$$

Since the partition was arbitrarily chosen,

$$\mathrm{Var}_{-1}^{+2}(m_f) \le \int_{-1}^{2} |Hf(x)|\,dx + C\,\|f\|_1 .$$

II. The proof of the right-hand side in (9) is more involved. First of all, consider sequence $\{f_k\}$ of simple functions as above. For this part of the proof we use (8). We have

$$\int_{-1}^{2} |Hf_k(x)|\,dx \le \sum_{i_0=-2^k+1}^{2^{k+1}-1} \left| \int_0^1 f_k(x) \ln \frac{|x-a_{i_0}|}{|x-a_{i_0+1}|}\,dx \right| + C\,\|f_k\|_1 ,$$

$$\sum_{i_0=-2^k+1}^{2^{k+1}-1} \left| \int_0^1 f_k(x) \ln \frac{|x-a_{i_0}|}{|x-a_{i_0+1}|}\,dx \right| \le A_k + B_k,$$

where

$$A_k = \sum_{i_0=-2^k+1}^{2^{k+1}-1} \left(\int_0^1 |f_k(x)-f(x)| \left| \ln \frac{|x-a_{i_0}|}{|x-a_{i_0+1}|} \right| dx \right),$$

$$B_k = \sum_{i_0=-2^k+1}^{2^{k+1}-1} \left| \int_0^1 f(x) \ln \frac{|x-a_{i_0}|}{|x-a_{i_0+1}|}\,dx \right| .$$

Clearly $B_k \le \mathrm{Var}_{-1}^{+2}(m_f) < \infty$. For A_k we use the estimate

$$A_k \le \sum_{i_0=-2^k+1}^{2^{k+1}-1} \sum_{|i-i_0|>1} \int_{a_i}^{a_{i+1}} |f_k(x)-f(x)| \left| \ln \frac{|x-a_{i_0}|}{|x-a_{i_0+1}|} \right| dx$$

$$+ \int_{a_{i_0-1}}^{a_{i_0+2}} |f_k(x)-f(x)| \left| \ln \frac{|x-a_{i_0}|}{|x-a_{i_0+1}|} \right| dx.$$

As in Lemma 1, if $x \in (a_i, a_{i+1})$ and $i \ne i_0-1, i_0, i_0+1$, then

$$\ln \frac{|x-a_{i_0}|}{|x-a_{i_0+1}|} = \frac{1}{i-i_0} + O\left(\frac{1}{(i-i_0)^2}\right).$$

Taking into account the choice of f_k's (condition 3), we obtain

$$\int_{a_i}^{a_{i+1}} |f_k(x)-f(x)| \left| \ln \frac{|x-a_{i_0}|}{|x-a_{i_0+1}|} \right| dx \le L\left(\frac{1}{|i-i_0|} + \frac{C}{(i-i_0)^2}\right) 2^{-2k}, \tag{12}$$

$$(13)\quad \int\limits_{a_{i_0-1}}^{a_{i_0+2}} |f_k(x)-f(x)| \left|\ln\frac{|x-a_{i_0}|}{|x-a_{i_0+1}|}\right| dx \le CL2^{-k}\int\limits_0^{2^{-k}} \ln\frac{1}{x}\,dx \le CLk2^{-2k}.$$

Thus

$$\begin{aligned} A_k &\le \sum_{i_0=-2^k+1}^{2^{k+1}-1} \sum_{i\neq i_0}\left(\frac{1}{|i-i_0|}+\frac{C}{(i-i_0)^2}\right)2^{-2k}L + \sum_{i_0=-2^k+1}^{2^{k+1}-1} Lk2^{-2k} \\ &\le C\sum_{i_0=-2^k+1}^{2^{k+1}} \left(Lk2^{-2k}\right) + CL2^{-k}k = CLk2^{-k} \to 0. \end{aligned}$$

Thus

$$\limsup_{k\to\infty}\int_{-1}^{2} |Hf_k(x)|\,dx \le \mathrm{Var}_{-1}^{+2}(m_f) + C\,\|f\|_1 .$$

The only fact that remains to be proved is

$$(14)\quad \int\limits_{-1}^{2} |Hf(x)|\,dx \le \limsup_{k\to\infty}\int\limits_{-1}^{2} |Hf_k(x)|\,dx.$$

But since $f_k \to f$ in L^1, we have that for every $\lambda > 0$

$$|\{x\colon |Hf_k(x)-Hf(x)| \ge \lambda\}| \le \frac{C}{\lambda}\,\|f_k - f\|_1 \to 0$$

and therefore $Hf_k \to Hf$ in measure. Then select a subsequence $\{f_{k_n}\}$ so that $Hf_{k_n} \to Hf$ a.e. The assertion follows from the Fatou's lemma. □

The next lemma follows from Lemma 3 by dilation and translation.

LEMMA 4. *Let f be a Lipschitz continuous function with compact support, such that $\int f(x)\,dx = 0$. Then*

$$(15)\quad \mathrm{Var}_{-\infty}^{+\infty}(m_f) - c\,\|f\|_1 \le \int_{-\infty}^{+\infty} |Hf(x)|\,dx \le \mathrm{Var}_{-\infty}^{+\infty}(m_f) + c\,\|f\|_1 .$$

Proof. If $\operatorname{supp} f \subseteq [0,1]$, then it is easy to see that

$$|Hf(x)|\chi_{(x\notin[-1,2])} \le C\,\|f\|_1\,|x|^{-2}.$$

Also observe that m_f is differentiable outside $[-1,2]$ and

$$|m_f'(a)|\chi_{(a\notin[-1,2])} \le C\,\|f\|_1\,a^{-2}.$$

Therefore

$$0 \le \left(\int_{-\infty}^{-1} + \int_{2}^{\infty} \right) |Hf(x)|\, dx \le C \left(\int_{-\infty}^{-1} + \int_{2}^{\infty} \right) \|f\|_1 / |x|^2\, dx \le C \|f\|_{L^1},$$

$$0 \le \mathrm{Var}_{-\infty}^{-1}(m_f) + \mathrm{Var}_{+2}^{+\infty}(m_f) \le C \|f\|_{L^1},$$

and combining the last two inequalities with Lemma 3 we get (15).

The general case follows by an easy dilation and translation argument. To this end, notice that every compactly supported function f can be recovered from a function $\bar{f}$ with dilation δ_t and translation τ_a, where $\bar{f}$: $\operatorname{supp}\bar{f} \subseteq [0, 1]$. Observe that

$$H(\delta_t \circ f) = \delta_t \circ (Hf),$$
$$m_{\delta_t \circ f}(a) = m_f(a/t)$$

where the second identity holds because of the mean value zero property of f. Having these in mind one easily gets

$$\|H(\delta_t \circ f)\|_1 = \|\delta_t \circ (Hf)\|_1 = \|Hf\|_1,$$
$$\mathrm{Var}_{-\infty}^{+\infty}\left(m_{\delta_t \circ f}\right) = \mathrm{Var}_{-\infty}^{+\infty}\left(m_f\left(\cdot/t\right)\right) = \mathrm{Var}_{-\infty}^{+\infty}(m_f).$$

It is also straightforward to check that the translation does not affect the expressions involved in (15) and therefore the lemma follows. □

We are finally ready to proceed with the proof of Theorem 1.

Proof of Theorem 1. Let Φ be a C^∞ function supported in $[0, 1]$ and $\int |\Phi(y)|\, dy = 1$. Define $f_k = f * \Phi_{1/k}$. We know that f_k is a smooth function with mean value zero ($\hat{f}_k(0) = 0$) and has compact support. Thus, we are in a position to apply Lemma 4. We have

$$\|Hf_k\|_{L^1} \le \mathrm{Var}_{-\infty}^{+\infty}(m_{f_k}) + C \|f_k\|_{L^1}, \tag{16}$$

where $m_{f_k}(a) = \int_R \left(f * \Phi_{1/k}\right)(x) \ln \frac{1}{|x-a|}\, dx$. But

$$\begin{aligned} m_{f_k}(a) &= \int \left(\int \Phi_{1/k}(y) f(x-y)\, dy \right) \ln \frac{1}{|x-a|}\, dx \\ &= \int \left(\int_{a-A_f} \Phi_{1/k}(y) f(z) \ln \frac{1}{|z-(a-y)|}\, dy \right) dz \\ &= \int \Phi_{1/k}(y) m_f(a-y)\, dy, \end{aligned}$$

where we used Fubini's theorem (m_f is defined and an absolutely convergent integral on A_f, which is a full measure subset of $\mathbf{R}^1$).

So

$$m_{f_k}(a) = \int \Phi_{1/k}(y) m_f(a-y)\,dy. \tag{17}$$

Take any sequence of points $a_1 < a_2 \ldots < a_n$. Notice that since A_f is a full measure set, so is $\bigcap_{i=1}^n (a_i - A_f)$. Thus

$$\begin{aligned}
\sum_{i=1}^{n-1} &|m_{f_k}(a_{i+1}) - m_{f_k}(a_i)| \\
&= \sum_{i=1}^{n-1} \left| \int \Phi_{1/k}(y)(m_f(a_{i+1}-y) - m_f(a_i - y))\,dy \right| \\
&\leq \int_{\bigcap (a_i - A_f)} \left|\Phi_{1/k}(y)\right| \left[\sum_{i=1}^{n-1} |m_f(a_{i+1}-y) - m_f(a_i-y)| \right] dy \\
&\leq \int \left|\Phi_{1/k}(y)\right| \operatorname{Var}_{A_f}(m_f)\,dy = \operatorname{Var}_{A_f}(m_f).
\end{aligned}$$

Therefore

$$\operatorname{Var}_{-\infty}^{+\infty}(m_{f_k}) \leq \operatorname{Var}_{A_f}(m_f)$$

and in particular, by (16),

$$\|Hf_k\|_{L^1} \leq \operatorname{Var}_{A_f}(m_f) + C\,\|f_k\|_{L^1}. \tag{18}$$

We now finish with an argument similar to the one in Lemma 3. Select a subsequence such that $Hf_{n_k} \to Hf$ a.e. An application of Fatou's lemma then gives

$$\|Hf\|_{L^1} - C\,\|f\|_{L^1} \leq \operatorname{Var}_{A_f}(m_f)$$

and this proves the first assertion in Theorem 1.

The converse direction of the proof is in fact easier, since everything is correctly defined via the $H^1 - BMO$ duality. Let $f \in H^1$ and $f_k = f * \Phi_{1/k}$ as above. Apply (15) to get

$$\operatorname{Var}_{-\infty}^{+\infty}(m_{f_k}) - C\,\|f_k\|_{L^1} \leq \|Hf_k\|_{L^1}. \tag{19}$$

Observe that $f_k = f * \Phi_{1/k} \to f$ in H^1 sense. (See [11], p. 127, 5.1(c).) Therefore

$$\begin{aligned}
&Hf_k \to Hf \quad \text{in} \quad L^1, \\
&m_{f_k}(a) = \left\langle f_k, \ln\frac{1}{|\cdot - a|}\right\rangle \to \left\langle f, \ln\frac{1}{|\cdot - a|}\right\rangle = m_f(a).
\end{aligned}$$

Taking limits in k in (19), we obtain

$$\operatorname{Var}_{-\infty}^{+\infty}(m_f) \leq \|Hf\|_{L^1} + C\,\|f\|_{L^1} \leq C\,\|f\|_{H^1}. \qquad \square$$

3. Characterization of $H^1(\mathbf{S}^{n-1})$

In this section we prove Theorem 2, which gives equivalence between various notions of $H^1(\mathbf{S}^{n-1})$. Let us recall that (a) $\Leftrightarrow$ (d) in Theorem 2 is the Ricci-Weiss result in [9].

Proof of Theorem 2. (a) $\Leftrightarrow$ (b) Let $\Omega(\theta) = \sum_i \lambda_i a_i(\theta)$, where a_i are $H^1(\mathbf{S}^{n-1})$ atoms. It clearly suffices to check that

$$\left\| \frac{a(x/|x|)}{|x|^n} \chi_{(1<|x|<2)} \right\|_{H^1} \leq C,$$

where C is uniform on H^1 atoms. Let $b(x) = \frac{a(x/|x|)}{|x|^n}\chi_{(1<|x|<2)}$. Also, denote by $J(\varphi_1, \ldots, \varphi_{n-1})$ the Jacobian factor of the unit sphere surface measure as a function of the polar angles. In what follows we will always assume that a vector $x \in \mathbf{R}^n$ is represented by its polar coordinates $(\varphi_1, \ldots, \varphi_{n-1}, r)$, where the first $(n-1)$ components are the polar angles and the last one is the polar radius. Hence

$$J(\varphi_1, \ldots, \varphi_{n-1}) = |(\sin^{n-2}\varphi_1)(\sin^{n-3}\varphi_2)\ldots(\sin\varphi_{n-2})|.$$

Clearly, $J(\varphi_1, \ldots, \varphi_{n-1})$ does not vanish on an open set on the unit sphere and therefore without loss of generality we may assume that

$$\text{diam(supp a)} < \delta_n,$$
$$0 < c_n \leq J(\varphi_1, \ldots, \varphi_{n-1}) \leq 1 \text{ on the support of } a$$

where the small constants δ_n, c_n depend only on the dimension n. These can be achieved; otherwise, if diam(supp a) $\geq \delta_n$, we use the trivial estimate $\|b\|_{H^1} \leq \delta_n^{-n+1}$. Define the auxiliary function

$$\tilde{b}(\varphi_1, \ldots, \varphi_{n-1}, r) = \frac{a(\varphi_1, \ldots, \varphi_{n-1})J(\varphi_1, \ldots, \varphi_{n-1})}{r}\chi_{(1<r<2)}.$$

We prove that $\|\tilde{b}\|_{H^1(\mathbf{R}^n)} \leq C$. Observe that $a(\varphi_1, \ldots, \varphi_{n-1})J(\varphi_1, \ldots, \varphi_{n-1})$ is a fixed multiple of an $H^1(\mathbf{R}^{n-1})$ atom. Consider $g(\varphi_1, \ldots, \varphi_{n-1}, r) = \psi(\varphi_1, \ldots, \varphi_{n-1})h(r)$, where

(1) $\psi \geq 0$; $h \geq 0$,
(2) supp $\psi \subset B(0, 1)$; supp $h \subset B(0, 1)$,
(3) $\int \psi = 1$, $\int h = 1$.

Define

$$M_g\tilde{b}(x) = \sup_{t>0} |\tilde{b} * g_t|(x).$$

The idea is to show that this maximal function has enough decay at infinity (due only to the fact that $\tilde{b}$ has mean value zero and is an L^1 function), while around the origin we exploit the structure of $\tilde{b}$. Obviously supp $\tilde{b} \subset [0, 2\pi]^{n-1} \times [1, 2]$. Choose a compact set K, so that supp $\tilde{b} \subset K$ and dist$(K^c, \text{supp}\, \tilde{b}) \geq 1$. This can be done independently of a, so every constant that appear later on, depending on K is actually uniform with respect to a. For $x \in K^c$ we have

$$\begin{aligned} \tilde{b} * g_t(x) &= \frac{1}{t^n} \int \tilde{b}(z) g\left(\frac{x-z}{t}\right) dz \\ &= \frac{1}{t^{n+1}} \int \tilde{b}(z) \nabla g\left(\frac{x - s_z z}{t}\right) dz. \end{aligned}$$

Since supp $g \subset B(0, 2)$ and $x \in K^c$ we have

$$\frac{1}{t} \leq \frac{2}{|x - s_z z|} \leq \frac{C}{|x|}.$$

thus we get

$$M_g \tilde{b}(x) = \sup_{t>0} |\tilde{b} * g_t|(x) \leq C/|x|^{n+1}, \tag{20}$$

which is the desired estimate at infinity. For $x \in K$,

$$|\tilde{b} * g_t|(\varphi_1, \dots, \varphi_{n-1}, r) = |(a\ J) * \psi_t|(\varphi_1, \dots, \varphi_{n-1}) \left| \frac{1}{r} \chi_{(1<r<2)} * h_t \right| (r).$$

which yields

$$M_g \tilde{b}(\varphi_1, \dots, \varphi_{n-1}, r) \leq M_\psi(a\ J) M\left(\chi_{(1<r<2)}/r\right).$$

Thus by Cauchy-Schwartz we obtain

$$\begin{aligned} \int_K M_g \tilde{b}(x)\, dx &\leq \left\|M_\psi(aJ)\right\|_{L^1(\mathbf{R}^{n-1})} \sqrt{\pi_2(K)} \left\|M\left(\chi_{(1<r<2)}/r\right)\right\|_{L^2} \\ &\leq C \left\|M_\psi(aJ)\right\|_{L^1(\mathbf{R}^{n-1})} \sqrt{\pi_2(K)} \left\|\chi_{(1<r<2)}/r\right\|_{L^2} \leq C, \end{aligned}$$

where $\pi_2(K)$ is the measure of the projection of K on e_2. This, together with (20), implies $\|\tilde{b}\|_{H^1(\mathbf{R}^n)} \leq C$. By the atomic decomposition in [3], we can represent $\tilde{b}$ as a sum of atoms whose supports are in a small neighborhood of the support of the function. Let $\tilde{b}(\varphi_1, \dots, \varphi_{n-1}, r) = \sum \lambda_i \tilde{b}_i(\varphi_1, \dots, \varphi_{n-1}, r)$, where the angular parts of the supports of the $\tilde{b}_i$'s are supported in a small neighborhood of supp a. Let

$$b_i(x) = \frac{1}{r^{n-1}} \frac{\tilde{b}_i(\varphi_1, \dots, \varphi_{n-1}, r)}{J(\varphi_1, \dots, \varphi_{n-1})}.$$

Observe that $b(x) = \sum_i \lambda_i b_i(x)$ and

$$\int b_i(x)\,dx = \int \tilde{b}_i(\varphi_1, \ldots, \varphi_{n-1}, r) d\varphi_1 \ldots d\varphi_{n-1} dr = 0,$$

$$|\text{supp}\, b_i| \sim |\text{supp}\, \tilde{b}_i|,$$

$$|b_i(x)| \le \sup_{\varphi,r} \frac{|\tilde{b}_i(\varphi_1, \ldots, \varphi_{n-1}, r)|}{|J(\varphi_1, \ldots, \varphi_{n-1})|} \le \frac{C}{|\text{supp}\, \tilde{b}_i|} \le \frac{C}{|\text{supp}\, b_i|}.$$

This implies that $\|\tilde{b}_i\|_{H^1} \le C$ and

$$\|b\|_{H^1} \le C \sum |\lambda_i| \le C \|\tilde{b}_i\|_{H^1} \le C. \quad \square$$

(b) $\Rightarrow$ (c) The proof of this direction consists of the trivial observation that the Riesz transforms map H^1 into L^1.

(c) $\Rightarrow$ (d) We have

$$\begin{aligned} V_j(x)/|x|^n &= R_j\left(\Omega(x)/|x|^n\right) \\ &= R_j\left(\frac{\Omega(x)}{|x|^n}\chi_{(|x|\notin[1,2])}\right) + R_j\left(\frac{\Omega(x)}{|x|^n}\chi_{(1\le|x|\le 2)}\right). \end{aligned}$$

For $4/3 \le |x| \le 3/2$, we will prove that the first function is bounded, whereas the second one is integrable by assumption. Indeed,

$$\begin{aligned} \left|R_j\left(\frac{\Omega(x)}{|x|^n}\chi_{(|\cdot|<1)}\right)(x)\right| &= c_n \lim_{\varepsilon\to 0}\left|\int_{\varepsilon<|y|<1} \frac{x_j - y_j}{|x-y|^{n+1}} \frac{\Omega(y)}{|y|^n}\,dy\right| \\ &= c_n \left|\int_{|y|<1}\left(\frac{x_j - y_j}{|x-y|^{n+1}} - \frac{x_j}{|x|^{n+1}}\right)\frac{\Omega(y)}{|y|^n}\,dy\right| \\ &\le C\,\|\Omega\|_{L^1} \max_{1/3\le|x|\le 5/2}\left|\nabla\left(\frac{x_j}{|x|^{n+1}}\right)\right| \le C\,\|\Omega\|_{L^1}\,. \end{aligned}$$

Similarly,

$$R_j\left(\frac{\Omega(\cdot)}{|\cdot|^n}\chi_{(|\cdot|>2)}\right)(x) = c_n \int_2^\infty \int_{S^{n-1}} \frac{x_j - \rho\theta_j}{|x-\rho\theta|^{n+1}} \frac{\Omega(\theta)}{\rho}\,d\theta d\rho,$$

and again for $4/3 \le |x| \le 3/2$, we get

$$R_j\left(\frac{\Omega(\cdot)}{|\cdot|^n}\chi_{(|\cdot|>2)}\right)(x) \le C \int_2^\infty \int_{S^{n-1}} |\Omega(\theta)| \frac{1}{\rho^{n+1}}\,d\theta d\rho \le C\,\|\Omega\|_{L^1}\,.$$

Observe that we actually proved

$$\frac{V_j(x)}{|x|^n}\chi_{(4/3<|x|<3/2)} \in L^1(\mathbf{R}^n),$$

which is the same as $V_j \in L^1(\mathbf{S}^{n-1})$ as stated.

(d) $\Rightarrow$ (a) This direction is contained in Theorem 2 in [9].

We pass now to the two dimensional case. First of all observe that (a) $\Leftrightarrow$ (f) is just a restatement of the fact that the Hilbert transform determines $H^1(\mathbf{R}^1)$. For the equivalence (e) $\Leftrightarrow$ (b), we use Theorem 1 in a crucial way.

(e) $\Rightarrow$ (b) By (1), we have

$$\begin{aligned} m_\Omega(e^{2\pi i a}) &= \int_0^1 \Omega(\cos 2\pi x, \sin 2\pi x) \ln \frac{1}{|\langle e^{2\pi i x}, e^{2\pi i a}\rangle|}\, dx \\ &= \int_0^1 \Omega(\cos 2\pi x, \sin 2\pi x) \ln \frac{1}{|\cos 2\pi |x-a||}\, dx. \end{aligned}$$

In what follows we shall identify $\mathbf{S}^1$ and [0, 1] via the usual exponential map. In particular we use $\mathrm{Var}_{\mathbf{S}^1}(m_\Omega)$ instead of $\mathrm{Var}_0^1(m_\Omega)$. Now, let us define

$$\tilde{m}(a) = m(e^{2\pi i(a-\frac{1}{4})}) = \int_0^1 \Omega(\cos 2\pi x, \sin 2\pi x) \ln \frac{1}{|\sin 2\pi |x-a||}\, dx.$$

Obviously

$$\mathrm{Var}_{-2}^{+2}(\tilde{m}) \le \mathrm{Var}_{\mathbf{S}^1}(m_\Omega) + C\, \|\Omega\|_{L^1(\mathbf{S}^1)}, \tag{21}$$

$$\mathrm{Var}_{\mathbf{S}^1}(m_\Omega) \le \mathrm{Var}_{-2}^{+2}(\tilde{m}) + C\, \|\Omega\|_{L^1(\mathbf{S}^1)}. \tag{22}$$

It is not difficult to see that $\Omega \in H^1(\mathbf{S}^1)$ iff $\Omega(\cos 2\pi x, \sin 2\pi x)\chi_{(0<|x|<1)} \in H^1(\mathbf{R}^1)$. Now consider

$$m(a) = \int_0^1 \Omega(\cos 2\pi x, \sin 2\pi x) \ln \frac{1}{|x-a|}\, dx.$$

By Theorem 1 and the previous observation, $\Omega \in H^1(\mathbf{S}^1)$ iff m has bounded variation. We will now show that m has bounded variation iff m_Ω has bounded variation, and Theorem 2 will be proved. Indeed, let

$$h(a) = m(a) - \tilde{m}(a) = \int_0^1 \Omega(\cos 2\pi x, \sin 2\pi x) \ln \left(\frac{\sin 2\pi |x-a|}{|x-a|}\right) dx.$$

It is straightforward to check that $|h'(a)| \leq C \|\Omega\|_{L^1(\mathbf{S}^1)}$, $a \in (-2, 2)$ and so by (21),

$$\begin{aligned}\mathrm{Var}_{-2}^{+2}(m) &\leq \mathrm{Var}_{-2}^{+2}(h) + \mathrm{Var}_{-2}^{+2}(\tilde{m}) \\ &\leq C \|\Omega\|_{L^1(\mathbf{S}^1)} + \mathrm{Var}_{\mathbf{S}^1}(m_\Omega).\end{aligned}$$

It is also clear that away from $[-2, 2]$, we have $|m'(a)| \leq C \|\Omega\|_{L^1(\mathbf{S}^1)} / a^2$ and therefore

$$\mathrm{Var}_{-\infty}^{+\infty}(m) \leq \mathrm{Var}_{\mathbf{S}^1}(m_\Omega) + C \|\Omega\|_{L^1(\mathbf{S}^1)}.$$

Similarly we get the other inequality,

$$\mathrm{Var}_{\mathbf{S}^1}(m_\Omega) \leq \mathrm{Var}_{-\infty}^{+\infty}(m) + C \|\Omega\|_{L^1(\mathbf{S}^1)},$$

which finishes the proof.

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References

[1] A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. **88** (1952), 85–139.

[2] ———, *On singular integrals*, Amer. J. Math. **78** (1956), 289–309.

[3] D. Chang, S. Krantz, and E. M. Stein, *Hardy spaces and elliptic boundary value problems*, Contemp. Math. **137** (1992), 119–131.

[4] R. R. Coifman, G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc., **83** (1977), 569–645.

[5] W. C. Connett, *Singular integrals near L^1*, *Proc. Sympos. Pure Math., Amer. Math. Soc.* (S. Wainger and G. Weiss, eds), vol. 35 I, 1979, pp. 163–165.

[6] J. Duandikoetxea and J. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Inv. Math., **84** (1986), 541–561.

[7] D. Fan and Y. Pan, *Singular integral operators with rough kernels supported by subvarieties*, Amer. J. Math. **119** (1997), 799–839.

[8] L. Grafakos and A. Stefanov, *Convolution Calderón-Zygmund singular integral operators with rough kernels*, Proc. 7th IWAA, Orono, Maine, 1997 to appear.

[9] F. Ricci and G. Weiss, A characterization of $H^1(\Sigma_{n-1})$, Proc. Sympos. Pure Math., Amer. Math. Soc., vol. 35 I, 1979, pp. 289–294.

[10] A. Seeger, *Singular integral operators with rough convolution kernels*, J. Amer. Math. Soc. **9** (1996), 95–105.

[11] E. M. Stein, *Harmonic analysis: Real variable methods, orthogonality, and oscillatory integrals*, Princeton University Press, Princeton NJ, 1993.

[12] ———, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton NJ, 1970.

[13] H. Triebel, *Interpolation theory, function spaces, differential operators*, 2nd edition, JAB Verlag, 1995.

[14] M. Weiss and A. Zygmund, *An example in the theory of singular integrals*, Studia Math. **26** (1965), 101–111.
[15] A. Zygmund, *Trigonometric series*, Cambridge University Press, Cambridge UK, 1959.

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