APPLICATIONS OF A COMPACTIFICATION FOR BOUNDED OPERATOR SEMIGROUPS

BY

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Introduction and Summary

In this paper[1] a compactification for bounded semigroups of linear operators in a Banach space is studied and some applications to abstract ergodic theory and invariant means are given. In Sec. 1 the compactification in question is described and in Sec. 2 its ideal theory is developed. Sec. 3 contains a discussion of ergodic elements for arbitrary bounded, not necessarily "ergodic," operator semigroups and is very close in spirit to Eberlein [6]. The connection between the compactification and the convolution semigroup of means introduced by Day in [4] is established (in (4.3)) and the following theorem is proved: the space $m(\Sigma)$ of all bounded real functions on an abstract semigroup Σ with unit contains a largest right amenable right introverted subspace Z which, moreover, lies in every maximal right amenable subspace of $m(\Sigma)$.

The following notations will be used throughout: If B_1, B_2 are Banach spaces then B_1^* is the conjugate space of B_1 and $L(B_1, B_2)$ is the Banach space of all bounded linear operators of B_1 into B_2; if $S \subset L(B_1, B_2)$ and $x \in B_1$ then $O_S(x)$ is the orbit of x under S and defined by $O_S(x) = \{Ax : A \in S\}$. The closure of a set S is denoted by S^-, and composition is indicated by juxtaposition or brackets.

1. Compactification of a bounded operator semigroup

We need the following two devices.

I. Suppose X is a linear topological space and S is a semigroup (under composition) of continuous linear operators in X. Let S^- be the closure of S in the product space X^X. We have

(i) S^- is a semigroup (under composition) of linear operators in X, and

(ii) for fixed $A \in S$ and $B \in S^-$ the maps $F \to AF$ and $F \to FB$ $(F \in S^-)$ are continuous in the product topology of X^X.

II. Suppose B is a Banach space. B can be regarded as a subspace of B^{**} and hence $L(B, B)$ can be regarded as a subspace of $L(B, B^{**})$. Let η be the mapping which takes each $U \in L(B, B^{**})$ into the function $F = \eta(U) \in L(B^*, B^*)$ defined by

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$$(1) \qquad (F\beta)x = (Ux)\beta \qquad (x \in B,\ \beta \in B^*).$$

We have

(i) η is a linear isometry of $L(B, B^{**})$ onto $L(B^*, B^*)$ and its restriction to $L(B, B)$ is simply the adjoint operation which takes each $U \in L(B, B)$ into its adjoint $U^* \in L(B^*, B^*)$, and

(ii) η is a weak$_*$-weak* operator[2] homeomorphism.

Both I and II are essentially known; the proofs are straightforward.

Now suppose B is a Banach space, η is the mapping defined in II, and S is a (uniformly) bounded semigroup lying in $L(B, B)$. Let A be the adjoint semigroup of S, $A = \eta(S)$, and let A^- be the closure of A in the product space X^X, where $X = B^*$ with the weak* topology. Then A^- is a weak* operator compact subset of $L(B^*, B^*)$ and, by I, a semigroup under composition. Define $S_0 = \eta^{-1}(A^-)$. Then $S \subset S_0$ and since η is already an anti-isomorphism of S onto A we can extend the multiplication from S to S_0 in such a way that η becomes an anti-isomorphism of S_0 onto A^-:

$$(2) \qquad UV = \eta^{-1}(\eta(V)[\eta(U)]) \qquad (U, V \in S_0).$$

The semigroup S_0 with multiplication as defined in (2) and the weak$_*$ topology is the desired compactification of S and has the following properties:

(1.1) (i) *S_0 is a closed (weak$_*$) compact subset of $L(B, B^{**})$;*

(ii) *S_0 is a semigroup and S is a (weak$_*$) dense subsemigroup of S_0, and*

(iii) *if $U \in S$ and $V \in S_0$ then the mappings $W \to VW$ and $W \to WU$ ($W \in S_0$) are (weak$_*$) continuous.*

Observe:

(1.2) *If $U, V \in S_0$ and $x \in B$ such that $Vx \in B$ then*

$$(UV)x = U(Vx).$$

Proof. If U, V and x are as described then for each $\beta \in B^*$,

$$((UV)x)\beta = (\eta(UV)\beta)x = (\eta(V)\eta(U)\beta)x$$
$$= (Vx)(\eta(U)\beta) = (\eta(U)\beta)(Vx) = U(Vx)\beta.$$

A semigroup S of bounded linear operators in a Banach space B is *weakly almost periodic* (w.a.p.) iff for each $x \in B$, $0_S(x)^-$ is weakly compact. S is w.a.p. iff $S_0 \subset L(B, B)$ and in this case S_0 coincides with the compactification of S introduced by DeLeeuw and Glicksberg [5]. Our construction is seen to be a simple extension of their procedure to the case of an arbitrary bounded, not necessarily w.a.p. operator semigroup.

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[2] A net U_α in $L(B, B^{**})$ converges to a point $U \in L(B, B^{**})$ in the *weak$_*$ topology* iff $(U_\alpha x)\beta \to (Ux)\beta$ ($x \in B$, $\beta \in B^*$). A net F_α in $(L(B^*, B^*)$ converges to point $F \in L(B^*, B^*)$ in the *weak* operator topology* iff $(F_\alpha \beta)x \to (F\beta)x$ ($\beta \in B^*$, $x \in X$).

2. Ideal theory

We now develop the ideal theory for the compactification semigroup S_0 constructed in the preceding section. In contrast to the situation in [5], Sec. 2, multiplication by V on the right, i.e., the mapping $U \to UV$ ($U \in S_0$), is not, in general, continuous if $V \in S_0$ and $V \notin S$ (cf. (1.1), (ii)). However, in many cases of interest S_0 is convex and its essential structure is much simpler.

Let B, S, and S_0 be as in the preceding section. A *right* [*left*] *ideal* of S_0 is a subset I of S_0 such that $IS_0 \subset I$ [$S_0 I \subset I$].

(2.1) *There is a smallest closed two-sided ideal of* S_0.

This ideal will be called the *kernel* of S_0 and denoted by $K(S_0)$.

(2.2) *Each closed right* [*left*] *ideal of* S_0 *contains a minimal closed right* [*left*] *ideal of* S_0. *The minimal closed right* [*left*] *ideals are pairwise disjoint and lie in the kernel.*

In both (2.1) and (2.2) the existence of the ideals in question follows from the compactness of S_0 by the Hausdorff Maximal Principle.

(2.3) *If* R *is a minimal (closed) right ideal of* S_0 *and* $U \in S_0$ *then* UR *is again a minimal (closed) right ideal of* S_0, *and if* L *is a minimal closed left ideal of* S_0 *and* $U \in S$ *then* LU *is again a minimal closed left ideal of* S_0.

This is essentially Lemma 2.1 of Clifford [3].

(2.4) *A right ideal of* S_0 *is minimal closed iff it is minimal. The union of all minimal right ideals is a two-sided ideal* K_C, *the smallest two-sided ideal of* S_0.

The first statement is proved in [5, p. 65], the second is Theorem 2.1 of Clifford [3].

(2.5) Lemma. *The closure of a right* [*left*] *ideal of* S_0 *is again a right* [*left*] *ideal of* S_0.

Proof. Suppose R is a minimal right ideal of S_0 and $U \in R^-$. Let U_α be a net in R such that $U_\alpha \to U$. If $V \in S$ then $U_\alpha V \to U_0 V$ so $UV \in R^-$; thus $US \subset R^-$ and hence $US_0 \subset R^-$. For left ideals the proof is even simpler.

In view of (2.5), $K_C^- = K(S_0)$, i.e., the minimal right ideals are dense in the kernel. For left ideals this can be somewhat improved.

(2.6) Lemma. *If* L *is a minimal closed left ideal of* S_0 *then* LS *is dense in* $K(S_0)$.

Proof. Let L be as indicated and put $M = LS$. By (2.2), $M \subset K(S_0)$ so $M^- \subset K(S_0)$. To prove the reverse inclusion we show that M^- is a two-sided ideal. By (2.5), M^- is a left ideal. Given $U \in M^-$ there are nets $\{U_\alpha\}$ in L and $\{V_\alpha\}$ in S (both over the same index set) such that $U_\alpha V_\alpha \to U$.

For each $W \in S$ we have $(U_\alpha V_\alpha)W \to UW$ and $U_\alpha(V_\alpha W) \in M$ and therefore $UW \in M^-$. Thus $US \subset M^-$, hence $US_0 \subset M^-$, and since $U \in M^-$ was arbitrary, we are finished.

Let T be the convex hull of S. Then T is also a bounded semigroup and its compactification, T_0, is convex.

(2.7) LEMMA. *Every minimal closed right [left] ideal of T_0 is convex.*

Proof. Suppose L is a minimal closed left ideal of T_0 and choose $V \in L$. Then $T_0 V$ is a convex left ideal of T_0 which lies in L and hence $(T_0 V)^- = L$. Thus L is the closure of a convex set and therefore convex. For right ideals the proof is even simpler.

The following theorem characterizes those right ideals of T_0 which are minimal.

(2.8) THEOREM. *A right ideal R of T_0 is minimal iff $AV = V(A, V \in R)$.*

Proof. Suppose R is a minimal right ideal of T_0, take $A \in R$ and put $F_A = \{V : V \in T_0 \text{ and } AV = V\}$. F_A is the set of all fixed points of the continuous linear map $U \to AU$ $(U \in T_0)$ and hence, by any one of the standard fixed point theorems, F_A is nonempty. If $V \in F_A$ we see from $AV = V$ and $A \in R$ that $V \in R$ so $F_A \subset R$. Since F_A is a right ideal of T_0 and R is minimal, $F_A = R$. Conversely, if R is a right ideal of T_0 so that $AV = V$ $(A, V \in R)$ and if R' is a right ideal of T_0 lying in R, let $A \in R'$ and get $R' \supset R'T_0 \supset AR = R$.

The next result states that all minimal right ideals are essentially "congruent."

(2.9) THEOREM. *Suppose R_1 and R_2 are distinct minimal right ideals of T_0 and $V_0 \in R_2$. The mapping $\phi : U \to V_0 U$ is a convex-linear homeomorphism of R_1 and R_2 which does not depend on the choice of $V_0 \in R_2$.*

Proof. Suppose R_1, R_2, V, and ϕ are as described. The mapping ϕ is convex-linear and continuous and by (2.3), $\phi(R_1) = R_2$. Let $U_0 \in R_1$ and let ψ be the mapping $V \to U_0 V$ $(V \in R_2)$. We have

$$\phi\psi(V) = V_0(U_0 V) = (V_0 U_0)V = V \qquad (V \in R_2)$$

(by (2.8)); i.e., $\phi[\psi]$ is the identity map on R_2, and similarly $\psi[\phi]$ is the identity map on R_1. Therefore $\psi = \phi^{-1}$. If $V_0' \in R_2$ and $\phi' : U \to V_0' U$ $(U \in R_1)$ then the same argument gives $\psi = (\phi')^{-1}$ so that $\phi' = \phi$.

A more precise picture of the multiplication operation in $K(T_0)$ is furnished by the next result.

(2.10) THEOREM. *If R is a minimal right and L a minimal closed left ideal of T_0 then $L \cap R$ is a singleton set. For fixed R the map $L \to L \cap R$ defines a 1-1 correspondence between all minimal closed left ideals of T_0 and all points of R.*

Proof. With R and L as in the first half of the theorem choose U and

$V \epsilon L \cap R$. Since LU is a left ideal $\subset L$ we have $(LU)^{-} = L$ and hence there is a net $\{V_\alpha\}$ in LU such that $V_\alpha \to V$. Write $V_\alpha = W_\alpha U$ with $W_\alpha \epsilon L$. Now $VV_\alpha \to VV = V$ (by (2.8)), but also for each α, $VV_\alpha = V(W_\alpha U) = (VW_\alpha)U = U$ (by (2.8) again). Thus $V = U$ and this proves the first half of the theorem. The essential assertion of the second half is that the union of all minimal right ideals (i.e., the set K_C of (2.4)) is a subset of the union of all minimal closed left ideals. Suppose R is a minimal closed right ideal of T_0 and $U \epsilon R$. Let M be any minimal closed left ideal of T_0 and put $L = (MU)^{-}$. L is a closed left ideal of T_0 and since $R(MU) = (RM)U = \{U\}$, a singleton set, we have $RL = \{U\}$. Letting L_0 be any minimal closed left ideal of T_0 which lies in L we see that $\emptyset \neq RL_0 \subset RL = \{U\}$, so

$$RL_0 = \{U\} \subset L_0 .$$

A *left zero* of T_0 is a degenerate right ideal, i.e., an element E of T_0 such that $EU = E$ $(U \epsilon T_0)$.

(2.11) *The following statements are equivalent:*[3]
 (i) *T_0 has a left zero;*
 (ii) *$K(T_0)$ is the set of all left zeros of T_0 ;*
 (iii) *$K(T_0)$ is the only minimal (closed) left ideal of T_0 .*

Proof. (i) $\Rightarrow$ (ii). The set of all left zeros of T_0 is a two-sided ideal which lies in $K(T_0)$; since it is also closed it coincides with $K(T_0)$.

(ii) $\Rightarrow$ (iii). Let L be a minimal closed left ideal of T_0 and observe that $K(T_0) = K(T_0)L \subset L$.

(iii) $\Rightarrow$ (ii). The argument is similar to that for (2.8). Let $A \epsilon T$ and put $F_A = \{U : U \epsilon T_0 \text{ and } UA = U\}$. Again by fixed point theory F_A is nonempty, in fact it is a (closed) left ideal of T_0 and hence includes $K(T_0)$. Since $A \epsilon T$ was arbitrary, (ii) is proved.

If E is a left zero of T_0 and $\{A_\alpha\}$ is a net in T such that $A_\alpha \to E$ then $(A_\alpha A - A_\alpha) \to 0$ weak$_*$ $(A \epsilon T)$ and conversely if $\{A_\alpha\}$ is a net in T such that $(A_\alpha A - A_\alpha) \to 0$ weak$_*$ $(A \epsilon T)$ then each cluster point of $\{A_\alpha\}$ is a left zero of T_0. Thus left zeros of T_0 replace in our setting the nets of almost right invariant averages first introduced abstractly by Eberlein [6][4] (i.e., nets $\{A_\alpha\}$ in T satisfying $(A_\alpha A - A_\alpha) \to 0$ weak$_*$ $(A \epsilon T)$). A *partial right zero* of T_0 is an element E of T_0 such that $AE = E$ $(A \epsilon T)$. Partial right zeros of T_0 are related to nets of almost left invariant averages (i.e., nets $\{A_\alpha\}$ in T satisfying $(AA_\alpha - A_\alpha) \to 0$ weak$_*$ $(A \epsilon T)$) in the same manner as left zeros of T_0 are related to nets of almost right invariant averages. If partial right zeros of T_0 exist they form a (closed) right ideal of T_0 which must intersect $K(T_0)$; hence if, in addition, left zeros of T_0 exist then some

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[3] Cf. Theorem 1 of [3].

[4] Eberlein considered more general averages A_α which satisfy both $(A_\alpha A - A_\alpha) \to 0$ and $(AA_\alpha - A_\alpha) \to 0$ $(A \epsilon T)$.

$E \in K(T_0)$ is both a left and partial right zero of T_0 and any net $\{A_\alpha\}$ in T such that $A_\alpha \to E$ is a net of two sided almost invariant averages. Under these circumstances S is said to be restrictedly weak$_*$ ergodic (Day [3]).

3. Ergodic elements

As a first application of the preceding ideas we indicate in this section how abstract ergodic theory[5] is to some extent possible in an *arbitrary* bounded semigroup of linear operators in a Banach space (without any ergodicity assumptions on the semigroup). Essentially this consists in a consideration of the notion of an ergodic element.[6] S, B, T, and T_0 have the same meaning as in Section 2.

(3.1) THEOREM. *If $x \in B$ and $y \in B$ then the following statements are equivalent*:

(i) $y \in O_T(x)^-$ *and* $Fy = y$ $(F \in S)$;

(ii) *there is a closed left ideal L of T_0 such that*

$$Vx = y \ (V \in L).$$

If S is restrictedly weak* ergodic (so that by (2.11) $K(T_0)$ consists of all left zeros of T_0 and is the unique minimal closed left ideal of T_0), then condition (ii) of (3.1) simply means that each left zero of T_0 has the value y at x. If in this case $\{A_\alpha\}$ is a net of almost two sided invariant averages then (i) and (ii) of (3.1) are easily seen to be equivalent to

(iii) *$A_\alpha x$ clusters weakly at y*,

and this is Eberlein's ergodic theorem for a restrictedly weak* ergodic semigroup [6, Theorem 3.1]. We will not prove (3.1) but state and prove instead a small generalization.

(3.2) THEOREM. *If $x \in B$ and $y \in B^{**}$ then the following statements are equivalent*:

(i) (a) *there is a net $\{A_\alpha\}$ in T such that $\beta(A_\alpha x) \to y(\beta)$ $(\beta \in B^*)$ and* (b) $y(\eta(U)\beta) = y(\beta)$ $(U \in T_0, \beta \in B^*)$, *where η is the map defined in* I, *Sec.* 1;

(ii) *there is a closed left ideal L of T_0 such that*

$$Vx = y \qquad (V \in L).$$

If $y \in B$ then condition (i) of (3.2) reduces to condition (i) of (3.1). For in this case (i)(a) says that there is a net $\{A_\alpha\}$ in T such that $A_\alpha x \to y$ weakly and by the Mazur-Bourgin theorem this is equivalent to $y \in O_T(x)^-$; (i)(b) certainly implies $\beta(y) = (F^*\beta)y = \beta(Fy)$ $(\beta \in B^*, F \in S)$, i.e., y is a fixed point of S, and conversely a fixed point of S is also a fixed point of T_0. Thus (3.1) is a special case of (3.2).

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[5] Cf. [6, part I], and the summary in [3, p. 279].

[6] Cf. [6, Definitions 3.1 and 8.1].

Proof. (i) $\Rightarrow$ (ii) Given $\{A_\alpha\}$ as in (i)(a) let $\{A_{\alpha'}\}$ be a subnet of $\{A_\alpha\}$ and let $A \in T_0$ such that $A_{\alpha'} \to A$. We still have $\beta(A_{\alpha'} x) \to y(\beta)$ $(\beta \in B^*)$ and also $\beta(A_{\alpha'} x) \to (Ax)\beta$ $(\beta \in B^*)$ and so $y = Ax$. Let

$$L = \{V : V \in T_0 \text{ and } Vx = y\};$$

we will show that L is a (closed) left ideal of T_0. $A \in L$ so L is nonempty and if $U \in T_0$ and $V \in L$ then by the computation in (1.2)

$$\begin{aligned}(UVx)\beta &= (Vx)(\eta(U)\beta) = y(\eta(U)\beta)\\ &= y(\beta) \qquad (\beta \in B^*)\end{aligned}$$

i.e., $UVx = y$ or $UV \in L$.

(ii) $\Rightarrow$ (i) Given L as in (ii) choose $V \in L$: any net $\{A_\alpha\}$ in T such that $A_\alpha \to V$ will do for (i)(a) and a computation like that in (i) $\Rightarrow$ (ii) can be used to prove (i)(b).

We note (and will use tacitly below) that L in (ii) can be assumed minimal (i.e., a minimal closed left ideal of T_0).

If x, y, and L satisfy (i) and (ii) of (3.2) we will say that (x, y) is an *ergodic pair*, x being an *ergodic element* and y a *generalized fixed point*, and that *L is constant* ($= y$) *at x*, y being *the value which L assumes at x*. If S is restrictedly weak* ergodic then the set of all ergodic pairs is (the graph of) a function (because there is only one minimal closed left ideal), say p; the set $\{x : x \in B,\ p(x) \in B\}$ is the "ergodic subspace" E of Eberlein [6], and the restriction of p to E is a projection of E onto the space of all fixed points of S; thus p is a "B^{**} valued projection" of the space of all ergodic elements onto the space of all generalized fixed points. If S is not assumed to be restrictedly weak* ergodic, (3.2) suggests that the minimal closed left ideals of T_0 can be used to regard the set of all ergodic pairs as the union of (the graphs of) B^{**} valued projections from spaces of ergodic elements onto spaces of generalized fixed points, in the following manner. Given a minimal closed left ideal L of T_0 let $D_L = \{x : x \in B \text{ and } L \text{ is constant at } x\}$ and let p_L be the map which takes each $x \in D_L$ into the value which L assumes at x. D_L is a closed linear subspace of B consisting only of ergodic elements and containing the fixed points of S. p_L is simply the restriction of any member of L to D_L, its range consists only of generalized fixed points, and its restriction to the space $D_L' = \{x : x \in D_L \text{ and } p_L(x) \in B\}$ is a projection of D_L' onto the space of all fixed points of S. In general there may be many minimal closed left ideals L and an element x of B may have many fixed points $p_L(x) \in O_T(x)^-$.

Two subspaces of B are of interest which can be regarded as defining more restricted notions of an ergodic element.

(3.3) THEOREM. *If $x \in B$ and L_0 is a fixed minimal closed left ideal of T_0 then the following statements are equivalent*:

(i) *for each $A \in T$, $Ax \in D_{L_0}$*;

(ii) *for each minimal closed left ideal L of T_0, $x \in D_L$*;

(iii) *for each $A \in T$ and each minimal closed left ideal L of T_0, $Ax \in D_L$*.

Proof. Let x and L_0 be as in the statement of the theorem.

(i) $\Rightarrow$ (ii) $Ax \in D_{L_0}$ iff $x \in D_{L_0 A}$ $(A \in T)$ and so (i) amounts to $x \in D_L$ for each minimal closed left ideal of T_0 of the form $L_0 A$ $(A \in T)$; that such ideals are dense in $K(T_0)$, see (2.6), is what essentially gives us (ii).

Suppose L is a minimal closed left ideal of T_0 and fix $V \in L$. By (2.6) there are nets $\{A_\alpha\}$ in T and $\{V_\alpha\}$ in L_0, both over the same index set, such that $V_\alpha A_\alpha \to V$. Writing $y_\alpha = p_{L_0}(A_\alpha x)$ we have

$$(*) \qquad y_\alpha(\beta) = (V_\alpha(A_\alpha x))\beta = (V_\alpha A_\alpha x)\beta \to (Vx)\beta \qquad (\beta \in B^*).$$

Now take $U \in T_0$. From $(*)$, with $\eta(U)\beta$ in place of β, we get

$$y_\alpha(\eta(U)\beta) \to (Vx)(\eta(U)\beta) = (UVx)\beta \qquad (\beta \in B^*),$$

by the computation in the proof of (3.2). But since each y_α is a generalized fixed point ($y_\alpha = p_{L_0 A_\alpha}(x)$) we have

$$y_\alpha(\eta(U)\beta) = y_\alpha(\beta) \to (Vx)\beta \qquad (\beta \in B^*).$$

Hence $UVx = Vx$. We have proved that the left ideal $T_0 V$ is constant $= Vx$ at x. Hence $(T_0 V)^-$ is constant $= Vx$ at x. Since $T_0 V \subset L$ we have $(T_0 V)^- = L$ and the proof is finished.

(ii) $\Rightarrow$ (iii) Given $A \in T$ repeat the argument of (i) $\Rightarrow$ (ii), with Ax in place of x.

Let E be the intersection of all spaces D_L (L a minimal closed left ideal of T_0). E is an invariant[7] closed linear subspace of B. Theorem 3.3 states that for each D_L the set $\{x : x \in B \text{ and for each } F \in S,\ Fx \in D_L\}$ coincides with E; in other words, all the spaces D_L have one and the same largest invariant subspace, namely E. An interesting question is how the various spaces D_L are related to each other or how they might be generated from E.

(3.4) *If $x \in B$ and $y \in B^{**}$ then the following statements are equivalent*:

(i) *$x \in E$ and all p_L (L a minimal closed left ideal of T_0) have the value y at x;*

(ii) *$x \in E$ and (x, y) is the only ergodic pair whose first term is x;*

(iii) *$K(T_0)$ is constant $= y$ at x.*

(iii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii) are trivial and the converses follow from (2.6).

Let $E_0 = \{x : x \in E \text{ and } K(T_0) \text{ is constant at } x\}$. E_0 is an invariant closed linear subspace of E. If S is restrictedly weak* ergodic then $E = E_0 = D'_{K(T_0)}$.

4. Means on spaces of bounded functions on a semigroup

In this section it is shown (in (4.3)) that in a certain special case the compactification semigroup of Sec. 1 can be identified with the convolution semigroup of means on a space of bounded real-valued functions on an abstract semigroup (discussed first by Day in [4]). With this identification the

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[7] A linear subspace M of B is *invariant* under S iff $x \in M$ and $F \in S$ imply $Fx \in M$.

ergodic theorem (3.1) provides a partial extension (in (4.4)) of the Lorentz-Day theorem on almost convergent functions ([7, Sec. 1], and [4, Sec. 9]).

Notations and Terminology. Throughout the remainder of this paper Σ is a fixed semigroup with unit e, $m(\Sigma)$ is the Banach space of all bounded real valued functions on Σ with the sup norm, and r_σ $(\sigma \in \Sigma)$ is the right shift operator induced on $m(\Sigma)$ by multiplication by σ on the right: $r_\sigma\, x(\tau) = x(\tau\sigma)$ $(x \in m(\Sigma),\ \tau \in \Sigma)$. A linear subspace X of $m(\Sigma)$ is *right invariant* iff $x \in X$ implies $r_\sigma\, x \in X$ $(\sigma \in \Sigma)$, X is *admissible* iff X is right invariant, uniformly closed, and contains the constant functions, and X is *right introverted* iff $x \in X$ and $\beta \in m(\Sigma)^*$ imply that the function $\beta(r.\, x) : \sigma \rightarrow \beta(r_\sigma\, x)$ $(\sigma \in \Sigma)$ is again in X. A *finite mean* is a convex combination of evaluation functionals. A *mean* on an admissible subspace X is a functional $\mu \in X^*$ such that $\| \mu \| = \mu(1) = 1$ (equivalently, $\| \mu \| = 1$ and $\mu \geq 0$); μ is *right invariant on* X iff $x \in X$ implies $\mu(x) = \mu(r_\sigma\, x)$ $(\sigma \in \Sigma)$. The *convolution* of two means μ and ν on a right introverted admissible subspace X is the mean $\mu * \nu$ defined by $\mu * \nu(x) = \mu\,(\nu(r.\, x))$ $(x \in X)$.

(4.1) Lemma. *If X is an admissible subspace of $m(\Sigma)$ then the smallest right introverted admissible subspace of $m(\Sigma)$ which contains X, denoted by X^r, coincides with the closed linear span of $Z = \{\beta(r.\, X) : \beta \in m(\Sigma)^*,\ x \in X\}$.*

Proof. Writing β_e for the evaluation functional at e, the identity $\beta_e(r.\, x) = x$ $(x \in X)$ shows that $X \subset Z$. Since Z is also right introverted, its closed linear span contains X^r. The reverse inclusion is clear.

(4.2) Lemma. *If X is an admissible subspace of $m(\Sigma)$ then the set X_r defined by $X_r = \{x : x \in X$ and for each $\beta \in m(\Sigma)^*,\ \beta(r.\, x) \in X\}$ is the largest right introverted admissible subspace of $m(\Sigma)$ contained in X.*

Proof. Given X and X_r as in the statement of the lemma it is not hard to see that X_r is an admissible subspace of $m(\Sigma)$ which lies in X. By (4.1) and construction, $(X_r)^r \subset X$. But also by construction X_r contains every right introverted subspace of $m(\Sigma)$ which lies in X and hence it must contain $(X_r)^r$.

(4.3) Theorem. *Suppose X is an admissible subspace of $m(\Sigma)$; let S be the semigroup of right shift operators r_σ $(\sigma \in \Sigma)$ restricted to X; put $T =$ the convex hull of S and $T_0 = T_0(X) =$ the compactification of T; and let $M = M(X^r)$ be the set of means on X^r. Then there is a mapping ϕ of M onto T_0 such that*

(a) *ϕ is a convex-linear weak*-weak$_*$ homeomorphism and anti-isomorphism, and*

(b) *corresponding elements $\mu \in M$ and $U = \phi(\mu) \in T_0$ satisfy*

$$\mu(\beta(r.\, x)) = (Ux)\beta \qquad (x \in X,\ \beta \in X^*). \tag{1}$$

Proof. Given $\mu \in M$ and using (4.1) and $| \mu(\beta(r.\, x))| \leq \| \beta \|\, \| x\|$ $(x \in X,\ \beta \in X^*)$, formula (1) defines an element $U = \phi(\mu)$ of $L(X, X^{**})$.

The mapping ϕ is clearly convex-linear and weak*-weak$_*$ continuous and since it takes the evaluation functionals into right shift operators, it takes their weak* closed convex hull, namely M, onto T_0. By (4.1) again, ϕ is 1-1 and hence, by compactness, ϕ is a homeomorphism. It remains to show that ϕ is an anti-isomorphism. Suppose $\mu, \nu \in M$; write $U = \phi(\mu)$ and $V = \phi(\nu)$, and choose $\beta \in X^*$ and $x \in X$. We must show that $(VUx)\beta = \mu * \nu(\beta(r.\,x))$. By definition of multiplication in T_0 (cf. Sec. 1) and formula (1),

$$(VUx)\beta = (\eta(U)\eta(V)\beta)x = (Ux)\gamma = \mu(\gamma(r.\,x))$$

where $\gamma = \eta(V)\beta \in X$. Thus we must show that $\gamma(r.\,x) = \nu(r.\,z)$ where $z = \beta(r.\,x)$. Fix $\sigma \in \Sigma$. Since $\beta(r.(r_\sigma x)) = r_\sigma z$ we have

$$\gamma(r_\sigma x) = (\eta(V)\beta)(r_\sigma x) = V(r_\sigma x)\beta = \nu(\beta(r.(r_\sigma x))) = \nu(r_\sigma x).$$

This completes the proof.

Theorems (3.1) and (4.3) together imply the following:

(4.4) THEOREM. *If $x \in m(\Sigma)$ and c is a real number then the following statements are equivalent*:

(i) *Some sequence of convex combinations of right translates of x converges uniformly to $c1$ ($=$ the constant c function on Σ)*;

(ii) *there is a closed left ideal L of $T_0(m(\Sigma))$ such that*

$$\mu(x) = c \qquad (\mu \in \phi^{-1}(L));$$

(iii) *there is a closed left ideal L of $T_0(m(\Sigma))$ such that*

$$Ux = c1 \qquad (U \in L).$$

Proof. By (3.1), (i) and (iii) are equivalent and by formula (1) of (4.3), with β_e (as in the proof of (4.1)) in place of β, (iii) implies (ii). It only remains to show that (ii) implies (iii). Let L be as in (ii) and let $U \in L$, with $\mu = \phi^{-1}(U)$. If β is a *mean* on $m(\Sigma)$ then

$$(Ux)\beta = \mu(\beta(r.\,x)) = \mu * \beta(x) = (VUx)\beta_e = c \tag{2}$$

where $V = \phi(\beta)$. If $\beta \in m(\Sigma)^*$ is arbitrary write $\beta = \lambda_1\beta_1 - \lambda_2\beta_2$ where $\lambda_i \geq 0$ and β_i is a mean on $m(\Sigma)$ ($i = 1, 2$); by (2),

$$(Ux)\beta = \lambda_1(Ux)\beta_1 - \lambda_2(Ux)\beta_2 = \lambda_1 c - \lambda_2 c = \beta(c1).$$

Therefore $Ux = c1$.

A mean μ on $m(\Sigma)$ is right invariant iff $\phi(\mu)$ is a left zero of $T_0(m(\Sigma))$. Thus if right invariant means on $m(\Sigma)$ exist they correspond to elements of the kernel of $T_0(m(\Sigma))$ (cf. (2.8)) and (ii) states that all right invariant means have the value c at x, i.e., x is an almost convergent function.

5. Right amenable subspaces

An admissible subspace X of $m(\Sigma)$ is *right amenable* iff there is a right invariant mean on X. In general $m(\Sigma)$ can have many different maximal right amenable subspaces. The purpose of this section is to prove the following theorem:

THEOREM. *There is a right introverted linear subspace Z of $m(\Sigma)$ such that*
(a) *Z is right amenable;*
(b) *Z contains every right introverted right amenable subspace of $m(\Sigma)$;*
(c) *Z lies in every maximal right amenable subspace of $m(\Sigma)$.*

The proof will be given in several parts; the notations used are the same as those in Theorem (4.3).

Suppose X and Y are admissible subspaces of $m(\Sigma)$ and $X \subset Y$. Let p be the mapping which assigns to each $\mu \in M(Y^r)$ its restriction to X^r, $p(\mu) = \mu \mid X^r \in M(X^r)$. Then p is a weak* continuous convex-linear map of $M(Y^r)$ onto $M(X^r)$ which preserves convolution and can be transferred, via (4.3), to a mapping g of $T_0(Y)$ onto $T_0(X)$.

(5.1) *If $U \in T_0(Y)$, $x \in X$, and $\beta \in Y^*$, then*

$$(Ux)\beta = (g(U)x)\beta' \tag{3}$$

where $\beta' = \beta \mid X$.

Proof. By (4.3), with Y in place of X, equation (3) states that $\mu(\beta(r.\, x) = \nu(\beta'(r.\, x))$ where $\mu = \phi^{-1}(U)$ and $\nu = \mu \mid X^r$, and this is obvious.

The mapping g is the *canonical mapping* of $T_0(Y)$ onto $T_0(X)$; it is weak$_*$ continuous, convex-linear and a homomorphism and hence it maps (closed) ideals into (closed) ideals and minimal (closed) ideals onto minimal (closed) ideals.

Now let Z_0 be the set $\{x : x \in m(\Sigma)$ and every minimal right ideal of $T_0(m(\Sigma))$ is constant at $x\}$. It is not hard to see that Z_0 is an admissible subspace of $m(\Sigma)$. Define $Z = (Z_0)_r$.

Proof of (a). Suppose R is a minimal right ideal of $T_0(Z)$. Writing g for the canonical map of $T_0(m(\Sigma))$ onto $T_0(Z)$, let R' be a minimal right ideal of $T_0(m(\Sigma))$ such that $g(R') = R$. If $z \in Z$ then R' is constant at z and hence, by (5.1), R is constant at z. Since $z \in Z$ was arbitrary, R is a singleton set. Thus $T_0(Z)$ contains a left zero and Z is right amenable.

Proof of (b). Suppose X is a right introverted right amenable subspace of $m(\Sigma)$ and let g be the canonical map of $T_0(m(\Sigma))$ onto $T_0(X)$. Suppose R is a minimal right ideal of $T_0(m(\Sigma))$ and choose $U_1, U_2 \in R$. We know that $g(R)$ is a singleton subset of $T_0(X)$ so $(g(U_1)x)\gamma = (g(U_2)x)\gamma$ $(x \in X, \gamma \in X^*)$ and by (5.1) again this implies $(U_1 x)\beta = (U_2 x)\beta$ $(x \in X, \beta \in m(\Sigma)^*)$. Thus $U_1 x = U_2 x$ $(x \in X)$. Since $U_1, U_2 \in R$ and R itself were arbitrary we have proved that $X \subset Z_0$ and by (4.2) this means $X \subset Z$.

For the proof of (c) we need the following basic fact:

(5.2) LEMMA. *Suppose X is an admissible subspace of $m(\Sigma)$ and μ is a mean on X. Define $R(\mu) = \{U : U \in T_0(X)$ and $\phi^{-1}(U) \mid X = \mu\}$. Then μ is right invariant (on X) iff $R(\mu)$ is a right ideal of $T_0(X)$.*

Proof. Given X and μ as in the statement of the lemma, we have by (1) of (4.3) that $U \in R(\mu)$ iff $(Ux)\beta_e = \mu(x)$ $(x \in X)$. If μ is right invariant choose $U \in R(\mu)$ and $\sigma \in \Sigma$ and get

$$((Ur_\sigma)x)\beta_e = U(r_\sigma x)\beta_e = \mu(r_\sigma x) = \mu(x) \qquad (x \in X),$$

i.e., $Ur_\sigma \in R(\mu)$; since $R(\mu)$ is closed and convex, this implies that $R(\mu)$ is a right ideal. The other implication can be proved by simply reversing the argument.

Proof of (c). Suppose X is a maximal right amenable subspace of $m(\Sigma)$ and μ is a mean on X. Let g be the canonical map of $T_0(m(\Sigma))$ onto $T_0(X)$. Since $R(\mu)$ is a right ideal of $T_0(X)$ (by (5.2)), there is a minimal right ideal R of $T_0(m(\Sigma))$ such that $g(R) \subset R(\mu)$. Let $\nu \in M(m(\Sigma))$ such that $\phi(\nu) \in R$. Then $\nu \mid Z$ is right invariant and $\nu \mid X$ $(= \mu)$ is likewise and hence ν is right invariant on $(X + Z)^-$ (= the closure of the vector sum of X and Z). Since X is a maximal right amenable subspace, this implies $Z \subset X$.

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