THE CONSTRUCTION OF A CLASS OF DIFFUSIONS

BY
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1. Introduction

E. B. Dynkin [4] has shown that the generator of a diffusion on a locally compact, separable space Q has a canonical representation in terms of the mean hitting times and hitting probabilities. Let $x(t)$ be a strict Markov process with generator $\mathfrak{G}$ whose domain is $D(\mathfrak{G})$. Let $f \in D(\mathfrak{G})$, $\xi \in Q$, U be a neighborhood of ξ with compact closure and nonnull boundary and τ^U be defined as inf $(t : x(t) \notin U)$. Then

$$(\mathfrak{G}f)(\xi) = \lim_{U \downarrow \xi} \frac{E_\xi(f(x(\tau^U))) - f(\xi)}{E_\xi(\tau^U)} .$$

It is easy to show that $\mathfrak{G}$ satisfies a maximum property and is a local operator on $C(Q)$. W. Feller [6] has posed the converse question, namely, does every local operator on $C(Q)$ which satisfies the maximum property generate a diffusion. As a partial solution of this problem it will be shown that every such operator arising from a set of mean hitting times and hitting probabilities having certain smoothness properties does indeed generate a diffusion. The method employed is the construction of a sequence of approximating random walks which will be shown to converge to a limit process which is a diffusion. This is an extension of the construction of F. B. Knight [10], [11] for the one-dimensional case.

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2. Some definitions and the main result

Let Q be a locally compact, separable Hausdorff space with metric $\rho(\cdot, \cdot)$. Let $\mathbf{C}$ be the class of all compact subsets of the state space Q and $\mathbf{S}$ be the σ-field generated by $\mathbf{C}$. The sets of $\mathbf{S}$ are called the *Borel sets* of Q [7].

Let Δ be a collection of open sets with nonnull boundaries of the space Q such that

i. the closure of any set of Δ is a compact subset of Q,
ii. Δ is a base for the topology of Q, and
iii. if $D_1, D_2 \in \Delta$, then $D_1 \cup D_2$, $D_1 - \bar{D}_2$, and $D_1 \cap D_2 \in \Delta$ if they are nonempty.

For $D \in \Delta$, let $\mathbf{B}(\partial D)$ be the class of Borel subsets of ∂D, the boundary of D.

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We will assume that Q is noncompact so that we can write $Q = \bigcup_{m=1}^{\infty} \Gamma_m$, $\Gamma_m \supset \Gamma_{m-1}$ and $\Gamma_m \in \Delta$.

A collection $\{h_{\partial D}(\cdot, \cdot), D \in \Delta\}$ of real-valued functions on $\bar{D} \times \mathbf{B}(\partial D)$ is called *a family of smooth hitting probabilities* if:

i. $h_{\partial D}(\cdot, A)$, $A \in \mathbf{B}(\partial D)$, is measurable with respect to $\mathbf{S}$,
ii. $h_{\partial D}(\xi, \cdot)$, $\xi \in D$, is a probability measure on $\mathbf{B}(\partial D)$,
iii. if $\xi \in \partial D$, $h_{\partial D}(\xi, \{\xi\}) = 1$,
iv. if f is a nonnegative function measurable with respect to $\mathbf{B}(\partial D)$, then $u(z) \equiv \int_{\partial D} f(\xi) h_{\partial D}(z, d\xi)$ is continuous if finite-valued on D and is continuous on $\bar{D}$ if f is continuous on ∂D, and
v. if D_1, $D_2 \in \Delta$ and $D_1 \subset D_2$, $\xi \in D_1$, then

$$h_{\partial D_2}(\xi, A) = \int_{\partial D_1} h_{\partial D_1}(\xi, d\eta) h_{\partial D_2}(\eta, A).$$

A finite real-valued function, v, is said to be *subharmonic relative to the family* $\{h_{\partial D}(\cdot, \cdot), D \in \Delta\}$ if it is upper semi-continuous and if

$$v(z) \leq \int_{\partial D} h_{\partial D}(z, d\eta) v(\eta) \qquad \text{for every } D \in \Delta.$$

v is said to be *superharmonic* if $-v$ is subharmonic.

The *fine topology* induced on Q by the family $\{h_{\partial D}(\cdot, \cdot), D \in \Delta\}$ is the least fine topology such that all superharmonic functions are continuous in this topology, that is, it is generated by sets of the form

$$N_\xi = \{\eta : \eta \in Q, \mid v(\eta) - v(\xi) \mid < \varepsilon, \varepsilon > 0, v \text{ superharmonic}\}.$$

N_ξ is called a fine neighborhood of ξ.

A collection $\{e_D(\cdot) : D \in \Delta\}$ of real-valued functions on D is called *a family of smooth mean hitting times with respect to* $\{h_{\partial D}(\cdot, \cdot), D \in \Delta\}$ if:

i. $0 < e_D(\xi) < \infty$, $\xi \in D$,
ii. if D_1, $D_2 \in \Delta$, $D_1 \subset D_2$, $\xi \in D_1$, then

$$e_{D_2}(\xi) = e_{D_1}(\xi) + \int_{\partial D_1 - \partial D_2} h_{\partial D_1}(\xi, d\eta) e_{D_2}(\eta),$$

iii. if $D_m \uparrow D$, D_m, $D \in \Delta$, $\xi \in D_m$ for every m, then $e_{D_m}(\xi) \uparrow e_D(\xi)$, and
iv. $e_D(\xi)$ is a continuous function of $\xi \in D$.

A family of smooth mean hitting times is said to satisfy the *fine neighborhood condition* if for any fine neighborhood N_ξ of ξ and $D_m \in \Delta$, $D_m \downarrow N_\xi$ there exists an $\varepsilon > 0$ such that $e_{D_m}(\xi) \geq \varepsilon$ for every m.

Given a class of regular sets Δ of a space Q, a family of smooth hitting probabilities $\{h_{\partial D}(\cdot, \cdot) : D \in \Delta\}$ and a family of smooth mean hitting times, $\{e_D(\cdot) : D \in \Delta\}$, satisfying the fine neighborhood condition we construct an operator $\mathfrak{G}$ as follows. If $u \in C(Q)$,

$$\mathfrak{G}u(\xi) \equiv \lim_{\substack{D \in \Delta \\ D \downarrow \xi}} \frac{\int_{\partial D} h_{\partial D}(\xi, d\eta)u(\eta) - u(\xi)}{e_D(\xi)} \quad \text{if the limit exists,}$$
$$\equiv +\infty \quad \text{otherwise.}$$

Then $\mathfrak{G}$ is a linear operator on the domain

$$D(\mathfrak{G}) \equiv \{u : u \in C(Q), \mathfrak{G}u \in C(Q)\}$$

and is called *a generalized differential operator.*

Following [5] the definitions of Markov process, strict Markov process and stationary Markov process will now be given.

Consider

a. a space Ω and a function $\zeta(w)$, $\zeta : \Omega \to [0, \infty]$,
b. a function $x(t, w) = x_t(w)$ defined for $w \in \Omega$, $t \in [0, \zeta(w)]$, whose range is the measure space $(Q, \mathbf{S})$ (by convention we say that $x(t, w) \notin Q$ for $t > \zeta(w)$),
c. for each $0 \leq s \leq t$ a σ-field F_t^s in the space $\Omega_t = \{w : \zeta(w) > t\}$, such that $F_{t_1}^{s_1} \subset F_{t_2}^{s_2}$ if $s_1 > s_2$, and $t_1 \leq t_2$, and
d. for each $s \geq 0$, $x \in Q$, a function $P_{s,x}(\cdot)$ on the smallest σ-field F^s which contains F_t^s for each $t \geq s$.

These elements define a *Markov process* $X = (x_t, \zeta, F_t^s, P_{s,x})$ on the space Q if the following conditions are satisfied:

1. $s \leq t \leq u$ and $A \in F_t^s$ implies that $\{A, \zeta > u\} \in F_u^s$,
2. $\{x_t \in \Gamma\} \in F_t^s$ for any $0 \leq s \leq t$, $\Gamma \in \mathbf{S}$
3. $P_{s,x}$ is a probability measure on the σ-field F^s,
4. for any $0 \leq s \leq t$, $\Gamma \in \mathbf{S}$, $P(s, x; t, \Gamma) \equiv P_{s,x}\{x_t \in \Gamma\}$ is an $\mathbf{S}$-measurable function of x,
5. $P(s, x; s, Q - \{x\}) = 0$, and
6. if $0 \leq s \leq t \leq u$, $x \in Q$, $\Gamma \in \mathbf{S}$, then

$$P_{s,x}\{x_u \in \Gamma \mid F_t^s\} = P(t, x_t; u, \Gamma), \qquad \text{a.e. } [\Omega_t, P_{s,x}].$$

The function $x(u, w) = x_u(w)$ induces a mapping of the measure space $([s, t] \times \Omega_t, B_t^s \times F_t^s)$ into $(Q, \mathbf{S})$. (B_t^s is the σ-field of subsets of $[s, t]$ generated by intervals.) The Markov process is said to be *measurable* if this mapping is measurable for any $0 \leq s \leq t$.

A nonnegative function $\tau(w)$ is an *s-Markov time* if

i. $s \leq \tau(w) \leq \max [s, \zeta(w)]$, $w \in \Omega$, and
ii. $\{w : \tau(w) < t < \zeta(w)\} \in F_t^s$, $s \leq t$.

The subsets $A \subset \Omega_\tau$ such that for any $t \geq s$, $\{A, \tau < t < \zeta\} \in F_t^s$ form a σ-field in Ω_τ denoted by $F_{\tau+}^s$.

The Markov process $X = (x_t, \zeta, F_t^s, P_{s,x})$ in $(Q, \mathbf{S})$ is said to be *strict Markov* if it is measurable and satisfies:

i. for any $t \geq 0$, $\Gamma \in \mathbf{S}$, $P(s, x; t, \Gamma) = P_{s,x}\{x_t \in \Gamma\}$ is a $B_t^0 \times \mathbf{S}$ measurable function of s and x, and
ii. if τ is an s-Markov time, we have for any $F_{\tau+}^s$ measurable function $\eta(w) \geq \tau(w)$ and for any $x \in Q$, $\Gamma \in \mathbf{S}$,

$$P_{s,x}\{x_\eta \in \Gamma \mid F_{\tau+}^s\} = P(\tau, x_\tau : \eta, \Gamma), \qquad \text{a.e. } [\Omega_\tau, P_{s,x}].$$

Let F^* be the minimal system of subsets of the space $\Omega_0 = \{\zeta > 0\}$ that contains all the sets $\{x_t \in \Gamma\}$, $t \geq 0$, $\Gamma \in \mathbf{S}$, and is closed with respect to the operations of taking complements and countable unions and intersections.

The Markov process $X = (x_t, \zeta, F_t^s, P_{s,x})$ is said to be *stationary* if for any $t \geq 0$ there is a field homomorphism $\theta_t : F^* \to F^*$ such that
i. $\theta_t \Omega_0 = \Omega_t$,
ii. $\theta_t\{x_h \in \Gamma\} = \{x_{t+h} \in \Gamma\}$, $h \geq 0$, $\Gamma \in \mathbf{S}$, and
iii. for any $A \in F^*$, $P_{t,x}(\theta_t A) = P_{0,x}(A)$.
For a stationary process it suffices to consider the measures $P_x(\cdot) \equiv P_{0,x}(\cdot)$ and 0-Markov times which will be called simply *Markov times.*

A stationary strict Markov process $X = (x_t, \zeta, F_t^s, P_x)$ is a *diffusion* if

$$P_\xi(x(t, w) \text{ is a continuous function of } t \in [0, \zeta]) = 1 \qquad \text{for all } \xi \in Q.$$

MAIN RESULT. *Let Δ be a class of regular subsets of Q, $\{h_{\partial D}(\cdot, \cdot), D \in \Delta\}$ be a family of smooth hitting probabilities and $\{e_D(\cdot), D \in \Delta\}$ be a family of smooth mean hitting times which satisfy the fine neighborhood condition. Then for any Γ_n there exists a diffusion $\tilde{X} = (\tilde{x}_t, \zeta^n, F_t^s, P_x)$ such that if*

$$\tilde{\tau}^D(w) \equiv \inf (t : \tilde{x}(t) \notin D), \qquad D \in \Delta, \qquad D \subset \Gamma_n, \qquad \xi_0 \in D,$$

then

$$P_{\xi_0}(\tilde{x}(\tilde{\tau}^D) \in \cdot) = h_{\partial D}(\xi_0, \cdot), \tag{2.1}$$

$$E_{\xi_0}(\tilde{\tau}^D) = e_D(\xi_0), \tag{2.2}$$

and

$$\zeta^n(w) = \tilde{\tau}^{\Gamma_n}(w). \tag{2.3}$$

This implies that a restriction of the generalized differential operator $\mathfrak{G}$, arising from the families $\{h_{\partial D}(\cdot, \cdot), D \in \Delta\}$ and $\{e_D(\cdot), D \in \Delta\}$, to some linear subspace of $C(Q)$ is the generator of the diffusion $\tilde{X}$.

Let us first give an outline of the proof of this result. We begin by defining a sequence, $\{\mathcal{C}_m\}$, of open coverings of the space Q. Next we construct a sequence of generalized random walks in which at each step a jump is made to the closest boundary of a set of $\mathcal{C}_m$ in accordance with the given hitting probabilities. Then by mapping the ordered set of jumps into [0, 1] we construct a continuous, strict, nonstationary Markov process $x(s)$. To obtain the probability structure of $x(s)$ we make use of the projective limit of the generalized random walks. It is then shown that we can define a natural time parameter for the paths of $x(s)$ by means of limits of sums of mean hitting times associated with the successive steps of the random walks. This natural time parameter is shown to be a continous, strictly increasing function of s. Finally by reparameterizing the paths of $x(s)$ with the natural time parameter it is verified that we obtain the required diffusion.

3. The sequence of generalized random walks

In this section a sequence of generalized random walks will be constructed. By a generalized random walk is meant a random walk with the time parameter ranging over the ordinals. Following [1, pp. 71–76] the ordinals will be designated by $1, 2, \cdots, \omega, \cdots, 2\omega, \cdots, \omega^2, \cdots, \omega^\omega, \cdots, \omega^{\omega^\omega}, \cdots, \varepsilon_0, \cdots$, and if X is a set of ordinals $h(X)$ is the ordinal number of the well ordered set of all ordinals smaller than or equal to some ordinal of X.

We first define a sequence of coverings, $\{\mathcal{C}_m\}$, of the space Q. We require the following lemma.

LEMMA 3.1. *Given* $\varepsilon > 0$, $\xi \in Q$, *there exists a* $D \in \Delta$ *such that* $\xi \in D$ *and* $\| e_D(\cdot) \| \leq \varepsilon$ *where* $\| \cdot \|$ *is the sup norm.*

Proof. Choose a set D' such that $\xi \in D'$ and $D' \in \Delta$. Then there is a set $D \subset D'$ such that $\xi \in D$, $D \in \Delta$ and such that

$$| e_{D'}(\xi) - e_{D'}(\eta) | < \varepsilon \qquad \text{for } \eta \in \bar{D}.$$

Then

$$e_D(\eta) = e_{D'}(\eta) - \int_{\partial D} h_{\partial D}(\eta, dz) e_{D'}(z) < 2\varepsilon,$$

Q.E.D.

Since Q is separable there is a countable class of sets $\{\tilde{\Gamma}_m, m \in Z^+\}$, $Z^+ = \{0, 1, 2, \cdots\}$, such that $\{\tilde{\Gamma}_m, m \in Z^+\}$ is a base for the topology on Q and each $\tilde{\Gamma}_m \in \Delta$. Moreover, there is a subclass $\mathcal{C}_0$ of $\{\tilde{\Gamma}_m, m \in Z^+\}$ such that (i) if $D \in \mathcal{C}_0$, $\| e_D \| \leq 1$, (ii) $\mathcal{C}_0$ is locally finite, that is, any compact subset of Q has nonnull intersection with only finitely many sets of $\mathcal{C}_0$, and (iii) $\mathcal{C}_0$ is an open covering of Q which is a refinement of $\{\Gamma_i : i \in Z^+\}$ where $\Gamma_P \equiv \bigcup_{m=1}^{P} \tilde{\Gamma}_m$. Also, $Q = \bigcup_{m=1}^{\infty} \Gamma_m$ and $\Gamma_m \supset \Gamma_{m-1}$ [9, Chapter 5].

Hence for any $K \in Z^+$ and $\xi \in Q$ there is a set $D \in \Delta$ such that $\xi \in D$ and $0 < \| e_D \| \leq 2^{-K}$. Let us make some selection of such sets for each ξ and K (axiom of choice), $\tilde{D}(K, \xi)$, such that diam $(\tilde{D}(K, \xi)) \downarrow 0$ uniformly for $\xi \in Q$ as $K \to \infty$. Let $\mathfrak{D}_K \equiv \{\tilde{D}(K, \xi), \xi \in Q\}$.

We now define the sequence of coverings $\{\mathcal{C}_m\}$ inductively, starting with $\mathcal{C}_0$. Given $\mathcal{C}_{m-1}$, we obtain $\mathcal{C}_m$ as follows. For any set $D \in \mathcal{C}_{m-1}$ the collection of sets $\{\tilde{D}(m, \xi) : \tilde{D}(m, \xi) \in \mathfrak{D}_m, \xi \in \bar{D}\}$ forms an open covering of the compact set $\bar{D}$ and hence there is a finite subcovering, $\mathcal{C}'_m(\bar{D})$. $\mathcal{C}_m$ is the collection of open sets obtained by considering all intersections of sets of $\mathcal{C}_{m-1}$ and sets of $\{\mathcal{C}'_m(\bar{D}), D \in \mathcal{C}_{m-1}\}$. $\mathcal{C}_m$ is locally finite. Furthermore, $\mathcal{C}_m$ is a refinement of $\mathcal{C}_{m-1}$, that is, every set of $\mathcal{C}_m$ is a subset of a set of $\mathcal{C}_{m-1}$. Let

$\mathcal{C} \equiv \bigcup_{m=1}^{\infty} \mathcal{C}_m$, $\mathcal{C}_m^* \equiv \Delta \cap$ the field generated by $\mathcal{C}_m$ and the closures of sets of $\mathcal{C}_m$, and $\mathcal{C}^* \equiv \bigcup_{m=1}^{\infty} \mathcal{C}_m^*$.

Given $\xi \in Q$, $K \in Z^+$ we assign a set $D(K, \xi)$ as follows:

$$D(K, \xi) \equiv \bigcap\{D : \xi \in D,\ D \in \mathcal{C}_K\} - \bigcup\{D,\ \xi \notin D,\ D \in \mathcal{C}_K\}.$$

Since $\mathcal{C}_K$ is locally finite, $D(K, \xi)$ is nonnull and $D(K, \xi) \in \Delta$. Furthermore, $\xi \in D(K, \xi)$ and $0 < \| e_{D(K,\xi)} \| \leq 2^{-K}$. In other words, $D(K, \xi)$ is the smallest open set containing ξ in $\mathcal{C}_K^*$.

We can now construct the space of paths of the random walks. Since Q is noncompact, we can adjoin to Q the point ∞, define Q' as $Q \cup \{\infty\}$, and topologize Q' so that it is the one point compactification of Q. The open sets of Q' are the open sets of Q and the complements in Q' of the compacts subsets of Q.

A *K-path* starting at ξ_0, designated by $w_K(\cdot)$, is a mapping from the set $\{1, 2, \cdots, \varepsilon_0\}$ into Q' such that:

i. $w_K(0) = \xi_0$,

ii. $w_K(\alpha + 1) \in \partial D(K, w_K(\alpha))$ unless $w_K(\alpha) = \infty$, if $w_K(\alpha) = \infty$, then $w_K(\alpha + 1) = \infty$, and

iii. if $\{w_K(\alpha_m),\ m \in Z^+\}$ has ∞ as a cluster point, then $w_K(h(\{\alpha_m\})) = \infty$.

The class of all such K-paths is designated by $\Omega_K^{\xi_0}$. $w_K(\cdot)$ has K generalized subsequences $w_K^i(\cdot)$, $0 \leq i \leq K - 1$. $w_K^i(\cdot)$ is the generalized subsequence of $w_K(\cdot)$ such that $w_K^i(0) = w_K(0)$, $w_K^i(\alpha + 1) \in \partial D(i, w_K^i(\alpha))$, and if $w_K^i(\alpha_m)$ corresponds to $w_K(\beta_m)$ for $m \in Z^+$, then $w_K^i(h(\{\alpha_m\})) = w_K(h(\{\beta_m\}))$. Clearly $w_K^i(\cdot) \in \Omega_i^{\xi_0}$.

Let $\Omega_K \equiv \bigcup_{\xi_0 \in Q} \Omega_K^{\xi_0}$. We can define a mapping

$$M_K : \Omega_K \to \Omega_{K-1} \text{ (onto)} \quad \text{by} \quad M_K(w_K(\cdot)) = w_K^{K-1}(\cdot).$$

Under this mapping we can construct the total inverse image in Ω_K of any $w_{K-1}(\cdot)$.

We are now going to interpret Ω_K as the sample space of a generalized random walk having the one step transition probabilities induced by the hitting probabilities.

Let $F_K^{(\alpha_0)}$ be the least σ-field of subset of Ω_K generated by sets of the form

$$\{w_K(\cdot) : w_K(\alpha) \in A,\ A \in \mathbf{C},\ \alpha \leq \alpha_0\}$$

and

$$\{w_K(\cdot) : w_K(\alpha) = \infty,\ \alpha \leq \alpha_0\}.$$

Similarly, let $F_K^{(\alpha_0-)}$ be the least σ-field of subsets of Ω_K generated by sets of the form

$$\{w_K(\cdot) : w_K(\alpha) \in A,\ A \in \mathbf{C},\ \alpha < \alpha_0\}$$

and

$$\{w_K(\cdot) : w_K(\alpha) = \infty,\ \alpha < \alpha_0\}.$$

Let $\tilde{F}_K^{(\alpha_0)}$ and $\tilde{F}_K^{(\alpha_0-)}$ be the corresponding fields of subsets of Ω_K.

For each $\xi_0 \in Q$, by the iteration of the transition probabilities

$$P_{w_K(\alpha)}(w_K(\alpha+1) \in A) = h_{\partial D(K, w_K(\alpha))}(w_K(\alpha), A), \qquad A \in \mathbf{S},$$

we can obtain a probability measure on the field $\tilde{F}_K^{(\omega-)}$ which assigns zero measure to the set $\{w_K(\cdot) : w_K(0) \neq \xi_0\}$. By the Kolmogorov extension theorem this measure can be extended to a probability measure on $F_K^{(\omega-)}$. The probability measure space thus obtained will be denoted by

$$(\Omega_K^{(\omega-)}, F_K^{(\omega-)}, P_{K,\xi_0}^{(\omega-)}).$$

If $\xi_0 \in \Gamma_n$ and $w_K(\cdot) \in \Omega_K^{\xi_0}$, let

$$\delta_n^K(w_K(\cdot)) \equiv \text{glb}\,\{\alpha : w_K(\alpha) \notin \Gamma_n\}.$$

For $\xi_0 \in \Gamma_n$, let

$$B_n^{\xi_0} \equiv \{w_K(\cdot) : w_K(\cdot) \in \Omega_K^{\xi_0}, \delta_n^K(w_K(\cdot)) \geq \omega\}.$$

Clearly $B_n^{\xi_0} \in F_K^{(\omega-)}$. Let $D_K^{m+1} \equiv D(K, w_K(m))$ and $e_K^m \equiv e_{D_K^m}(w_K(m))$.

We next proceed to extend the probability measure $P_{K,\xi_0}^{(\omega-)}$ to $F_K^{(\omega)}$ and then to $F_K^{(\varepsilon_0)}$.

LEMMA 3.2. *If $\xi \in \Gamma_n$, then $E_{K,\xi}(\sum_{m=1}^{\omega-} \tilde{e}_K^m) \leq e_{\Gamma_n}(\xi)$* where

$$\begin{aligned} \tilde{e}_K^m &\equiv e_K^m, \qquad m < \delta_n^K, \\ &\equiv 0, \qquad m \geq \delta_n^K. \end{aligned}$$

Proof.

$$\begin{aligned} e_{\Gamma_n}(\xi) &= \tilde{e}_K^1 + \int_{\partial D_K^1} h_{\partial D_K^1}(\xi, d\eta_1) e_{\Gamma_n}(\eta_1) \\ &= \tilde{e}_K^1 + \int_{\partial D_K^1} h_{\partial D_K^1}(\xi, d\eta) \left[\tilde{e}_K^2 + \int_{\partial D_K^2} h_{\partial D_K^2}(\eta_1, d\eta_2) e_{\Gamma_n}(\eta_2) \right] \\ &= E_{K,\xi}[\tilde{e}_K^1 + \tilde{e}_K^2] + \int_{\Gamma_n} e_{\Gamma_n}(\eta) P_{K,\xi}[w_K(2) \in d\eta]. \end{aligned}$$

By the continuation of this procedure for a finite number of steps we obtain

$$e_{\Gamma_n}(\xi) = E_{K,\xi}[\tilde{e}_K^1 + \cdots + \tilde{e}_K^m] + \int_{\Gamma_n} e_{\Gamma_n}(\eta) P_{K,\xi}(w_K(m) \in d\eta).$$

Hence, $E_{K,\xi}(\sum_{m=1}^{\omega-} \tilde{e}_K^m) \leq e_{\Gamma_n}(\xi)$, Q.E.D.

If $w_K(\cdot) \in B_n^{\xi_0}$, $w_K(1)$, $w_K(2)$, $\cdots$ form a countable set of points in a compact space and hence has a nonempty set of cluster points $\{y_1, y_2, \cdots\}$.

THEOREM 3.1. *The $P_{K,\xi}^{(\omega-)}$- probability that a path belonging to $B_n^{\xi_0}$ has more than one cluster point is zero.*

Proof. The cluster points belong to the closed set

$$\bigcup\{\partial D : D \in \mathcal{C}_K(\Gamma_n)\} \quad \text{where} \quad \mathcal{C}_K(\Gamma_n) \equiv \mathcal{C}_K \cap \Gamma_n.$$

Let $\{G_j : j \in Z^+\}$ be the subsets of $\{\tilde{\Gamma}_m\}$ that have nonnull intersections with Γ_n. Then it suffices to show that Λ_n^{st} which is defined to be the set of paths in $B_n^{\xi_0}$ which have cluster points in G_s and G_t, $\rho(G_s, G_t) \neq 0$, has $P_{K,\xi_0}^{(\omega-)}$-probability zero. $\Lambda_n^{st} \in F_K^{(\omega-)}$. Since $\rho(\partial G_s, \partial G_t) \neq 0$, there is a set $G_t' \supset G_t$ such that $\rho(\partial G_s, \partial G_t') \neq 0$ and such that $e_{G_t'}(\xi) > \eta$ for some $\eta > 0$ and every $\xi \in G_t$. Every path in Λ_n^{st} makes infinitely many exits from the interior of G_t to the complement of G_t'. If $P_{K,\xi_0}^{(\omega-)}(\Lambda_n^{st}) = \alpha > 0$, it follows that

$$e_{\Gamma_n}(\xi_0) \geq \alpha \cdot N \cdot \eta$$

for arbitrary N by an argument similar to that given in the proof of Lemma 3.2. However this yields a contradiction, whence $P_{K,\xi_0}^{(\omega-)}(\Lambda_n^{st}) = 0$, Q.E.D.

LEMMA 3.3. *$P_{K,\xi_0}^{(\omega-)}(\cdot)$ can be extended to a probability measure on the σ-field $F_K^{(\omega)}$ by continuity.*

Proof. This is accomplished by considering the regular content $P_K^{(\omega)}(w_K(\omega) \in \cdot)$ defined on the class of compact sets by the continuity of the $w_K(\alpha)$ as $\alpha \to \omega$ and then extending this to a regular Borel measure on $F_K^{(\omega)}$ by the method of Halmos [7, chapt. 10]. In more detail the content is defined as follows. We can write $C = \bigcap_{m=1}^{\infty} O_m$ where the O_m are open sets and $O_m \supset O_{m+1}$ for any $C \in \mathbf{C}$. Let

$$P_K^{(\omega)}(w_K(\omega) \in C) \equiv \lim_{p\to\infty} P_{K,\xi_0}^{(\omega-)}[\bigcup_{r=1}^{\infty} \bigcap_{m=r}^{\infty}\{w_K(\cdot) : w_K(m) \in O_p\}].$$

The limit exists and by a result of Halmos [7, Theorem C, p. 238] it follows that $P_K^{(\omega)}(w_K(\omega) \in \cdot)$ is a regular content on $\mathbf{C}$, Q.E.D.

Since $e_{\Gamma_n}(\cdot)$ is continuous it is easy to show that

$$e_{\Gamma_n}(\xi) = E_{K,\xi}\left[\sum_{m=1}^{\omega-} \tilde{e}_K^m\right] + \int_{\Gamma_n} e_{\Gamma_n}(\eta) P_{K,\xi}^{(\omega)}(w_K(\omega) \in d\eta).$$

This construction can be extended to obtain $P_{K,\xi_0}^{(\alpha)}$ for $\alpha = \omega + 1, \omega + 2, \cdots, \omega \cdot 2, \cdots, \varepsilon_0$ (Principle of transfinite induction [1]). That is, we can define $(\Omega_K^{\varepsilon_0}, F_K^{(\varepsilon_0)}, P_{K,\xi_0}^{(\varepsilon_0)})$. Note that $P_{K,\xi_0}^{(\varepsilon_0)}$ is concentrated on paths which are left continuous at ∞ in the sense of condition iii of the definition of K-paths. Furthermore, an argument similar to that of Theorem 3.1 yields the following result.

THEOREM 3.2. *Let $\{\alpha_m\}$ be a sequence of ordinals less than ε_0. Then, except for a set of paths of $P_{K,\xi_0}^{(\varepsilon_0)}$- probability zero, either $w_K(\alpha_m)$ converges to a point of Q as $m \to \omega-$ or else $w_K(h\{\alpha_m\}) = \infty$.*

THEOREM 3.3. *Except for a set of paths of $P_{K,\xi_0}^{(\varepsilon_0)}$-probability zero, $w_K(\varepsilon_0) = \infty$.*

Proof. If $\delta_n^K(w_K(\cdot)) \geq \omega$, $w_K(m) \to x_0$ as $m \to \omega$ where

$$x_0 \in \bigcup\{\partial D : D \in \mathcal{C}_K(\Gamma_n)\}.$$

Moreover for points $x_m \equiv w_K(m)$ sufficiently near x_0, $e_{D(K,x_m)}(x_m) \downarrow 0$ except for a set of paths of $P_{K,\xi_0}^{(\varepsilon_0)}$-probability zero. Otherwise as in the proof of Theorem 3.1 we can show that $e_{\Gamma_n}(\xi_0)$ is arbitrarily large yielding a contradiction. Since $e_{D(K,\xi)}(\cdot)$ is bounded away from zero in a neighborhood of ξ for $\xi \notin \bigcup\{\partial D, D \in \mathcal{C}_K(\Gamma_n)\}$, this implies that x_0 must lie on the intersection of the boundaries of two distinct sets, that is $x_0 \in \partial D_1 \cap \partial D_2$, $D_1 \neq D_2$, D_1 and $D_2 \in \mathcal{C}_K(\Gamma_n)$. In the same way we conclude that if $\delta_n^K(w_K(\cdot)) \geq \omega^\omega$, $w_K(\omega^\omega)$ must lie on the intersection of the boundaries of at least three distinct sets. Let ω_m be defined recursively by $\omega_m = \omega_{m-1}^\omega$ and $\omega_0 = \omega$. Then in general $w_K(\omega_m)$ must lie on the intersection of the boundaries of at least $m + 1$ different sets. However since there are only finitely many sets in $\mathcal{C}_K(\Gamma_n)$, $\delta_n^K(w_K(\cdot)) < \varepsilon_0$. Hence $w_K(\varepsilon_0) = \infty$ since $P_{K,\xi_0}^{(\varepsilon_0)}$ is concentrated on paths left continuous at ∞, Q.E.D.

COROLLARY. $e_{\Gamma_n}(\xi) = E_{K,\xi}(\sum_{m=1}^{\varepsilon_0} \tilde{e}_K^m)$.

The generalized random walk which has been constructed is designated by R_K. The importance of the fact that $w_K(\varepsilon_0) = \infty$ is that the cardinal number of ε_0 is $\aleph_0$ so that all the subsets of $\Omega_K^{\varepsilon_0}$ determined by conditions on the successive steps of the random walk are measurable.

4. The projective limit of $(\Omega_K^{\varepsilon_0}, F_K^{(\varepsilon_0)}, P_{K,\xi_0}^{(\varepsilon_0)})$

Let the topology on $\Omega_K^{\varepsilon_0}$ be the product topology induced by the topology of Q' in the space $\prod\{Q'_\alpha, \alpha \leq \varepsilon_0\}$.

LEMMA 4.1. *Let $D \in \mathcal{C}_K^*$. Then if $\xi \in D$,*

$$P_{K,\xi}^{(\varepsilon_0)}(w_K(\delta_K^D) \in A) = h_{\partial D}(\xi, A)$$

where $\delta_K^D = \inf\{\alpha : w_K(\alpha) \notin D\}$.

Proof. Proceeding stepwise we obtain

$$h_{\partial D}(\xi, A) = h_{\partial D_K^1}(\xi, A) + \Bigg\{\int_{\partial D_K^1 - \partial D} h_{\partial D_K^1}(\xi, d\eta_1)$$
$$\cdot\Bigg[\cdots + h_{\partial D_K^p}(\eta_p, A) + \int_{\partial D_K^p - \partial D} h_{\partial D_K^p}(\eta_p, d\eta_{p+1}) h_{\partial D}(\eta_{p+1}, A)\Bigg]\Bigg\}.$$

The contribution of the last term on the right hand side goes to zero as $p \to \varepsilon_0$. Hence the result follows since the remaining terms represent $P_{K,\xi}^{(\varepsilon_0)}(w_K(\delta_K^D) \in A)$, Q.E.D.

THEOREM 4.1. *The measure spaces $(\Omega_K^{\varepsilon_0}, F_K^{(\varepsilon_0)}, P_{K,\xi_0}^{(\varepsilon_0)})$ form a stochastic process in the sense of Bochner* [2] *with mappings $M_{K+1} \cdots M_L : \Omega_L^{\varepsilon_0} \to \Omega_K^{\varepsilon_0}$ for $L > K$.*

Proof. We will first show that the spaces $(\Omega_K^{\varepsilon_0}, F_K^{(\varepsilon_0)}, P_{K,\xi_0}^{(\varepsilon_0)})$ are *regular*, that is, any measurable set can be approximated in measure by a compact set.

By the approximation theorem [3, Theorem 2.3, p. 605] it suffices to show that a set of the field $\tilde{F}_K^{(\varepsilon_0)}$ may be approximated in measure by a compact set. But the latter follows immediately from Tychonoff's theorem since if A_α, $\alpha < m$, are Borel subsets of Q' then

$$P_{K,\xi_0}^{(\varepsilon_0)}(w_K(m) \in \cdot\,, w_K(\alpha) \in A_\alpha\,, \alpha < m)$$

is a regular Borel measure.

The mappings M_K are continuous in the product topology of

$$\prod\{Q'_\alpha : \alpha \leq \varepsilon_0\}.$$

We next prove that for each $K > 0$ the total inverse mapping M_K^{-1} of M_K is a measure preserving mapping from

$$(\Omega_{K-1}^{\varepsilon_0}, F_{K-1}^{(\varepsilon_0)}, P_{K-1,\xi_0}^{(\varepsilon_0)}) \quad \text{onto} \quad (\Omega_K^{\varepsilon_0}, F_K^{(\varepsilon_0)}, P_{K,\xi_0}^{(\varepsilon_0)}).$$

It suffices to demonstrate that M_K^{-1} preserves the measure of a set of the field $\tilde{F}_{K-1}^{(\varepsilon_0)}$. But this follows immediately from Lemma 4.1.

The theorem then follows from a result of Bochner [2, Theorem 5.1.1], Q.E.D.

The projective limit process obtained will be denoted by R_∞ with probability measure space $(\Omega_\infty, F_\infty\ P_{\infty,\xi_0})$. The projective limit Ω_∞ of the spaces $\{\Omega_K^{\varepsilon_0}\}$ is the set of all sequences $(w_1(\cdot), w_2(\cdot), \cdots)$ such that $M_K(w_K(\cdot)) = w_{K-1}(\cdot)$ for each $K > 0$. Each set $B_K \in F_K^{(\varepsilon_0)}$ is the projection onto $\Omega_K^{\varepsilon_0}$ of the set of all elements of whose K^{th} components are in B_K. The theorem means that the finitely additive measure induced on Ω_∞ by the projective inverses of all $B_K \in F_K^{(\varepsilon_0)}$, $P_{K,\xi_0}^{(\varepsilon_0)}(B_K)$, $K \in Z^+$, can be extended to a countably additive measure $P_{\infty,\xi_0}(\cdot)$ on the least σ-field containing the projective inverses of each such B_K. The elements of Ω_∞ will be called R_∞-paths and will be denoted by w_∞.

For each $\alpha \leq \varepsilon_0$, $K > 0$, let $E_K(\alpha, w_\infty)$ or simply $E_K(\alpha)$ be the least ordinal such that if $M_K\, w_K(\cdot) = w_{K-1}(\cdot)$, then the ordered set $w_K(1), \cdots, w_K(E_K(\alpha))$ contains $w_{K-1}(1), \cdots, w_{K-1}(\alpha)$ as an ordered subset. $E_K(\alpha, \cdot)$ is a random variable on $(\Omega_\infty, F_\infty)$ whose range is the set of ordinals $\{1, 2, \cdots, \varepsilon_0\}$.

5. The nonstationary Markov process, X

We shall now introduce a nonstationary strict Markov process,

$$X = (x_s, 1, F_{st}^\bullet, P_{s,x}),$$

up to the boundary of Γ_n. It will later be shown that X can be reparameterized to yield the required diffusion.

Let

$$B_2^n \equiv \{K/2^n, K = 0, 1, \cdots, 2^n\}, \qquad B_2 \equiv \bigcup_{n=1}^\infty B_2^n,$$

and

$$B_2^{s_0} \equiv \{t : t \in B_2, t > s_0\} \cup \{s_0\}.$$

DEFINITION 5.1. If $p \in Z^+$ or $p = \omega$ and t_1 and $t_2 \in B_2$, a *2-partition of* $[t_1, t_2]$ *of length* p is the ordered subset of B_2,

$$\{t_1, t_1 + (t_2 - t_1)/2, \cdots, t_1 + (t_2 - t_1)/2 + \cdots + (t_2 - t_1)/2^{p-2}, t_2\}.$$

DEFINITION 5.2. If $p \in Z^+$, $t_2 \in B_2$ and $s_0 \in [0, t_2]$, a *2-partition of length* p *of* $[s_0, t_2]$ is the set of points consisting of s_0 together with the elements which are greater than s_0 of the 2-partition of $[0, t_2]$ which contains exactly $(p - 1)$ points greater than s_0.

DEFINITION 5.3. If $t_2 \in B_2$ and $s_0 \in [0, t_2]$, a *2-partition of* $[s_0, t_2]$ *of length* ω^m is obtained as follows. We first take a 2-partition of $[0, t_2]$ of length ω, partition each of the subintervals so obtained by a 2-partition of length ω and iterate this procedure a total of m times. The required 2-partition then consists of s_0 together with the elements of all the above two partitions which are greater than s_0.

DEFINITION 5.4. If $t_2 \in B_2$, $s_0 \in [0, t_2]$ and $\alpha < \varepsilon_0$ a *2-partition of length* α is constructed as follows. If $\alpha < \varepsilon_0$ it must be of the form

$$\alpha = a_m \omega^m + \cdots + a_0$$

with $a_m \neq 0, a_0, \cdots, a_m \in Z^+$. If $m = 0$ the 2-partition of length a_0 is the 2-partition of finite length a_0 of Definition 5.2. If $m > 0$ and $a_i = 0$ for all $i < m$, then take a 2-partition of $[s_0, t_2]$ of length $a_m + 1$ and partition the a_m intervals so obtained by 2-partitions of length ω^m. In this case we are finished. If $m > 0$ and $a_i \neq 0$ for some $i < m$, take a 2-partition of $[s_0, t_2]$ of length $a_m + 2$ and partition each of the first a_m intervals so obtained by 2-partitions of length ω^m. We then repeat this procedure for the $(a_m + 1)^{st}$ interval with respect to the ordinal $a_{m'} \omega^{m'} + \cdots + a_0$ where m' is the largest integer less than m such that $a_{m'} \neq 0$. Working inductively, we obtain the required 2-partition of length α in at most m steps.

We shall now define a natural ordering on the elements of w_∞. Let $w_K(m)$ and $w_p(q)$ belong to w_∞ and suppose that $K \geq p$. Then we say that $w_K(m) \geq w_p(q)$ if $m \geq E_K \cdots E_p(q)$. Let $\{w_K(m) : K \in Z^+, m \leq \varepsilon_0\}$ considered as an ordered set of elements under this ordering be denoted by $\theta(w_\infty)$. The elements of $\theta(w_\infty)$ will be designated by $\binom{m}{K}$, $m \leq \varepsilon_0$ and $K \in Z^+$ where $\binom{m}{K}$ corresponds to $w_K(m)$. $\theta(w_\infty)$ is a chain which has no gaps [9].

Let $\theta_n(w_\infty)$ be defined to be the ordered subset of elements of $\theta(w_\infty)$ which are less than or equal to $\binom{\delta_n^0}{0}$, that is, the set corresponding to jumps up to the boundary of Γ_n.

If $w_\infty = (\{w_1(\cdot)\}, \{w_2(\cdot)\}, \cdots)$, we define $w_{\infty+K,m}$ to be the same sequence of generalized sequences with the elements of each $\{w_r(\cdot)\}$ which are less than $\binom{m}{K}$ deleted and letting $\binom{0}{K'}_{\infty+K,m} \equiv \binom{m}{K}$ for all K'.

The first step in the construction of the Markov process X is the definition of an order isomorphism $\Lambda : \theta_n(w_\infty) \to B_2^{s_0}$. Since w_∞ can be considered to be a mapping $w_\infty : \theta(w_\infty) \to Q'$, Λ induces a mapping from $B_2^{s_0} \to Q'$. For a fixed $w_\infty \in \Omega_\infty$, $\xi_0 \in \Gamma_n$ and $s_0 \in [0, 1)$ this induced mapping will be designated by $w(\cdot, w_\infty, s_0, \xi_0)$.

Λ is defined inductively as follows.

$$\Lambda : \left\{ \binom{0}{0}, \cdots, \binom{\delta_n{}^0}{0} \right\} \to \{s_0, s_0^1, \cdots, s_0^{\delta_n - 1^0}, 1\}$$

where $\{s_0, s_0^1, \cdots, s_0^{\delta_n-1^0}, 1\}$ are the successive elements of the 2-partition of $[s_0, 1]$ of length δ_n^0. Given the mapping

$$\Lambda : \left\{ \binom{0}{K}, \cdots, \binom{\delta_n{}^K}{K} \right\} \to \{s_K^0, \cdots, s_K^{\delta_n{}^K}\}$$

where $s_K^0 = s_0$ and $s_K^{\delta_n{}^K} = 1$ we obtain the mapping

$$\Lambda : \left\{ \binom{0}{K+1}, \cdots, \binom{\delta_n{}^{K+1}}{K+1} \right\} \to \{s_{K+1}^0, \cdots, s_{K+1}^{\delta_n{}^{K+1}}\}$$

as follows. We map

$$\Lambda : \left\{ \binom{E_{K+1}(\alpha)}{K+1}, \cdots, \binom{E_{K+1}(\alpha+1)}{K+1} \right\} \to \{s_{K+1}^{E_{K+1}(\alpha)}, \cdots, s_{K+1}^{E_{K+1}(\alpha+1)}\}$$

where $\{s_{K+1}^{E_{K+1}(\alpha)}, \cdots, s_{K+1}^{E_{K+1}(\alpha+1)}\}$ are the successive elements of a 2-partition of $[s_K^\alpha, s_K^{\alpha+1}]$ of length p where p is the ordinal number of the well ordered set

$$\left\{ \binom{E_{K+1}(\alpha)}{K+1}, \cdots, \binom{E_{K+1}(\alpha+1)}{K+1} \right\}.$$

Since $\theta_n(w_\infty)$ has no gaps, it can easily be shown that the mapping is onto $B_2^{s_0}$. Furthermore, if $s \geq t \geq s_0$, $s, t \in B_2$ and $\Lambda\binom{m}{K} = t$, then

$$w(s, w_\infty, s_0, \xi_0) = w(s, w_{\infty+K,m}, t, w(t, w_\infty, s_0, \xi_0)).$$

An s_0-path from $\xi_0 \in \Gamma_n$ to $\partial\Gamma_n$ is a continuous mapping from $[s_0, 1]$, $0 \leq s_0 < 1$ to $\bar{\Gamma}_n$, denoted by $x(s)$, such that (i) $x(s_0) = \xi_0$, (ii) $x(s) \in \Gamma_n$, $s_0 \leq s < 1$, and (iii) $x(1) \in \partial\Gamma_n$.

THEOREM 5.1. *The mapping $w(s, w_\infty, s_0, \xi_0)$ can be uniquely extended to an s_0-path $x(s)$ for almost every w_∞. The class of 0-paths $x(s)$ will be designated by $\tilde{\Omega}$.*

Proof. If $r \in [s_0, 1]$, $r_m \to r$ and $r_m \in B_2^{s_0}$, then $\{w(r_m)\}$ is a infinite set of points in the compact set Γ_n and thus has at least one limit point, say $w(r)$. Either of the following cases can occur. The first case is that in which for any p there is an M such that $w(r_m) \in A$, $A \in \mathcal{C}_p(\Gamma_n)$ for every $m \geq M$ and $w(r) \in A$. But since

$$\sup \{\operatorname{diam} A : A \in \mathcal{C}_p(\Gamma_n)\} \to 0 \quad \text{as} \quad p \to \infty,$$

given $\varepsilon > 0$ there is an M' such that for $m \geq M'$, $\rho(w(r_m), w(r)) < \varepsilon$. Hence $w(r_m) \to w(r)$. The second case is that in which for all sufficiently large K, $r_{p_i} = \Lambda\binom{m_i}{K}$ for infinitely many $r_{p_i} < r$. But by Theorem 3.2, $w(r_{p_i})$ converges to a single limit point with probability one. But then since

$$\sup \{\operatorname{diam} A : A \in \mathcal{C}_K(\Gamma_n)\} \downarrow 0 \quad \text{as} \quad K \to \infty,$$

we can conclude that $w(r_m)$ converges to a single limit point, namely $w(r)$. In this case $r \in B_2^{s_0}$. Hence for $r \in [s_0, 1]$, $r \notin B_2$, we can uniquely define $x(r, w_\infty)$ or simply

$$x(r) \equiv w(r, w_\infty, s_0, \xi_0) \equiv \lim_{p\to\infty} w(r_p, w_\infty, s_0, \xi_0),$$

where $r_p \to r$ and $r_p \in B_2$, for almost every $w_\infty \in \Omega_\infty$, Q.E.D.

There is a one to one injection of Ω_∞ onto $\tilde{\Omega}$. Let the σ-field $F^\bullet_{st} \subset F_\infty$ be defined as follows:

i. if $0 \leq s \leq t \leq 1$ and s and t belong to B_2, $F^\bullet_{st}$ is the smallest σ-subfield of F_∞ generated by the projective inverses of sets of the form

$$(w_\infty : x(m/2^r, w_\infty) \in A_r^m) \qquad \text{for } s \leq m/2^r \leq t,\ A_r^m \in \mathbf{C};$$

ii. if $0 \leq s \leq t \leq 1$, $F^\bullet_{st}$ is defined to be

$$\bigcap\{F^\bullet_{s_p t_m},\ p \in Z^+,\ m \in Z^+\}$$

where $\{s_p\}$ and $\{t_m\}$ are sequences of points of B_2 such that $s_p \downarrow s$ and $t_m \uparrow t$.

The definition is consistent for $s, t \in B_2$ since the paths are continuous.

LEMMA 5.1. *$\{x(t) \in \Gamma\} \in F^\bullet_{st}$ where $0 \leq s \leq t$ and $\Gamma \in \mathbf{S}$.*

Proof. It suffices to show this for $\Gamma \in \mathbf{C}$. We can then write $\Gamma = \bigcup_{m=1}^\infty U_m$ where $U_{m+1} \subset U_m$ and the U_m are open sets. If $t_r \downarrow t$ and $t_r \in B_2$, then because of the continuity of the paths

$$\{x(t) \in \Gamma\} = \bigcap_p \bigcup_K \bigcap_{r \geq K} \{x(t_r) \in U_p\}$$

which belongs to $F^\bullet_{st}$, Q.E.D.

$P_{\infty,\xi}$ induces a probability measure $P_{s,\xi}$ on the sets of the form

$$\{x(t) \in A,\ t \in B_2,\ A \in \mathbf{S},\ t \geq s\}.$$

By continuity this can be extended to sets of the form $\{x(t) \in A,\ A \in \mathbf{S}\}$ for $t \in [s, 1]$ by first defining a regular content on $\mathbf{C}$ and then extending it to a measure as in Lemma 3.3. The same can be done for any finite set of times $t_1, \cdots, t_p$. Then by the Kolmogorov extension theorem $P_{s,\xi}$ can be extended to $F^\bullet_{s1}$. We now wish to show that the process $X = (x(t), 1, F^\bullet_{st}, P_{s,x})$ is actually a (nonstationary) strict Markov process.

LEMMA 5.2. $P_{s_0,\xi_0}[x(s) \in \partial D_K^1$, some $s \in (s_0, s_K^1)] = 0$, *that is, with probability one* $\tau^{D_K{}^1} \equiv \inf\{s : x(s) \notin D_K^1\} = s_K^1$.

Proof. If $x(s) = x(s_K^1)$, $s \in (s_0, s_K^1)$, then there is a $K^* > K$ and $s_0^* \in B_2$ such that $x(s) \notin D^1_{K^*}$ and $x(s) \neq x(s^1_{K^*})$ for $s \in (s_0^*, s_K^1)$. That is, if

$$P_{s_0,\xi_0}[x(s) = x(s_K^1),\ s \in (s_0, s_K^1)] > 0$$

then there is a $K^* \geq K$ and $s_0^* \in B_2 \cap (s_0, s_K^1)$ such that

$$P_{s_0^*, x(s_0^*)}[x(s) \in \partial D^1_{K^*},\ x(s) \neq x(s_K^1),\ s \in (s_0^*, s_K^1)] > 0.$$

Hence it suffices to show that

$$P_{s_0,\xi_0}[x(s) \in \partial D_K^1, x(s) \neq x(s_K^1), s \in (s_0, s_K^1)] = 0.$$

Since ∂D_K^1 is separable, it suffices to show that if G_1 and G_2 are disjoint sets of $\Delta \cap \partial D_K^1$, then

$$P_{s_0,\xi_0}[x(s) \in G_1, \text{some } s \in (s_0, s_K^1), x(s_K^1) \in G_2] = 0.$$

Let $A_m \equiv \{x : h_{\partial D_K^1}(x, G_1) > 1 - 2^{-m}\}$. Because of the property iv of the hitting probabilities,

$$\{x(s) \in G_1, \quad \text{some} \quad s \in (s_0, s_K^1), x(s_K^1) \in G_2\}$$
$$\subset \bigcap_{m=1}^{\infty}\{x(s) \in A_m, \quad \text{some} \quad s \in (s_0, s_K^1), x(s_K^1) \in G_2, s \in B_2\}.$$

But

$$P_{s_0,\xi_0}\{x(s) \in A_m, s \in B_2 \cap (s_0, s_K^1), x(s_K^1) \in G_2\} \leq 2^{-m},$$

so that by an application of the Borel-Cantelli Lemma we obtain the result, Q.E.D.

Let $N_K(s, w)$ or simply $N_K(s)$ be the lub $\{\alpha : \Lambda(\frac{\alpha}{K}) \leq s\}$ for $s \in B_2$. Let $r(\xi_0, p)$ be the smallest integer $m > p$ such that $\bar{D}(m, \xi_0) \subset D(p, \xi_0)$ and define $r_K(\xi)$ inductively by

$$r_0(\xi) \equiv r(\xi, 0) \quad \text{and} \quad r_K(\xi) \equiv r(\xi, r_{K-1}(\xi)).$$

LEMMA 5.3. *If* $U \in \Delta$, $U \subset \Gamma_n$, $\xi_0 \in \Gamma_n$, $s \in B_2$, $s \leq t$, *and* $\alpha_1, \cdots, \alpha_{r_K(\xi_0)-1}$ *are a given set of ordinals, then*

$$P_{s,\cdot}(E_1(1) = \alpha_1, \cdots, E_{r_K(\xi_0)-1}(1) = \alpha_{r_K(\xi_0)-1}, x(t) \in U)$$

is measurable on $\partial D(r_K(\xi_0), \xi_0)$.

Proof. Because of the continuity of the paths (Theorem 5.1), if $f(\cdot)$ is a continuous function with support in Γ_n, then

$$\int P_{s,\cdot}(E_1(1) = \alpha_1, \cdots, E_{r_K(\xi_0)-1} = \alpha_{r_K(\xi_0)-1}, x(t) \in dy)f(y)$$
$$= \lim_{K_0 \to \infty} \int P_{s,\cdot}(E_1(1) = \alpha_1, \cdots, E_{r_K(\xi_0)-1} = \alpha_{r_K(\xi_0)-1}, w_{K_0}(N_{K_0}(t)) \in dy)f(y).$$

Hence it suffices to show that $P_{s,\cdot}(w_{K'}(1) \in B_1, \cdots, w_{K'}(p) \in B_p)$, $B_i \in \mathbf{C}$, is measurable on $\partial D(r_K(\xi_0), \xi_0)$ for arbitrary $K' \geq K$. We prove this by induction on p. If p is finite the result follows from the property iv of the hitting probabilities. The result follows for $p = \omega-$ since

$$P_{s,\cdot}(w_{K'}(1) \in B_1, \cdots) = \lim_{m\to\infty} P_{s,\cdot}(w_{K'}(1) \in B_1, \cdots, w_{K'}(m) \in B_m).$$

Since $B_\omega \in C$, we can find a sequence $\{U_m\}$ of open sets such that $U_m \downarrow B_\omega$. But then the result follows for $p = \omega$ since

$$P_{s,\cdot}(w_{K'}(1) \in B_1, \cdots, w_{K'}(\omega) \in B_\omega)$$
$$= P_{s,\cdot}(\bigcap_m \bigcup_p \bigcap_{q \geq p}[w_{K'}(1) \in B_1, \cdots, w_{K'}(q) \in U_m]).$$

The last argument remains true if we replace $\{1, 2, \cdots\}$ by any sequence of ordinals. Hence the result follows for $p = \varepsilon_0$ by the principle of transfinite induction, Q.E.D.

THEOREM 5.2. *If* $0 \leq s \leq t$ *and* $\Gamma \subset \Gamma_n$, $\Gamma \in \mathbf{S}$, *then*

$$P(s, x; t, \Gamma) \equiv P_{s,x}(x(t) \in \Gamma)$$

is an $\mathbf{S}$-*measurable function of* x.

Proof. It suffices to show this for $\Gamma \in \mathbf{C}$ or $\Gamma \in \Delta$ and for $t - s > 0$. Let K be such that $2^{-(K-2)} < t - s$. Given $\xi_0 \in \Gamma_n$ either (i) no other sets of $\mathcal{C}_{r_K(\xi_0)}(\Gamma_n)$ intersect $D(r_K(\xi_0), \xi_0)$ or else (ii) ξ_0 lies on the boundaries of a finite collection of sets which do, that is, $\xi_0 \in \partial D_1 \cap \cdots \cap \partial D_m$.

Case (i). If $x(s) = \eta \in D(r_K(\xi_0), \xi_0)$, the time $\tau^* \equiv \tau^{D(r_K(\xi_0),\xi_0)}$ at which the boundary of $D(r_K(\xi_0), \xi_0)$ is first reached takes on at most countably many values $\{\tau_r, r \in Z^+\}$ with $\tau_r \in B_2$ and satisfies $s < \tau^* < s + 2^{-K} < t$ (Lemma 5.2). The value of $\tau^*(w_\infty) = \Lambda\left({}_{r_K(\xi_0)}^{1}\right)$ depends on the ordinals $E_m(1)$, $1 \leq m \leq r_K(\xi_0) - 1$. $\tau^*(w_\infty)$ is also determined by

$$w_{r_K(\xi_0)}(1) \quad \text{and} \quad E_m(1, w_{\infty + r_K(\xi_0), 1}), \qquad 1 \leq m \leq r_K(\xi_0) - 1.$$

Let E_r^* be the subset in $F_{s1}^{\bullet}$ which contains the paths, w_∞, for which $\tau^* = \tau_r$. Then $E_r^* \in F_{\tau_r 1}^{\bullet}$ and $P_{\tau r_0,\eta}(E_r^*, x(t) \in \Gamma)$ is measurable on $\partial D(r_K(\xi_0), \xi_0)$ by Lemma 5.3. Hence, if $\xi \in D(r_K(\xi_0), \xi_0)$,

$$P_{s,\xi}(x(t) \in \Gamma) = \int_{\partial D(r_K(\xi_0),\xi_0)} h_{\partial D(r_K(\xi_0),\xi_0)}(\xi, d\eta) \left[\sum_{r=1}^{\infty} P_{\tau_r,\eta}(E_r^*, x(t) \in \Gamma)\right]$$

which is continuous in $D(r_K(\xi_0), \xi_0)$ by property iv of the hitting probabilities. Hence if $P_{s,\xi_0}(x(t) \in \Gamma) < a$, then there is a neighborhood N_{ξ_0} of ξ_0 in which this is true.

Case (ii). In this case $\eta \in \partial D_1 \cap \cdots \cap \partial D_m$ and $x(s) = \eta$. A similar argument shows that if $P_{s,\xi_0}(x(t) \in \Gamma) < a$, then there is a relatively open subset N_{ξ_0} of ξ_0 contained in $D(r_K(\xi_0), \xi_0) \cap \partial D_1 \cap \cdots \cap \partial D_m$ in which this is true.

Hence if we let $\Lambda \equiv \{\xi : P_{s,\xi}(x(t) \in \Gamma) < a\}$, $\Lambda = \bigcup_{\xi \in \Lambda} N_\xi$ where N_ξ is either an open set or else a relatively open set in $\partial D_1 \cap \cdots \cap \partial D_m$. But since there is a countable base for the sets of the form N_ξ, Λ is the union of a countable class of measurable sets and is measurable, Q.E.D.

We need the following lemma.

LEMMA 5.4. *Let* $f(\cdot)$ *be a measurable function on* Q,

$$x_0 \notin \bigcap\{\partial D : D \in \mathcal{C} \cap \Gamma_n\} \quad \textit{and} \quad |f(\cdot)| \leq M.$$

Then $F(u, y) \equiv \int_Q P(u, y\,; t, dz)f(z)$ *satisfies* $\lim_{y\to x_0 u\downarrow s} F(u, y) = F(s, x_0)$ *for* $s < t$.

Proof. Given x_0 and $\varepsilon > 0$ we are required to find a neighborhood, N_{x_0}, of x_0 and a $\delta > 0$ such that if $\eta \in N_{x_0}$ and $0 \le u - s < \delta$, then

$$|F(u, \eta) - F(s, x_0)| < \varepsilon.$$

Choose K such that $2^{-K} < t - s$. If $x(s) = \eta \in D(r_K(x_0), x_0)$, then the time, τ^*, at which the boundary of the set $D(r_K(x_0), x_0)$ is first reached takes on at most countably many values $\{\tau_r, r \in Z^+\}$ and satisfies $s < \tau^* < s + 2^{-K} < t$. Given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$P_{s,\eta}(\tau^* < s + \delta) < \varepsilon/M$$

for all η in some neighborhood $N^*_{x_0} \subset D(r_K(x_0), x_0)$ of x_0. If $s \le u < s + \delta$ and $\eta \in N^*_{x_0}$, then

$$\left| F(u, \eta) - \int_{\partial D(r_K(x_0), x_0)} h_{\partial D(r_K(x_0), x_0)}(\eta, d\xi) \left[\sum_{r=1}^{\infty} \int_Q P_{\tau_r, \xi}(E^*_r, x(t) \in dz) f(z) \right] \right| \le (\varepsilon/M)\cdot M = \varepsilon.$$

But by the smoothness of the hitting probabilities the integral expression is a continuous function of $\eta \in D(r_K(x_0), x_0)$. Hence we can find a neighborhood of x_0, $N_{x_0} \subset N^*_{x_0}$, such that for $\eta \in N_{x_0}$ and $s \le u < s + \delta$,

$$|F(u, \eta) - F(s, x_0)| < 2\varepsilon,$$

Q.E.D.

THEOREM 5.3. $X = (x(t), 1, F^{\bullet}_{st}, P_{s,x})$ *is a simple Markov process.*

Proof. We must show that if $0 \le s \le t \le u$, then

$$P_{s,x}(x(u) \in \Gamma \mid F^{\bullet}_{st}) = P(t, x(t); u, \Gamma), \qquad \text{a.e. } \Omega_t, P_{s,x}.$$

By the definition of conditional probability it suffices to show that if $0 \le s < t < u$, $x \in D$ and $A \in F^{\bullet}_{st}$, then

$$P_{s,x}(A, x(u) \in \Gamma) = \int_A P(t, x(t); u, \Gamma) P_{s,x}(dw).$$

We first prove the result for $t \in B_2$. But if $\Lambda\binom{m}{K} = t$,

$$w(u, w_\infty\,; s, x) = w(u, w_{\infty+K,m}, t, w(t))$$

and hence

$$\int_A P(t, x(t); u, \Gamma) P_{s,x}(dw) = P_{s,x}(A, x(u) \in \Gamma)$$

because $\Lambda^{-1}(t)$ takes on at most countably many values [5, Theorem 5.2, p. 116]. Now let $t \in [0, 1] - B_2$, $t_m \downarrow t$ where $t_m \in B_2$ and $t_m < u$. $A \in F^{\bullet}_{st_m}$ and hence

$$(5.1) \qquad P_{s,x}(A, x(u) \in \Gamma) = \int_A P(t_m, x(t_m), u, \Gamma) P_{s,x}(dw).$$

But by the continuity of the paths and Lemma 5.4 we have

$$P(t_m, x(t_m), u, \Gamma) \to P(t, x(t), u, \Gamma) \quad \text{as} \quad m \to \infty.$$

Hence the result follows by passing to the limit in equation (5.1), Q.E.D.

THEOREM 5.4. *Let* $\{w : w(0) = \xi_0, \tau(w) \geq t\} \in F^{\bullet}_{0t}$ *for* $t \in [0, 1]$, *that is, let* $\tau(w)$ *be a* 0-*Markov time. Then if*

$$F^{\bullet}_{0\tau+} \equiv \{B : B \in F^{\bullet}_{01} ; B \cap (w : \tau(w) < t) \in F^{\bullet}_{0t}, t \in [0, 1]\},$$

$$P_{0,x}\{x(\eta) \in \Gamma \mid F^{\bullet}_{0\tau+}\} = P(\tau, x(\tau); \eta, \Gamma)$$

with probability one where $\eta(w)$ *is a* $F^{\bullet}_{0\tau+}$ *measurable function and* $\eta(w) \geq \tau(w)$.

Proof. (This is a slight modification of a result of E. B. Dynkin [5, Theorem 5.9, p. 134].)

Let $f(z)$ be measurable. By Lemma 5.4,

$$((u, y) : F(u, y) < a) \cap Q \times [s, t)$$

is a measurable subset of $Q \times [s, t)$, $0 \leq s < t \leq 1$. Thus following the argument of Dynkin it suffices to show that if $f \in C(Q)$ and $\tau \leq t$, then

$$E_{0,x}\{f(x(t)) \mid F^{\bullet}_{0\tau+}\} = E_{\tau,x(\tau)}(f(x(t))), \qquad \text{a.e. } \Omega_\tau, P_{s,x}.$$

Let the points $\{t_k^m, k \in Z^+\}$ define a sequence of subdivisions $\{\Delta_k^m\}$, $m \in Z^+$, of the interval $[0, t]$ such that

$$\max_k \operatorname{diam}(\Delta_k^m) \downarrow 0 \quad \text{as} \quad m \to \infty.$$

Let

$$\begin{aligned} \tau_m(w) &\equiv t_k^m \qquad \text{if} \quad \tau(w) \in \Delta_k^m \text{ and } \tau(w) \notin B_2 \\ &\equiv \tau(w) \quad \text{if} \quad \tau(w) \in B_2 . \end{aligned}$$

The random variable τ_m takes on only countably many values and therefore

$$E_{0,x}\{f(x(t)) \mid F^{\bullet}_{0\tau_m+}\} = F(\tau_m, x(\tau_m)), \qquad \text{a.e. } \Omega_{\tau_m}, P_{s,x}.$$

The restrictions of τ_m to Ω_τ are clearly $F^{\bullet}_{0\tau+}$ measurable and

$$\{A, \tau_m < 1\} \in F^{\bullet}_{0\tau_m+} \qquad \text{for each } A \in F^{\bullet}_{0\tau+}.$$

We thus have for $m \in Z^+$ and $A \in F^{\bullet}_{0\tau+}$,

$$(5.2) \qquad E_{0,x}\{\chi_A \chi_{\tau_m<1} f(x(t))\} = E_{0,x}\{\chi_A \chi_{\tau_m<1} F(\tau_m, x(\tau_m))\},$$

where χ_A is the characteristic function of the set A. But since $F(\tau, x(\tau))$ is $F^{\bullet}_{0\tau+}$ measurable, it suffices to show that

$$E_{0,x}\{\chi_A f(x(t))\} = E_{0,x}\{\chi_A F(\tau, x(\tau))\}.$$

Moreover, by Lemma 5.4 and the continuity of the paths,

$$F(\tau_m, x(\tau_m)) \to F(\tau, x(\tau)) \quad \text{as} \quad m \to \infty.$$

Hence by passing to the limit in equation (5.2) we obtain

$$E_{0,x}\{\chi_A f(x(t))\} = E_{0,x}\{\chi_A F(\tau, x(\tau))\},$$

Q.E.D.

COROLLARY. *Similarly, if $\eta(w)$ is $F^{\bullet}_{s\tau+}$ measurable, then*

$$P_{s,x}\{x(\eta) \in \Gamma \mid F^{\bullet}_{s\tau+}\} = P(\tau, x(\tau); \eta, \Gamma)$$

with probability one, that is, X is a strict Markov process.

6. Introduction of the natural time parameter

Following Knight [10], [11] we are going to introduce a continuous natural time parameter into each of the R_K and by carrying out a limiting process show that it is possible to define a single time parameter for R_∞.

Let

$$\tilde{e}_K^m \equiv e_K^m \quad \text{if} \quad m < \delta_n^K,$$
$$\equiv 0 \quad \text{if} \quad m \geq \delta_n^K,$$

where $\delta_n^K(w_K(\cdot)) \equiv \text{glb}\,\{\alpha : w_K(\alpha) \notin \Gamma_n\}$.

Given a path $w_K(\cdot)$ we construct a continuous parameter path $w_K^*(t)$ by setting $w_K^*(t) \equiv w_K(0)$ for $0 \leq t < \tilde{e}_K^1$ and $w_K^*(t) \equiv w_K(m)$ for $\sum_{p=1}^m \tilde{e}_K^p \leq t < \sum_{p=1}^{m+1} \tilde{e}_K^p$. The continuous parameter process thus constructed up to the boundary of Γ_n is designated by R_K^*. The associated projective limit space and process are designated by $(\Omega_\infty^*, F_\infty^*, P_{\infty,\xi}^*)$ and R_∞^*, respectively. $(\Omega_\infty^*, F_\infty^*, P_{\infty,\xi}^*)$ and $(\Omega_\infty, F_\infty, P_{\infty,\xi})$ are equivalent measure spaces.

The time lag $L_{K-1,K}(t)$ between R_{K-1}^* and R_K^* is defined by

$$L_{K-1,K}(w_\infty, t) \equiv L_{K-1,K}(t) \equiv \sum_{m=1}^{E_K(p)} \tilde{e}_K^m - \sum_{m=1}^{p} \tilde{e}_{K-1}^m$$

for $\sum_{m=1}^p \tilde{e}_{K-1}^m \leq t < \sum_{m=1}^{p+1} \tilde{e}_{K-1}^m$. By iterating $(K - r)$ times the operation of finding the time lag we can define a time lag between any pair R_K^* and R_r^*, $K < r$. Specifically,

$$L_{K,r}\left(\sum_{m=1}^p \tilde{e}_K^m\right) \equiv L_{K,K+1}\left(\sum_{m=1}^p \tilde{e}_K^m\right) + L_{K+1,K+2}\left(\sum_{m=1}^{E_{K+1}(p)} \tilde{e}_{K+1}^m\right)$$
$$+ \cdots + L_{r-1,r}\left(\sum_{m=1}^{E_{r-1}\cdots E_{K+1}(p)} \tilde{e}_{r-1}^m\right)$$

and

$$L_{K,r}(t) \equiv L_{K,r}\left(\sum_{m=1}^p \tilde{e}_K^m\right)$$

for $\sum_{m=1}^p \tilde{e}_K^m \leq t < \sum_{m=1}^{p+1} \tilde{e}_K^m$. From the corollary to Theorem 3.3 it follows that

$$E_{\infty,\xi}\left(L_{K,r}\left(\sum_{m=1}^p \tilde{e}_K^m\right)\right) = 0.$$

We will now prove a generalization of Theorem 1.3–2 of [10].

THEOREM 6.1.

$$P_{\infty,\xi}\{[\sup|L_{K,r}(t)| : 0 \leq t < \textstyle\sum_{m=1}^{\varepsilon_0} \tilde{e}_K^m] \geq \varepsilon\} \leq e_{\Gamma_n}(\xi)\varepsilon^{-2}2^{-(K-3)}.$$

Proof. We can write

$$(\textstyle\sum_{m=1}^{\delta_n K} \tilde{e}_K^m - \sum_{m=1}^{\delta_n K+r} \tilde{e}_{K+r}^m) = \sum_{m=1}^{\delta_n K}(\tilde{e}_K^m - M_{K,r}^m)$$

where

$$M_{K,r}^m \equiv \textstyle\sum\{\tilde{e}_{K+r}^s : E_{K+r} \cdots E_{K+1}(m) \leq s \leq E_{K+r} \cdots E_{K+1}(m+1) - 1\}.$$

Since $\delta_n^K(w_K(\cdot))$ is a stopping time [3], $\sum_{m=1}^{q}(\tilde{e}_K^m - M_{K,r}^m)$ is a martingale in q, $q \leq \varepsilon_0$. Hence by the Kolmogorov inequality for countable martingales [12],

$$\begin{aligned} A &\equiv P_{\infty,\xi}(\sup_{q\leq\varepsilon_0} |\textstyle\sum_{m=1}^{q}(\tilde{e}_K^m - M_{K,r}^m)| > \varepsilon) \\ &\leq \varepsilon^{-2}E_{\infty,\xi}(\textstyle\sum_{m=1}^{\varepsilon_0}(\tilde{e}_K^m - M_{K,r}^m))^2. \end{aligned}$$

If $m > p$

$$\begin{aligned} E_{\infty,\xi}\{(\tilde{e}_K^p - M_{K,r}^p)(\tilde{e}_K^m - M_{K,r}^m)\} &= E_{\infty,\xi}\{(\tilde{e}_K^p - M_{K,r}^p)(E_{\infty,\xi}(e_K^m - M_{K,r}^m)|F_K^{(p)}))\} \\ &= 0 \end{aligned}$$

and hence

$$A \leq \varepsilon^{-2}E_{\infty,\xi}(\textstyle\sum_{m=1}^{\varepsilon_0}[(\tilde{e}_K^m)^2 - 2\tilde{e}_K^m M_{K,r}^m + (M_{K,r}^m)^2]).$$

But since we may first take a conditional expectation with respect to the field $F_K^{(\varepsilon_0)}$, $A \leq \varepsilon^{-2}E_{\infty,\xi}(\sum_{m=1}^{\varepsilon_0}((M_{K,r}^m)^2 - (\tilde{e}_K^m)^2)$. However we have the result

$$\begin{aligned} E_{\infty,\xi}\int_0^{M_{K,r}^m}\left[E_{\infty,w_{K+r}^*(t)}\int_t^{M_{K,r}^m} ds\right]dt &= \int_0^\infty\left[\int_0^b\left[\int_t^b ds\right]dt\right]P_{\infty,\xi}(M_{K,r}^m \in db) \\ &= 2^{-1}E_{\infty,\xi}([M_{K,r}^m]^2). \end{aligned}$$

Moreover

$$\begin{aligned} E_{\infty,\xi}\int_0^{M_{K,r}^m}\left[E_{\infty,w_{K+r}^*(t)}\int_t^{M_{K,r}^m} ds\right]dt &= E_{\infty,\xi}\int_0^{M_{K,r}^m}[E_{\infty,w_{K+r}^*(t)}[M_{K,r}^m - t]\,dt] \\ &\leq E_{\infty,\xi}\int_0^{M_{K,r}^m}(2^{-(K+r)} + \tilde{e}_K^m(w_t^+))\,dt \\ &\leq E_{\infty,\xi}\int_0^{M_{K,r}^m}(2^{-(K+r)} + 2^{-K})\,dt \\ &\leq 2^{-(K-1)}\tilde{e}_K^m. \end{aligned}$$

Hence we obtain

$$\begin{aligned} E_{\infty,\xi}\{\textstyle\sum_{m=1}^{\varepsilon_0}((M_{K,r}^m)^2 - (\tilde{e}_K^m)^2)\} &\leq 2E_{\infty,\xi}\{\textstyle\sum_{m=1}^{\varepsilon_0}[2^{-(K-1)}\tilde{e}_K^m - (\tilde{e}_K^m)^2]\} \\ &\leq 2^{-(K-3)}E_{\infty,\xi}[\textstyle\sum_{m=1}^{\varepsilon_0}\tilde{e}_K^m] = e_{\Gamma_n}(\xi)\cdot 2^{-(K-3)}. \end{aligned}$$

Therefore

$$P_{\infty,\xi}\{[\sup \mid L_{K,r}(t)\mid : 0 \leq t < \textstyle\sum_{m=1}^{\varepsilon_0} \tilde{e}_K^m] \geq \varepsilon\}$$
$$= P_{\infty,\xi}(\sup_{q\leq\varepsilon_0} \mid \textstyle\sum_{m=1}^{q}(\tilde{e}_K^m - M_{K,r}^m)\mid \geq \varepsilon) \leq 2^{-(K-3)}\varepsilon^{-2}e_{\Gamma_n}(\xi),$$

Q.E.D.

THEOREM 6.2. *If*

$$L_{K,r}^* \left(\textstyle\sum_{m=1}^{\varepsilon_0} \tilde{e}_K^m\right) \equiv \sup \left\{\mid L_{K,r}(t)\mid, 0 \leq t \leq \textstyle\sum_{m=1}^{\varepsilon_0} \tilde{e}_K^m\right\},$$

then

$$P_{\infty,\xi}\{\sup_{r>K} L_{K,r}^*(\textstyle\sum_{m=1}^{\varepsilon_0} \tilde{e}_K^m) > 2\delta\}$$
$$< \{e_{\Gamma_n}(\xi)\delta^{-2}2^{-(K-3)}\cdot[1 - (e_{\Gamma_n}(\xi) + 2\delta)\delta^{-2}2^{-(K-3)}]^{-1}\}.$$

Proof. We will follow the method of Knight [10]. For a fixed $r > K$ take $s > r$. Then

$$P_{\infty,\xi}\{L_{K,s}^*(\textstyle\sum_{m=1}^{\varepsilon_0} \tilde{e}_K^m) > \delta \mid L_{K,r}^*(\textstyle\sum_{m=1}^{\varepsilon_0} \tilde{e}_K^m) > 2\delta\}$$
$$\geq P_{\infty,\xi}\{L_{r,s}^*(\textstyle\sum_{m=1}^{\varepsilon_0} \tilde{e}_K^m) \leq L_{K,r}^*(\textstyle\sum_{m=1}^{\varepsilon_0} \tilde{e}_K^m) - \delta \mid L_{K,r}^*(\textstyle\sum_{m=1}^{\varepsilon_0} \tilde{e}_K^m) > 2\delta\}$$
$$\geq E_{\infty,\xi}[1 - ((\textstyle\sum_{m=1}^{\varepsilon_0} \tilde{e}_K^m) + L_{K,r}(\textstyle\sum_{m=1}^{\varepsilon_0} \tilde{e}_K^m))(L_{K,r}^*(\textstyle\sum_{m=1}^{\varepsilon_0}(\tilde{e}_K^m - \delta)^2 2^{-(K-3)} \mid$$
$$L_{K,r}^*(\textstyle\sum_{m=1}^{\varepsilon_0} \tilde{e}_K^m) > 2\delta))]$$
$$> 1 - (e_{\Gamma_n}(\xi) + 2\delta)\delta^{-2}2^{-(K-3)}.$$

Hence

$$P_{\infty,\xi}\{L_{K,s}^*(\textstyle\sum_{m=1}^{\varepsilon_0} \tilde{e}_K^m) > \delta \mid \max_{K<r<s} L_{K,r}^*(\textstyle\sum_{m=1}^{\varepsilon_0} \tilde{e}_K^m) > 2\delta\}$$
$$> 1 - (e_{\Gamma_n}(\xi) + 2\delta)\delta^{-2}2^{-(K-3)}.$$

But recalling that $P(A \mid B) = P(A \cap B)/P(B)$ we obtain

$$P_{\infty,\xi}\{\text{lub}_{r>K} L_{K,r}^*(\textstyle\sum_{m=1}^{\varepsilon_0} \tilde{e}_K^m) > 2\delta\}$$
$$< \{(e_{\Gamma_n}(\xi))\delta^{-2}2^{-(K-3)}\cdot[1 - (e_{\Gamma_n}(\xi) + 2\delta)\delta^{-2}2^{-(K-3)}]^{-1}\}.$$

Hence the theorem is proved.

Theorem 6.2 implies that

$$T_{\xi_0}^{*K}[\left(\tfrac{m}{K'}\right), w_\infty] \equiv \textstyle\sum \{\tilde{e}_K^p : p \leq N_K(\Lambda(\tfrac{m}{K'}))\}$$

converges with P_{∞,ξ_0}-probability one as $K \to \infty$. Let

$$T_{\xi_0}^*[\left(\tfrac{m}{K'}\right), w_\infty] = \lim_{K\to\infty} T_{\xi_0}^{*K}[\left(\tfrac{m}{K'}\right), w_\infty].$$

Hence we obtain a continuous time parameter for the R_∞^* process.

7. Extension of the natural time parameter to X

Let $s \in B_2$ and consider the path $w(u, w_\infty ; s_0, \xi_0)$. Let

$$T_{s_0,\xi_0}^K(s, w) \equiv T_{\xi_0}^{*K}[\Lambda^{-1}(s), w_\infty]$$

and

$$T_{s_0,\xi_0}(s, w) \equiv T_{\xi_0}^*[\Lambda^{-1}(s), w_\infty].$$

Clearly $T_{s_0,\xi_0}(s, w) = \lim_{K\to\infty} T^K_{s_0,\xi_0}(s, w)$. $T_{s_0,\xi_0}(s, w)$ is called *the natural time parameter* for a path starting at $\xi_0 \in \Gamma_n$ at time s_0. We will now show that $T_{s_0,\xi_0}(s, w)$ can be extended to a continuous strictly increasing function of $s \in [s_0, 1]$ for almost every w.

THEOREM 7.1. *Except for a set of paths of measure zero, $T_{s_0,\xi_0}(s,w)$ can be extended uniquely to a continuous nondecreasing function of s.*

Proof. Given $\delta > 0$ it suffices to find a neighborhood

$$N_s(w) = (s - \varepsilon, s + \varepsilon)$$

such that

$$| T_{s_0,\xi_0}(s + \varepsilon, w) - T_{s_0,\xi_0}(s - \varepsilon, w)| < \delta.$$

Let

$$A_K \equiv \{w : \mathrm{lub}_{r>K} L^*_{K,r}(\textstyle\sum_{m=1}^{\varepsilon_0} \tilde{e}^m_K) > 2\delta, w(s_0) = \xi_0\}^C.$$

As $K \to \infty$, $P_{s_0,\xi_0}(A^C_K) \downarrow 0$ and $P_{s_0,\xi_0}(\bigcup_{K=1}^\infty A_K) = 1$ by Theorem 6.2.

If $w \in A_K$, $| T_{s_0,\xi_0}(s, w) - T^K_{s_0,\xi_0}(s, w)| \leq 2\delta$. Choose K_0 such that $2^{-(K_0-1)} < \delta$ and consider any $K > K_0$. There are two cases to be considered. The first case is that in which s lies between $\Lambda(\binom{\alpha}{K})$ and $\Lambda(\binom{\alpha+2}{K})$ for some α. Let $N_s = (\Lambda(\binom{\alpha}{K}), \Lambda(\binom{\alpha+2}{K}))$. Since $\tilde{e}^\alpha_K + \tilde{e}^{\alpha+1}_K < 2^{-(K-1)} < \delta$, if $s' \in N_s$, then

$$| T_{s_0,\xi_0}(s', w) - T_{s_0,\xi_0}(s, w)| < 5\delta.$$

Since $P_{s_0,\xi_0}(\bigcup_{K=1}^\infty A_K) = 1$, we have the result for this case. The second case is that in which $s \in B_2$ and $\Lambda^{-1}(s) = h((\binom{\alpha_p}{K}), p \in Z^+)$. By the corollary to Theorem 3.3, $\sum_{p=1}^{\Lambda^{-1}(s)} \tilde{e}^p_K$ converges with P_{s_0,ξ_0}-probability one and hence there is an $N < \Lambda^{-1}(s)$ such that $\sum_{p=N}^{\Lambda^{-1}(s)} \tilde{e}^p_K < \delta$. Then if

$$s' \in (\Lambda(\tbinom{N}{K}), \Lambda(\tbinom{\Lambda^{-1}(s)+1}{K})),$$

$$| T_{s_0,\xi_0}(s', w) - T_{s_0,\xi_0}(s, w)| < 5\delta.$$

Since there are at most countably many values of this type we are finished.

LEMMA 7.1. *If $s \in B_2$, $T^K_{s_0,\xi_0}(s, w)$ is measurable with respect to $F^\bullet_{s_0,s}$.*

Proof. Recall that $T^K_{s_0,\xi_0}(s, w) = \sum_{m=1}^{N_K(s)} \tilde{e}^m_K$. If $N_0(s, w_1) = N_0(s, w_2)$, then $\Lambda(\binom{m}{0}, w_1) = \Lambda(\binom{m}{0}, w_2)$ for $m \leq N_0(s, w_1)$. Hence

$$\{w : N_0(s, w) = \alpha_0\} = \{w(s^1_0) \in A^1_0, \cdots, w(s^{\alpha_0}_0) \in A^{\alpha_0}_0\}$$

which belongs to $F^\bullet_{0s}$. Now if $N_0(s) = \alpha_0$, $N_1(s)$ is the sum of at most countably many ordinals each of which is determined by conditions on at most countably many points of $[s_0, s]$ and so on. But then

$$\{w : \tilde{e}^m_K < a\} \cap \{w : N_r(s) = \alpha_r, r = 0, 1, \cdots, K, \alpha_K > m\} \in F^\bullet_{s_0 s}$$

since $\tilde{e}^m_K < a$ induces a condition of the form $[w(s^m_K) \in B]$, $B \in \mathbf{S}$. The result then easily follows.

LEMMA 7.2. *$T_{s_0,\xi_0}(s, w)$ is measurable with respect to $F^{\bullet}_{s_0 s}$.*

Proof. If $s \in B_2$, $T_{s_0,\xi_0}(s, w) = \lim_{K\to\infty} T^K_{s_0,\xi_0}(s, w)$ and the measurability follows from Lemma 7.1. If $s \notin B_2$, then

$$T_{s_0,\xi_0}(s, w) = \lim_{s_i \downarrow s, s_i \in B_2} T_{s_0,\xi_0}(s_i, w),$$

Q.E.D.

If $D \in \Delta$, let $\tau^D(w) \equiv \inf [s : x(s) \notin D]$.

THEOREM 7.2. *If $D \in \mathcal{C}^*_K$, $D \subset \Gamma_n$, then*

$$\int T_{s_0,\xi_0}(\tau^D(w), w) P_{s_0,\xi_0}(dw) = e_D(\xi_0).$$

Proof. Recall that by Theorem 6.2,

$$P_{s_0,\xi_0}\{\text{lub}_{r>K} L^*_{K,r}(\textstyle\sum_{m=1}^{\varepsilon_0} \tilde{e}^m_K) > 2\delta\}$$
$$< e_{\Gamma_n}(\xi_0)\delta^{-2}2^{-(K-3)}\cdot[1 - (e_{\Gamma_n}(\xi_0) + 2\delta)\delta^{-2}2^{-(K-3)}]^{-1}$$

and by Theorem 3.3 that if $K' \geq K$ then

$$\int T^{K'}_{s_0,\xi_0}(\tau^D(w), w) P_{s_0,\xi_0}(dw) = e_D(\xi_0).$$

Given $\delta_0 > 0$ choose K'' large enough so that

$$[1 - (e_{\Gamma_n}(\xi_0) + 2\delta)\delta^{-2}2^{-(K-3)}] > \tfrac{1}{2}$$

for all $\delta \geq \delta_0$, $K > K''$. Then

$$\left| \int T_{s_0,\xi_0}(\tau^D(w), w) P_{s_0,\xi_0}(dw) - e_D(\xi_0) \right|$$
$$< \{4e_{\Gamma_n}(\xi_0)\cdot 2^{-(K-3)}[\textstyle\sum_{m=1}^{\infty}(m + 1)(m^{-2} - (m + 1)^{-2})]\delta_0^{-2} + \delta_0\}$$
$$= \{4e_{\Gamma_n}(\xi_0)2^{-(K-3)}[\textstyle\sum_{m=1}^{\infty}(2m + 1)(m)^{-2}(m + 1)^{-1}]\delta_0^{-2} + \delta_0\}$$
$$< 2\delta_0$$

for sufficiently large K and the result is proved.

THEOREM 7.3. *Let $U \in \Delta$, $U \subset \Gamma_n$. Then*
(i) $\{w : w(0) = \xi_0 \in U, \tau^U(w) > t\} \in F^{\bullet}_{0t}$, *and*
(ii) $\int T_{s_0,\xi_0}(\tau^U(w), w) P_{s_0,\xi_0}(dw) = e_U(\xi_0)$.

Proof. By a theorem due to Hunt [8], [5, Theorem 2, p. 185] there is a sequence of closed sets $C_m \uparrow U$ such that $\tau^{C_m} \uparrow \tau^U$. Hence there is a sequence $\{U_m\}$ of sets of $\mathcal{C}^*$ such that $U_m \uparrow U$ and $\tau^{U_m} \uparrow \tau^U$.

(i) is true for U_m since τ^{U_m} takes on only countably many values. But (i) then follows since $\tau^{U_m} \uparrow \tau^U$.

(ii) is true for U_m by Theorem 7.2. Since both $T_{s_0,\xi_0}(\tau^{U_m}, w)$ and $T_{s_0,\xi_0}(\tau^U, w) \leq T_{s_0,\xi_0}(\tau^{\Gamma_n}, w)$,

$$\int T_{s_0,\xi_0}(\tau^{U_m}, w)P_{s_0,\xi_0}(dw) \uparrow \int T_{s_0,\xi_0}(\tau^{U}, w)P_{s_0,\xi_0}(dw)$$

by the dominated convergence theorem. But since $\lim_{m\to\infty} e_{U_m}(\xi_0) = e_U(\xi_0)$,

$$\int T_{s_0,\xi_0}(\tau^{U}, w)P_{s_0,\xi_0}(dw) = e_U(\xi_0)$$

and the theorem is proved.

COROLLARY. *Let* $\Gamma \subset \Gamma_n$ *and* $\Gamma \in \mathbf{S}$. *Then*

(i) $\{w : w(s_0) = \xi_0 \in \Gamma, \tau^{\Gamma}(w) > t\} \in F^{\bullet}_{s_0t}$, *and*

(ii) *if* $T_{s_0,\xi_0}(\tau^{\Gamma}(w), w) = 0$ *with* P_{s_0,ξ_0}*-probability one, then there exists a sequence* $\{U_m\}$ *of open sets,* $U_m \downarrow \Gamma$ *such that* $e_{U_m}(\xi_0) \downarrow 0$.

Proof. Using the result of Hunt on analytic sets [5, Theorem 3, p. 188] (i) follows by an argument similar to that used in proving (i) of the theorem.

From [5, Theorem 3, p. 188] there is a sequence of open sets $U_m \subset \Gamma_n$, $U_m \downarrow \Gamma$, such that $T_{s_0,\xi_0}(\tau^{U_m}(w), w) \downarrow 0$ with probability one. Hence

$$\int T_{s_0,\xi_0}(\tau^{U_m}(w), w)P_{s_0,\xi_0}(dw) = e_{U_m}(\xi_0) \downarrow 0,$$

Q.E.D.

THEOREM 7.4. *Let* $\tau(w)$ *be a* 0*-Markov time. Then*

$$T_{0,\xi_0}(s, w) = T_{0,\xi_0}(\tau, w) + T_{\tau,x(\tau)}(s, w_\tau^+)$$

if $\tau(w) \leq s$ *except for a set of paths of* P_{0,ξ_0}*-probability zero where* $w_\tau^+(t) = w(t + \tau)$ *for* $t \geq 0$.

Proof. Let $w(s) = w(s, w_\infty, 0, \xi_0)$. Consider the K^{th} random walk and the corresponding jump times $\{s_K^0, \cdots, s_K^{\delta_n K}\}$. Then either (i) $\tau = s_K^\alpha$ for some α, or (ii) $\tau \in (s_K^\alpha, s_K^{\alpha+1})$ for some α. Hence

$$(7.1) \qquad | T^{K'}_{0,\xi_0}(s, w) - [T^{K'}_{0,\xi_0}(\tau(w), w) + T^{K'}_{\tau,x(\tau)}(s, w_\tau^+)]| < 2^{-K}$$

for $K' \geq K$ except for a set of P_{0,ξ_0}-probability zero because of the strict Markov property. The theorem follows by passing to the limit in equation (7.1).

We will now show that $T_{s_0,\xi_0}(s, w)$ is a strictly increasing function of s for almost every path w. Let

$$\tau(w) \equiv \sup \{s : T_{0,\xi_0}(s, w) = 0\}$$

for paths such that $x(0) = \xi_0$.

THEOREM 7.5. $\{w : \tau(w) \geq t, w(0) = \xi_0\} \in F^{\bullet}_{0t}$, *that is,* $\tau(w)$ *is a* 0*-Markov time.*

Proof. $\{w : \tau(w) \geq t\} = \{w : T_{0,\xi_0}(t, w) = 0\}$ which belongs to $F^{\bullet}_{0t}$ by Lemma 7.2, Q.E.D.

If $D \in \mathcal{C}$, $D \subset \Gamma_n$, and $w(0) = \xi \in D$, let $\tilde{\tau}^D(w) \equiv T_{0,\xi}(\tau^D(w), w)$.

THEOREM 7.6. *$P_{0,\xi}(\tilde{\tau}^D = 0)$ is an upper semi-continuous function of $\xi \in D$.*

Proof. Let $\xi_m \to \xi_0$ with $\{\xi_m\}$ and ξ_0 in D. Let $a_m \equiv P_{0,\xi_m}(\tilde{\tau}^D = 0)$. Then it suffices to show that if $a_m \to a$ then $a_0 \geq a$. Note that

$$P_{0,\xi_0}(\tilde{\tau}^D = 0) = \lim_{\delta \downarrow 0} P_{0,\xi_0}(\tilde{\tau}^D \leq \delta).$$

Given $\delta > 0$, $\varepsilon > 0$ we may choose $D_1 \subset D$, $D_1 \in \mathcal{C}$, such that

$$P_{0,\xi_0}(\tilde{\tau}^{D_1} \geq \delta/2) < \varepsilon.$$

Then

$$P_{0,\xi_0}(\tilde{\tau}^D \leq \delta) \geq P_{0,\xi_0}((\tilde{\tau}^D - \tilde{\tau}^{D_1}) \leq \delta/2) - \varepsilon.$$

Choose N_1 such that $| a_m - a | < \varepsilon$ and $\xi_m \in D_1$ for $m \geq N_1$. Then if $m \geq N_1$,

$$P_{0,\xi_0}(\tilde{\tau}^D \leq \delta) \geq P_{0,\xi_0}(\tilde{\tau}^D - \tilde{\tau}^{D_1} = 0) - \varepsilon.$$

Therefore

$$P_{0,\xi_0}(\tilde{\tau}^D \leq \delta) \geq P_{0,\xi_m}(\tilde{\tau}^D - \tilde{\tau}^{D_1} = 0) - 2\varepsilon$$

for m sufficiently large, say $m \geq N_2 \geq N_1$, since property iv of the hitting probabilities implies that $P_{0,\xi}(\tilde{\tau}^D - \tilde{\tau}^{D_1} = 0)$ is a continuous function of ξ. Hence

$$P_{0,\xi_0}(\tilde{\tau}^D \leq \delta) \geq P_{0,\xi_m}(\tilde{\tau}^D = 0) - 2\varepsilon = a_m - 2\varepsilon \geq a - 3\varepsilon.$$

Therefore, $P_{0,\xi_0}(\tilde{\tau}^D \leq \delta) \geq a$ and $P_{0,\xi_0}(\tilde{\tau}^D = 0) \geq a$ and the proof is completed.

COROLLARY. *$P_{0,\xi}(\tilde{\tau}^D = 0)$ is a subharmonic function of $\xi \in D$.*

Proof. $P_{0,\xi}(\tilde{\tau}^D = 0) \leq \int_{\partial D_1} h_{\partial D_1}(\xi, d\eta) P_{0,\eta}(\tilde{\tau}^D = 0)$ if $D_1 \in \mathcal{C}$ and $D_1 \subset D$, Q.E.D.

THEOREM 7.7. *If $D_m \downarrow \xi_0$, $D_m \in \mathcal{C}$, $D_m \subset \Gamma_n$, then*

$$P_{0,\xi_0}(\tilde{\tau}^D - \tilde{\tau}^{D_m} = 0) \downarrow P_{0,\xi_0}(\tilde{\tau}^D = 0).$$

Proof. Since $P_{0,\xi}(\tilde{\tau}^D = 0)$ is an upper semi-continuous function there is a neighborhood of ξ_0, $N_{\xi_0} \in \mathcal{C}$, such that

$$P_{0,\xi}(\tilde{\tau}^D = 0) \leq P_{0,\xi_0}(\tilde{\tau}^D = 0) + \varepsilon$$

for $\xi \in N_{\xi_0}$. But then if $D_m \subset N_{\xi_0}$,

$$P_{0,\xi_0}(\tilde{\tau}^D - \tilde{\tau}^{D_m} = 0) = \int_{\partial D_m} h_{\partial D_m}(\xi_0, d\eta) P_{0,\eta}(\tilde{\tau}^D = 0) \leq P_{0,\xi_0}(\tilde{\tau}^D = 0) + \varepsilon$$

and the theorem is proved.

COROLLARY 1. *If $P_{0,\xi_0}(\tilde{\tau}^D = 0) > 0$, then*

$$P_{0,\xi_0}(\tilde{\tau}^{D_m} = 0) \uparrow 1 \quad \textit{as} \quad m \to \infty.$$

Proof.

$$P_{0,\xi_0}(\tilde{\tau}^D = 0)$$
$$= \int_{\partial D_m} P_{0,\xi_0}[x(\tau^{D_m}) \in d\eta, \tilde{\tau}^{D_m} = 0] P_{0,\eta}(\tilde{\tau}^D = 0)$$
$$\leq P_{0,\xi_0}(\tilde{\tau}^{D_m} = 0)(P_{0,\xi_0}(\tilde{\tau}^D = 0) + \varepsilon).$$

Hence

$$P_{0,\xi_0}(\tilde{\tau}^{D_m} = 0) \geq P_{0,\xi_0}(\tilde{\tau}^D = 0)[P_{0,\xi_0}(\tilde{\tau}^D = 0) + \varepsilon]^{-1}$$

and the result follows immediately.

COROLLARY 2. *If* $P_{0,\xi_0}(\tilde{\tau}^D = 0) > 0$, *then*

$$P_{0,\xi_0}(\tilde{\tau}^{D(K,\xi_0)} = 0 \quad \textit{for some} \quad K \in Z^+) = 1.$$

Proof. This result follows immediately by application of the Borel-Cantelli lemma.

THEOREM 7.8. *$T_{0,\xi_0}(s, w)$ is a strictly increasing function of s for almost every w.*

Proof. Since the open sets of [0, 1] have a countable base, it suffices to show that if $D \in \mathcal{C}$, $D \subset \Gamma_n$, $P_{0,\xi}(\tilde{\tau}^D = 0) = 0$. Assume that there is some $D \in \mathcal{C}$ and a $\xi_0 \in D$ such that $P_{0,\xi_0}(\tilde{\tau}^D = 0) = a > 0$. We will deduce a contradiction from this hypothesis and thus prove the result. Let

$$D^* \equiv \{\xi : \xi \in D, \mid P_{0,\xi}(\tilde{\tau}^D = 0) - P_{0,\xi_0}(\tilde{\tau}^D = 0) \mid < a/2\}.$$

Because $P_{0,\xi}(\tilde{\tau}^D = 0)$ is a subharmonic function of ξ, D^* is a fine neighborhood of ξ_0. Moreover, $D^* \in \mathbf{S}$ by a result of Saks [13]. If $\xi \in D^*$, $P_{0,\xi}(\tilde{\tau}^D = 0) > 0$.

We will now show that $\tau(w) \geq \tau^{D^*}(w)$ for almost every w by demonstrating that otherwise we obtain a contradiction. Assume that $\tau(w) < \tau^{D^*}(w)$ on a set B of positive probability. On B, except for a set of paths of P_{0,ξ_0}-probability zero, there is a $t(w) > \tau(w)$ such that $T_{\tau,x(\tau)}(t(w), w) = 0$ by Corollary 2 of Theorem 7.7. But then by Theorem 7.4, $T_{0,\xi_0}(t(w), w) = 0$ contradicting the definition of $\tau(w)$.

But then $T_{0,\xi_0}(\tilde{\tau}^{D^*}(w), w) = 0$ with P_{0,ξ_0}-probability one so that by the corollary to Theorem 7.3 there is a sequence $\{U_m\}$ of open sets $U_m \downarrow D^*$ such that $e_{U_m}(\xi_0) \downarrow 0$ as $m \to \infty$. But this is a contradiction of the fine neighborhood condition and so the theorem is proved.

8. The required diffusion

We will now show that if we reparameterize X with the natural time parameter the required diffusion $\tilde{X} = (\tilde{x}(t), \zeta^n, F_t^s, P_x)$ is obtained.

Since we have shown that for almost every path, w, $T_{0,\xi_0}(s, w)$ is a continuous, strictly increasing function of s, $T_{0,\xi_0}(s, w)$ has a continuous, strictly increasing inverse $T_{0,\xi_0}^{-1}(t, w)$.

Let F_t^0 be the smallest σ-subfield of $F_\omega^{(\xi_0)}$ containing all sets of the form

$$(s < \zeta^n) \cap (w : \tilde{x}(s, w) \in A), \qquad A \in \mathbf{S}, 0 \le s \le t,$$

where if $x(0) = \xi_0$,

$$\tilde{x}(t) \equiv x(T_{0,\xi_0}^{-1}(t, w)), \qquad t \le \zeta^n(w) \equiv T_{0,\xi_0}^{-1}(1, w).$$

Let $F \equiv \bigcup_{m=1}^\infty F_m^0$.

$T_{0,\xi_0}^{-1}(t, w)$ is a 0-Markov time for the X process since

$$(w : T_{0,\xi_0}^{-1}(t, w) \ge s) = (w : T_{0,\xi_0}(s, w) \le t)$$

which belongs to $F_{0s}^{\bullet}$ by Lemma 7.2.

If we define $\Theta_t[\tilde{x}(\cdot, w)] \equiv \tilde{x}^t(s, w)$ where $\tilde{x}^t(t + s, w) \equiv \tilde{x}(s, w)$, $s \ge 0$, then we obtain with P_{0,ξ_0}-probability one

$$\tilde{x}^t(s, \overset{+}{w}_{T_{0,\xi_0}^{-1}(t,w)}) = x(s, w)$$

for $s \ge t$. Moreover by Theorem 7.4 it can be shown that Θ_t induces a field homomorphism on F.

Hence $\tilde{x}(t)$ can be described by a set of stationary transition probabilities

$$\begin{aligned} P_{\xi_0}(\tilde{x}(t) \in A) &\equiv P_{t_0,\xi_0}(\tilde{x}(t + t_0) \in A) \\ &\equiv P_{0,\xi_0}(x(T_{0,\xi_0}^{-1}(t, w)) \in A), \qquad A \in \mathbf{S}. \end{aligned}$$

Up to subsets of a set of zero measure, namely, the set of paths having discontinuities, $T_{0,\xi_0}(s, w)$ induces a one to one measure preserving transformation, T^*, of $F_{01}^{\bullet}$ onto F,

$$T^*(A) \equiv \{\tilde{x}(T_{0,\xi_0}(s, w)) : w \in A\}.$$

$\tilde{x}(t, w)$ is a continuous function of t except for a set of paths of P_{ξ_0}-measure zero.

THEOREM 8.1 *The process* $\tilde{X} = (\tilde{x}(t), \zeta^n, F_t^s, P_x)$ *is a stationary strict Markov process.*

Proof. It suffices to show that if $\tau(w)$ is a 0-Markov time, that is,

$$\{w : \tau(w) \ge t\} \in F_t^0,$$

then

$$P_{\xi_0}\{\tilde{x}(\eta, w) \in \Gamma \mid F_{\tau+}^0\} = P_{\tilde{x}(\tau)}\{\tilde{x}(\eta) \in \Gamma\} \qquad \text{a.e.},$$

where $F_{\tau+}^0 \equiv \{B : B \in F, B \cap (w : \tau(w) < t) \in F_t^0\}$, and $\eta(w)$ is an $F_{\tau+}^0$ measurable function such that $\eta(w) \ge \tau(w)$.

Let $\tau'(w) \equiv T_{0,\xi_0}^{-1}(\tau(w), w)$. We will now show that $\tau'(w)$ is a 0-Markov time for the process X.

$$(w : \tau'(w) < s, w(0) = \xi_0) = \bigcup_{r \in B_2}(w : T_{0,\xi_0}(s, w) \ge r, \tau(w) < r).$$

But $(w : T_{0,\xi_0}(s, w) \geq r, \tau(w) < r)$ is equal to the intersection of

$$(w : \tau(w) < r) \in F^{\bullet}_{0T_{0,\xi_0}^{-1}(r)+} \quad \text{and} \quad (w : T^{-1}_{0,\xi_0}(r, w) \leq s)$$

and therefore belongs to $F^{\bullet}_{0s}$. Hence $(w : \tau(w) < s) \in F^{\bullet}_{0s}$.

Moreover, T^* maps $F^{\bullet}_{0\tau'+}$ onto $F^{0}_{\tau+}$ one to one up to subsets of a set of P_{0,ξ_0}-measure zero. Therefore Theorem 5.4 implies that

$$P_{0,\xi_0}\{x(\eta') \in \Gamma \mid F^{\bullet}_{0\tau'+}\} = P_{\tau',x(\tau')}\{x(\eta') \in \Gamma\} \qquad \text{a.e.}$$

which then yields the result.

THEOREM 8.2. *For any set $D \in \Delta$, $D \subset \Gamma_n$, $\xi_0 \in D$, $A \in \mathbf{B}(\partial D)$,*
(i) $P_{\xi_0}(\tilde{x}(\tilde{\tau}^D) \in A) = h_{\partial D}(\xi_0, A)$ *and*
(ii) $E_{\xi_0}(\tilde{\tau}^D) = e_D(\xi_0)$ *where* $\tilde{\tau}^D = \inf\{t : \tilde{x}(t) \notin D\}$.

Proof. Let $D \in \mathfrak{C}^*$. Then because $T_{0,\xi_0}(s, w)$ is strictly increasing, $\inf\{t : \tilde{x}(t) \notin D\} = T_{0,\xi_0}(\tau^D(w), w)$, for paths for which $x(0) = \xi_0$. Furthermore,

$$P_{\xi_0}(\tilde{x}(\tilde{\tau}^D) \in A) = h_{\partial D}(\xi_0, A)$$

by Lemma 5.2 and $E_{\xi_0}(\tilde{\tau}^D) = e_D(\xi_0)$ by Theorem 7.2. Also, if $D \in \Delta$, then $E_{\xi_0}(\tilde{\tau}^D) = e_D(\xi_0)$ by Theorem 7.3.

Now say that there is a set $D \in \Delta$, $D \subset \Gamma_n$, such that $P_{\bullet}(\tilde{x}(\tilde{\tau}^D) \in \cdot) \neq h_{\partial D}(\cdot, \cdot)$. By adding D to $\mathfrak{C}_0$ and proceeding as above we can construct a new diffusion $\tilde{X}^* = (\tilde{x}^*(t), \zeta^{n*}, F^{s*}_t, P^*_x)$ such that $P^*_{\bullet}(\tilde{x}^*(\tilde{\tau}^D) \in \cdot) = h_{\partial D}(\cdot, \cdot)$. However the infinitesimal generator of $\tilde{X}^*$, $\mathfrak{G}^*$, is the same as the generator of $\tilde{X}$, $\mathfrak{G}$. Since the infinitesimal generator uniquely determines the process [12, Theorem A, p. 614] it follows that $X = X^*$ and hence

$$P_{\bullet}(\tilde{x}(\tilde{\tau}^D) \in \cdot) = P^*_{\bullet}(x^*(\tilde{\tau}^D) \in \cdot).$$

Hence the theorem is proved.

We have thus accomplished what we set out to do. That is, we have constructed a diffusion $\tilde{X} = (\tilde{x}(t), \zeta^n, F^s_t, P_x)$ up to the boundary of Γ_n with the specified mean hitting times and hitting probabilities.

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