CLOSURE PROPERTIES AND PARTIAL ENGEL CONDITIONS IN GROUPS

BY

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1. Introduction

Let $\mathfrak{p}$ be a set of primes and σ a partial ordering of $\mathfrak{p}$ so that $p \sigma p$ is false for every prime p in $\mathfrak{p}$, and $a \sigma b$, $b \sigma c$ implies $a \sigma c$. For convenience we write $q \bar{\sigma} p$ if either $q \notin \mathfrak{p}$ or else $q \sigma p$ is false. A σ-segment is a nonempty subset $\mathfrak{s}$ of $\mathfrak{p}$ defined by the following property: if q belongs to $\mathfrak{s}$ and $p \sigma q$, then p too belongs to $\mathfrak{s}$. The group G is called $\mathfrak{a}$-closed for $\mathfrak{a}$ a set of primes, if products of $\mathfrak{a}$-elements of G are $\mathfrak{a}$-elements. A torsion group is termed $(\mathfrak{p}, \sigma)$-dispersed if it is $\mathfrak{s}$-closed for every σ-segment $\mathfrak{s}$ of $\mathfrak{p}$ (cf. [1, Definition, p. 620]). Further we say that a group G possesses a σ-minimal prime p if p belongs to $\mathfrak{p}$, if G contains elements of order p and if p is minimal among the G-relevant primes in $\mathfrak{p}$ relative to the ordering σ.

Suppose that G is a finite group and that the set $G_{\mathfrak{p}}$ of all $\mathfrak{p}$-elements of G is a $(\mathfrak{p}, \sigma)$-dispersed subgroup of G. Evidently $G_{\mathfrak{p}}$ is a characteristic subgroup of G and G possesses the following properties:

1. *If x is a p-element of G with $p \in \mathfrak{p}$ and if y is a q-element of G with $q \notin \mathfrak{p}$, then for almost every positive integer i the order of $x^{(i)} \circ y$ is divisible by primes r with $r \sigma p$ only.*
2. *Every $\mathfrak{p}$-subgroup of G is $(\mathfrak{p}, \sigma)$-dispersed.*

It is our objective to investigate whether these properties of a torsion group G are sufficient to show that $G_{\mathfrak{p}}$ is a $(\mathfrak{p}, \sigma)$-dispersed subgroup of G. In the case that G is finite we can give a positive answer. But if G is only supposed to satisfy the local double chain condition for subgroups, which may or may not be equivalent to local finiteness, we have to impose additional—possibly superfluous—conditions, because our method does not go through otherwise.

2. Results

For the terminology used in the statement of our results the reader is referred to Section 3.

THEOREM. *If σ is a partial ordering of the set $\mathfrak{p}$ of primes, and if the group G satisfies the double chain condition locally, then the following properties of G are equivalent:*

(a)
1. *$G_{\mathfrak{p}}$ is a $(\mathfrak{p}, \sigma)$-dispersed subgroup of G.*
2. *If $p \in \mathfrak{p}$, then p-factors of G are locally finite.*

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(b) $G_{\mathfrak{p}}$ *is a locally finite and* $(\mathfrak{p}, \sigma)$*-dispersed subgroup of* G.

(c) 1. *If* $p \in \mathfrak{p}$, *then* p*-factors of* G *are locally finite.*
2. *If* p *and* q *are different primes with* $p \in \mathfrak{p}$ *and* $q \bar{\sigma} p$, *and if* a *is a* p*-element and* b *a* q*-element of* G, *then for almost every positive integer* i *the order of* $a^{(i)} \circ b$ *is divisible by primes* r *with* $r \sigma p$ *only.*
3. *Infinite factors* F *of* G *which can be generated by finitely many* $\mathfrak{p}$*-elements possess* σ*-minimal primes.*

(d) 1. $\mathfrak{p}$*-factors of* G *are locally finite and* $(\mathfrak{p}, \sigma)$*-dispersed.*
2. *If* p *and* q *are primes with* $p \in \mathfrak{p}$ *and* $q \notin \mathfrak{p}$, *and if* a *is a* p*-element and* b *is a* q*-element of* G, *then for almost every positive integer* i *the order of* $a^{(i)} \circ b$ *is divisible by primes* r *with* $r \sigma p$ *only.*
3. *Infinite factors* F *of* G *which can be generated by finitely many* $\mathfrak{p}$*-elements possess* σ*-minimal primes.*

From the theorem we shall derive two corollaries.

COROLLARY 1. *If* σ *is a partial ordering of the set* $\mathfrak{p}$ *of primes, and if the group* G *is locally finite, then the following properties are equivalent:*

(a) $G_{\mathfrak{p}}$ *is a* $(\mathfrak{p}, \sigma)$*-dispersed subgroup of* G.

(b) *If* p *and* q *are different primes with* $p \in \mathfrak{p}$ *and* $q \bar{\sigma} p$, *and if* a *is a* p*-element and* b *a* q*-element of* G, *then for almost every positive integer* i *the order of* $a^{(i)} \circ b$ *is divisible by primes* r *with* $r \sigma p$ *only.*

(c) 1. $\mathfrak{p}$*-subgroups of* G *are* $(\mathfrak{p}, \sigma)$*-dispersed.*
2. *If* p *and* q *are primes with* $p \in \mathfrak{p}$ *and* $q \notin \mathfrak{p}$, *and if* a *is a* p*-element and* b *a* q*-element of* G, *then for almost every positive integer* i *the order of* $a^{(i)} \circ b$ *is divisible by primes* r *with* $r \sigma p$ *only.*

It is an immediate consequence of our theorem that the properties (a) and (b) are equivalent and that (a) implies (c). The proof of the fact that (c) implies (a) will be carried out in Section 5.

If $\mathfrak{r}$ is the set of primes which are orders of the elements in the group H, then we term $\mathfrak{r} = \mathfrak{c}H$ the characteristic of H.

COROLLARY 2. *If the finite group* G *contains a nilpotent Hall-subgroup* H *of characteristic* $\mathfrak{r}$ *and if for every primary element* $x \in H$ *and every primary* $\mathfrak{r}'$*-element* y *of* G *the element* $x^{(i)} \circ y$ *equals* 1 *for almost every positive integer* i, *then* $H \triangleleft G$.

One easily derives Corollary 2 from Corollary 1. If τ is the trivial ordering of the set $\mathfrak{r}$ of primes, that is $q \bar{\tau} p$ for all p and q belonging to $\mathfrak{r}$, then every prime of $\mathfrak{r}$ is τ-minimal, and a finite $\mathfrak{r}$-group is nilpotent if, and only if, it is $(\mathfrak{r}, \tau)$-dispersed. If G contains a nilpotent Hall-subgroup of characteristic $\mathfrak{r}$, then applying a theorem of Wielandt ([3, Satz, S. 407]) every $\mathfrak{r}$-subgroup of G is nilpotent. It follows from this remark and the commutator condition of Corollary 2 that G satisfies the condition (c) of Corollary 1. Hence H is normal in G.

3. Definitions and notations

$o(x)$ = order of the group element x.

If $\mathfrak{p}$ is a set of primes, then x is termed a $\mathfrak{p}$-element if $o(x)$ is finite and divisible by primes in $\mathfrak{p}$ only.

A is termed a $\mathfrak{p}$-group, if all elements of A are $\mathfrak{p}$-elements.

$\mathfrak{p}'$ = set of all primes not in $\mathfrak{p}$.

The element x is called primary, if $o(x)$ is a prime power.

p is termed a G-relevant prime of the group G, if there are elements of order p in G.

$G_{\mathfrak{p}}$ = set of all $\mathfrak{p}$-elements of the group G.

Factor of a group G = epimorphic image of a subgroup of G.

$N(A, G)$ = normalizer of the subgroup A of G in G.

$A \lhd G$ means A is a normal subgroup of G.

$A \lhd\lhd G$ means A is a subnormal subgroup of G, i.e. there exist finitely many subgroups $A(i)$ with

$$A = A(0) \lhd \cdots \lhd A(k) \lhd A(k+1) \lhd \cdots \lhd A(n) = G.$$

Sylow subgroup of G = maximal p-subgroup of G for p a prime.

G satisfies the double chain condition means that G satisfies the maximum and minimum condition for its subgroups.

G satisfies the double chain condition locally means that finitely generated subgroups of G satisfy the double chain condition.

$x \circ y = x^{-1}y^{-1}xy$, $x^{(0)} \circ y = y$, $x^{(i+1)} \circ y = x \circ (x^{(i)} \circ y)$.

4. Reduction lemmas

LEMMA 1. *Let G be a group all of whose p-Sylow subgroups are finite. If every proper subgroup of G is p-closed, and if G contains no normal p-subgroup different from* 1, *then different p-Sylow subgroups of G have trivial intersection.*

Proof. Assume by way of contradiction the existence of two different p-Sylow subgroups A and C of G with non-trivial intersection. Fix A and consider the set $\mathfrak{D}$ of the intersections of A with all p-Sylow subgroups of G different from A. Because of the finiteness of A there is a maximal subgroup $D \neq 1$ in $\mathfrak{D}$. Hence there exists a p-Sylow subgroup B of G with $A \neq B$, $A \cap B = D$ and therefore $A \neq D \neq B$. It follows from a well known property of finite p-groups, that proper subgroups of p-subgroups of G are properly contained in their normalizers. Therefore

$$D \subset A \cap N(D, G) \quad \text{and} \quad D \subset B \cap N(D, G).$$

Consequently there are elements $a \in A \cap N(D, G)$ and $b \in B \cap N(D, G)$ which are not contained in D. From $D \neq 1$ and the fact that there is no normal

p-subgroup of G different from 1 it follows that $N(D, G)$ is a proper subgroup of G. Hence $N(D, G)$ is p-closed. It follows that $D\{a, b\}$ is a p-group with $D \subset D\{a, b\}$. Hence there is a p-Sylow subgroup T of G with $D\{a, b\} \subseteq T$ and $D \subset T \cap A$. From the maximality of D we deduce $T = A$. But then b would be an element of A contradicting the choice of b. This contradiction follows from our assumption of the existence of two different p-Sylow subgroups with non-trivial intersection, Q.E.D.

LEMMA 2. *Let $\mathfrak{r}$ be a set of primes and A a maximal $\mathfrak{r}$-subgroup of a group G. If $A \cap A^g = 1$ for $A \neq A^g$, and if b is an element of G such that there exists an element $a \neq 1$ of A with $a^{(i)} \circ b = 1$ for almost every positive integer i, then b is contained in the normalizer of A.*

Proof. Assume that A is a maximal $\mathfrak{r}$-subgroup of a group G and that the elements a and b are related in such a fashion as required in the lemma. If b were not contained in $N(A, G)$, then, because of $1 \in N(A, G)$ and $a^{(0)} \circ b = b \notin N(A, G)$, there would exist an integer $m \geq 0$ with

$$d = a^{(m)} \circ b \notin N(A, G) \quad \text{and} \quad a \circ d \in N(A, G).$$

Hence the element a^d would be contained in $N(A, G)$. Since A is a maximal $\mathfrak{r}$-subgroup of G, the $\mathfrak{r}$-element a^d is contained in $N(A, G)$ if and only if a^d is contained in A. Therefore

$$1 \neq a^d \in A \cap A^d$$

and it follows that $d \in N(A, G)$ from the condition of the lemma for the intersection of different conjugate subgroups of A. But this contradicts the choice of $d \notin N(A, G)$. Thus the lemma is proved.

LEMMA 3. *The p-Sylow subgroup S of G is a normal subgroup of G if and only if S is a subnormal subgroup of G.*

Note. The p-Sylow subgroup S of G is certainly subnormal in G whenever there exists a normal subgroup T of G which normalizes S and a normal subgroup P of G such that $ST \subseteq P$ and P/T is a finite p-group. For in this case S is a normal subgroup of ST and ST/T is a subnormal subgroup of P/T so that ST is a subnormal subgroup of P—this is the soc. normalizer property enjoyed by many classes of more or less nilpotent groups, in particular the finite p-groups. Hence $S \lhd \lhd G$ by transitivity of subnormality.

Proof. Obviously it suffices to prove the sufficiency of the condition. If S is subnormal in G, then there exist finitely many subgroups $S(i)$ of G, $i = 0, 1, \cdots, n$, satisfying $S = S(0)$, $G = S(n)$ and $S(i) \lhd S(i + 1)$. Assume, that S is a normal subgroup of $S(i)$. Since S is a p-Sylow subgroup of G, it is a p-Sylow subgroup of $S(i)$. Consequently S is a characteristic subgroup of the normal subgroup $S(i)$ of $S(i + 1)$ and as such S is a normal subgroup of $S(i + 1)$. Hence the validity of the lemma follows by induction.

5. Proof of the theorem

For the proof of our theorem we may assume that the group G under consideration satisfies the double chain condition for all subgroups, because in the case that G satisfies the double chain condition locally we can test the validity of any of the conditions (a)–(d) in the finitely generated subgroups of G. The fact that G satisfies the double chain condition implies that G is a torsion group all of whose subgroups are finitely generated. Therefore all locally finite groups occurring in the proof are finite. It is quite obvious that the properties (a)–(d) of a group are inherited by its factors.

Property (a) *implies* (b). Assume that this were false. Then, there exists among the groups with double chain condition a group G which possesses property (a) although $G_{\mathfrak{p}}$ is not finite. Because of the minimum condition there exists a minimal subgroup M of G, which does not possess property (b). Because of the maximum condition there exists an epimorphic image W of M with the following properties:

(1) The subgroup $W_{\mathfrak{p}}$ of W is not finite.
(2) If F is a proper factor of W, then $F_{\mathfrak{p}}$ is a finite subgroup of F.

From (2) one easily deduces

(3) $W_{\mathfrak{p}} = W$.

If W were a primary group it would be finite by (a, 2) and (3). This contradicts (1). Hence

(4) there are at least two W-relevant primes.

Since $W_{\mathfrak{p}} = W$ is $(\mathfrak{p}, \sigma)$-dispersed by (a, 1), it follows that W is not simple by (4). Hence there exists a normal subgroup K of W with $1 \subset K \subset W$. Because of (2) and (3) the proper factors K and W/K are finite. Therefore W is finite too. This contradicts (1) and it follows that (b) is a consequence of (a).

Property (b) *implies* (c). From (b) it follows that $F_{\mathfrak{p}}$ is a finite subgroup of every factor F of G. Hence (c, 1) follows trivially from (b).

If a is a p-element with $p \in \mathfrak{p}$, then a is contained in the finite and $(\mathfrak{p}, \sigma)$-dispersed $\mathfrak{p}$-subgroup $G_{\mathfrak{p}}$ of G. Denote by $\mathfrak{s}$ the set of all primes $r \in \mathfrak{p}$ with $r \,\sigma\, p$. Because of the $(\mathfrak{p}, \sigma)$-dispersion of G the set $G_{\mathfrak{s}}$ of all $\mathfrak{s}$-elements of G is a characteristic subgroup of G, since it is characteristic in $G_{\mathfrak{p}}$, and p is a σ-minimal prime of $G/G_{\mathfrak{s}}$. Therefore the set H_p of all p-elements of $G/G_{\mathfrak{s}}$ is a characteristic subgroup of $G/G_{\mathfrak{s}}$ since it is characteristic in $G_{\mathfrak{p}}/G_{\mathfrak{s}}$. Since H_p is a finite p-group, the terminal member of the lower central series equals 1. Since a is a p-element of G we have $aG_{\mathfrak{s}} \in H_p$. Hence, if b is an arbitrary element of G there is a positive integer m, with $a^{(m)} \circ b \in G_{\mathfrak{s}}$, because $H_p \lhd G/G_{\mathfrak{s}}$. Thus it is clear that (b) implies (c, 2).

Condition (c, 3) is satisfied vacuously, since a factor of G generated by $\mathfrak{p}$-elements is finite by (b).

Property (c) *implies* (d). Clearly (d, 2) and (d, 3) are immediate consequences of (c, 2) and (c, 3).

Assume that the group G satisfies condition (c), but does not satisfy (d, 1). Then there exists a $\mathfrak{p}$-factor U of G, which is not finite and $(\mathfrak{p}, \sigma)$-dispersed. Because of the double chain condition there exists a factor W of U with the following properties:

(1) W is a $\mathfrak{p}$-group and satisfies the condition (c).
(2) W does not satisfy (d, 1).
(3) Proper factors of W satisfy (d, 1).

Assume there exists a normal r-subgroup $R \neq 1$ of W with $r \in \mathfrak{p}$. Then (c, 1) yields the finiteness of R and from (1) and (3) follows the finiteness of W/R. Thus W is finite. If p is an arbitrary σ-minimal prime of W, and if a is a p-element and b an arbitrary q-element of W with $q \neq p$, then all but a finite number of elements $a^{(i)} \circ b$ equal 1 by (c, 2). Application of [2, Korollar 5, S. 242] yields that the set W_p of all p-elements of W is a characteristic subgroup of W. This is true for every σ-minimal prime of the finite group W. Combining this result with property (3) we see that W satisfies condition (d, 1) which contradicts (2). Hence

(4) there is no normal r-subgroup of W different from 1 with $r \in \mathfrak{p}$.

By (1) and the maximum condition W can be generated by a finite number of $\mathfrak{p}$-elements. Hence there is a σ-minimal prime p of W by (c, 3).

Now, applying Lemma 1, it follows by (c, 1), (3) and (4) that

(5) different p-Sylow subgroups of W have trivial intersection.

From (5), (c, 2) and Lemma 2 we deduce that the subgroup T of W generated by all p'-elements of W is contained in the normalizer of a p-Sylow subgroup A of W. Of course A is different from 1, because p is a W-relevant prime of W, and T is normal in W. From the construction of T it is clear that W/T is a p-group. From property (c, 1) follows the finiteness of W/T. Hence application of Lemma 3 yields $A \lhd W$ contradicting (4); and from this contradiction it follows that (d) is a consequence of (c).

Property (d) *implies* (a). Obviously (d, 1) implies (a, 2). Assume that there is a group G which meets the requirements of (d), but not those of (a). Then $G_{\mathfrak{p}}$ is not a subgroup of G by (d, 1). From the double chain condition follows the existence of a factor W of G with the following properties:

(1) W is not $\mathfrak{p}$-closed.
(2) Proper factors of W are $\mathfrak{p}$-closed.

From (1) we deduce the existence of two $\mathfrak{p}$-elements the product of which is no $\mathfrak{p}$-element. Because of (2) the group W is generated by these elements.

Hence

(3) W is generated by a finite number of $\mathfrak{p}$-elements.

From (d, 3) and (3) follows the existence of a σ-minimal prime p of W. Assume that W contains a normal p-subgroup $K \neq 1$ of W. Then by (2) the group W/K is $\mathfrak{p}$-closed. Hence $(W/K)_{\mathfrak{p}} = W_{\mathfrak{p}}/K$ and because of $p \in \mathfrak{p}$ follows the $\mathfrak{p}$-closure of W contradicting (1). Thus we have derived:

(4) There is no normal p-subgroup of W different from 1.

Now, applying Lemma 1, it follows by (d, 1), (2), (4) and the σ-minimality of p that

(5) different p-Sylow subgroups of W have trivial intersection.

From (5), (d, 2) and Lemma 2 we deduce that the characteristic subgroup T of W, generated by all $\mathfrak{p}'$-elements of W, is contained in the normalizer of a p-Sylow subgroup $A \neq 1$ of W. From (d, 1) and the construction of T it follows that W/T is a finite and $(\mathfrak{p}, \sigma)$-dispersed $\mathfrak{p}$-group. Since p is a σ-minimal prime of W it is σ-minimal in W/T too, and the p-elements of W/T form a finite normal subgroup of W/T which contains AT/T. Now application of Lemma 3 yields $A \triangleleft W$ which contradicts property (4). This proves that (d) implies (a).

Thus we have completed the proof of the equivalence of the properties (a), (b), (c) and (d) for groups with double chain condition, whereof follows the validity of the theorem from what we had remarked at the beginning of the proof.

Proof of Corollary 1. It remains only to prove that property (c) of Corollary 1 implies (a). We may confine ourselves to prove this for finite groups, since (a) is a local property.

If (a) were not a consequence of (c), then among the finite groups satisfying (c), but not (a) there is a group G of minimal order with the following properties:

(1) $G_{\mathfrak{p}}$ is not a subgroup of G.

For otherwise $G_{\mathfrak{p}}$ would be $(\mathfrak{p}, \sigma)$-dispersed by (c, 1) and (a) would be a consequence of (c).

(2) If U is a proper subgroup of G, then $U_{\mathfrak{p}}$ is a characteristic subgroup of U.

This follows immediately from the minimality of G, and the fact that (c) is inherited by subgroups.

(3) There is no normal $\mathfrak{p}$-subgroup of G different from 1.

For, if $K \neq 1$ is a normal $\mathfrak{p}$-subgroup of G, then G/K satisfies condition (c, 2). If U/K is a $\mathfrak{p}$-subgroup of G/K, then U is a $\mathfrak{p}$-subgroup of G. From (c, 1) it follows that U is $(\mathfrak{p}, \sigma)$-dispersed and so is U/K as an epimorphic image of U. Hence G/K satisfies condition (c, 1) as well. From the minimality of G and $K \neq 1$ we deduce that $(G/K)_{\mathfrak{p}}$ is a subgroup of G/K. Since K is a $\mathfrak{p}$-group, we have $(G/K)_{\mathfrak{p}} = G_{\mathfrak{p}}/K$. Therefore $G_{\mathfrak{p}}$ is a subgroup of G contradicting property (1). Thus (3) is valid.

Let K be a proper normal subgroup of G. From (2) it follows that $K_{\mathfrak{p}}$ is a characteristic subgroup of K and therefore normal in G. But there exists no normal $\mathfrak{p}$-subgroup of G different from 1 by (3), hence $K_{\mathfrak{p}} = 1$, and we have derived

(4) every proper normal subgroup of G is a $\mathfrak{p}'$-group.

Because of (1) the set $G_{\mathfrak{p}}$ is different from 1. Hence there are primes in $\mathfrak{p}$, which are the orders of elements of G. Since G is finite, we can find a σ-minimal prime with this property. Denote that prime by p.

Now, applying Lemma 1 we deduce from (c, 1), (2), (3) and the σ-minimality of p that

(5) different p-Sylow subgroups of G have trivial intersection.

If A is a p-Sylow subgroup of G, then $A \neq 1$, because p was chosen to be a prime dividing the order of G. Applying Lemma 2 we deduce from (c, 2), (5) and the σ-minimality of p, that the characteristic subgroup T of G generated by all $\mathfrak{p}'$-elements of G is contained in the normalizer of A in G. By (3) we have $N(A, G) \subset G$ and applying (4) we get that T is a normal $\mathfrak{p}'$-Hall-subgroup of G. Now a theorem of Schur [4, Theorem 25, S. 162] yields the existence of a subgroup V of G with $TV = G$ and $V \cap T = 1$. From the construction of T it follows that V is a $\mathfrak{p}$-group, which contains a p-Sylow subgroup of G. Since p-Sylow subgroups of finite groups are conjugate we can assume $A \subseteq V$. From (c, 1) we have that V is $(\mathfrak{p}, \sigma)$-dispersed and from the σ-minimality of p follows $A \lhd V$. Then $1 \neq A \lhd TV = G$, a contradiction which proves the corollary.

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