PARTIALLY ORDERED LINEAR SPACES AND LOCALLY CONVEX LINEAR TOPOLOGICAL SPACES

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1. Introduction

In a partially ordered set it is possible to define various kinds of convergence in terms of the partial order. A general discussion of convergence defined in terms of a partial order may be found in [1], [3], or [6]. In general, a convergence defined in this way cannot be topologized, i.e., cannot be defined as a convergence with respect to some topology. For example, convergence almost everywhere is of this type (see Section 2, Example 2). Since convergence defined in terms of a partial order cannot always be topologized, it is natural to ask whether convergence with respect to a topology can be obtained as a convergence defined in terms of a partial order. This is possible, for example, if we consider any compact Hausdorff space. One simply embeds this space in the product of unit intervals (where, as usual, the product is partially ordered componentwise) and then uses order convergence (i.e., componentwise convergence) in the product. The product is generally much larger than the original compact Hausdorff space and, therefore, the question of whether the space itself can be suitably partially ordered remains open. Nevertheless, the above example suggests that partially ordered spaces are reasonable generalizations of topological spaces.

In this paper we will show that any locally convex linear topological space (l.c.l.t.s.) over the reals can be suitably embedded in a partially ordered linear space (p.o.l.s.). This result is closely related to Grothendieck's result which states that every l.c.l.t.s. is isomorphic to a subspace of the product of normed linear spaces [2]. However, the generality of our result is that it allows one to consider locally convex linear topological spaces as subspaces of certain kinds of partially ordered linear spaces. This in turn allows one to consider the topological convergence as a special kind of convergence defined in terms of a partial order. It is hoped that this will provide for a unified treatment of topological and non-topological convergence in linear spaces, at least in the case of the more important types of convergence which are useful in analysis.

Finally, we shall prove two theorems showing that there is a connection between continuous linear operators and functionals and positive linear operators and functionals in certain cases.

2. Basic definitions

In this paper all linear spaces are assumed to be over the reals and topologies are assumed to be Hausdorff. A p.o.l. space X is a partially ordered set in

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which the partial order (denoted by $\leqq$) is consistent with the linear structure in the following sense: (a) if $x, y, z \in X$ and $x \leqq y$, then $x + z \leqq y + z$; (b) if $x \in X$ and $0 \leqq x$ and α is a non-negative real number, then $0 \leqq \alpha x$. We will use Greek letters to denote real numbers and will, as usual, use the same symbol $\leqq$ to denote inequality between real numbers. We now give the necessary definitions of convergence in a p.o.l.s. which will be used in this paper.

DEFINITION 1. A non-empty subset M of a p.o.l.s. X is said to be directed to 0 if for every $x, y \in M$ there exists $z \in M$ such that $z \leqq x$ and $z \leqq y$, and if $\inf M = 0$. When we write $\inf M = 0$, this means that $0 \leqq x$ for all $x \in M$ and if $u \leqq x$ for all $x \in M$, then $u \leqq 0$.

DEFINITION 2. A net $\{x_n, n \in D\}$ of elements from X is said to order converge (o-converge) to 0 if there exists non-empty $M \subset X$ which is directed to 0 such that for each $y \in M$ there exists $k \in D$ such that $-y \leqq x_n \leqq y$ for all $n > k$. More generally, the net $\{x_n, n \in D\}$ is said to o-converge to $x \in X$ if the net $\{x_n - x, n \in D\}$ o-converges to 0. (See [4, Chap. 2] for a general discussion of nets.)

DEFINITION 3. A net $\{x_n, n \in D\}$ of non-negative elements from X unboundedly order converges (uo-converges) to 0 if every bounded net $\{y_n, n \in D\}$, where $0 \leqq y_n \leqq x_n$, o-converges to 0. A net $\{y_n, n \in D\}$ is said to be bounded if there exists $u \in X$ such that $-u \leqq y_n \leqq u$ for all $n \in D$.

DEFINITION 4. An arbitrary net $\{x_n, n \in D\}$ uo-converges to 0 if there exists a net $\{z_n, n \in D\}$, where $-z_n \leqq x_n \leqq z_n$ (hence, $0 \leqq z_n$), which uo-converges to 0. Note that o-convergence implies uo-convergence.

The use of uo-convergence is quite natural as the following two examples show.

Example 1. If X is the linear space of all real-valued functions defined on an abstract set Ω and the partial order is defined pointwise, then uo-convergence is simply pointwise convergence. Note that in this case uo-convergence can be topologized. In this example o-convergence and uo-convergence coincide if and only if Ω is a finite set; we emphasize that this refers to convergence of arbitrary nets, because for sequences these types of convergence coincide even if Ω is infinite.

Example 2. Let X be the p.o.l.s. of real-valued integrable functions defined on [0, 1]. Here uo-convergence is simply convergence almost everywhere. In this example uo-convergence cannot be topologized [3, pp. 52–54]. In this case o-convergence and uo-convergence do not coincide even for sequences.

3. The main theorems

In the proof of the first theorem we shall use the following lemma.

LEMMA. *Let E be a linear space and let p be a semi-norm defined on E. Now let $E(p)$ be the collection of all ordered pairs (x, λ), where $x \in E$ and λ is a real number. In $E(p)$ we define equality as follows:*

$$(x, \lambda) = (y, \mu) \quad \textit{iff} \quad p(x - y) = \mu - \lambda = 0,$$

and the partial order is defined as follows:

$$(x, \lambda) \leqq (y, \mu) \quad \textit{iff} \quad p(x - y) \leqq \mu - \lambda.$$

Then $E(p)$ is a p.o.l.s. in which o-convergence and uo-convergence coincide. Furthermore, if $\{x_n, n \in D\}$ is a net of elements from E, then $\lim p(x_n) = 0$ *iff the net $\{(x_n, 0), n \in D\}$ o-converges to $(0, 0)$.*

Proof. The properties of the semi-norm p (i.e., $p(x) \geqq 0$, $p(\alpha x) = |\alpha| p(x)$, and $p(x + y) \leqq p(x) + p(y)$) can be applied directly to show that equality and the partial order as defined in $E(p)$ have the necessary properties and, hence, $E(p)$ is a p.o.l.s.

From the definitions of convergence given above it is clear that if every net $\{(x_n, \lambda_n), n \in D\}$ of non-negative elements from $E(p)$ which uo-converges to $(0, 0)$ is eventually bounded, then uo-convergence implies o-convergence and, hence, they are equivalent. Thus, let $\{(x_n, \lambda_n), n \in D\}$ be a net of non-negative elements from $E(p)$ which uo-converges to $(0, 0)$. Since $(0, 0) \leqq (x_n, \lambda_n)$, we have $p(x_n) \leqq \lambda_n = 2\lambda_n - \lambda_n$, which means $(x_n, \lambda_n) \leqq (0, 2\lambda_n)$. Now if we put $y_n = x_n/(1 + \lambda_n)$ and $\mu_n = \lambda_n/(1 + \lambda_n)$, then

$$(0, 0) \leqq (y_n, \mu_n) \leqq (x_n, \lambda_n) \quad \text{and} \quad (y_n, \mu_n) \leqq (0, 2) \quad \text{for all } n \in D.$$

Therefore, the net (y_n, μ_n), $n \in D$ must o-converge to $(0, 0)$.

Before proceeding, let us show that if $M \subset E(p)$ is non-empty and directed to $(0, 0)$, then $\eta_0 = \inf \{\eta : (z, \eta) \in M\} = 0$. Assume the contrary, i.e., that $\eta_0 = 2\beta > 0$. Hence, there exists $(z_1, \eta_1) \in M$ such that $\eta_1 \leqq 3\beta$. Now if $(z, \eta) \in M$ and $(z, \eta) \leqq (z_1, \eta_1)$, then $p(z - z_1) \leqq \eta_1 - \eta \leqq \beta \leqq \eta - \beta$, which means that $(z_1, \beta) \leqq (z, \eta)$. Since M is directed to $(0, 0)$, we see that $(z_1, \beta) \leqq (z, \eta)$ for all $(z, \eta) \in M$ and since $\inf M = (0, 0)$, we must have $(z_1, \beta) \leqq (0, 0)$, which is a contradiction because $\beta > 0$. Hence, by contradiction we have $\eta_0 = 0$.

Since the net $\{(y_n, \mu_n), n \in D\}$ o-converges to $(0, 0)$, there exists non-empty $M \subset E(p)$ which is directed to $(0, 0)$ such that for any $(z, \eta) \in M$ there exists $k \in D$ such that $-(z, \eta) \leqq (y_n, \mu_n) \leqq (z, \eta)$ for all $n > k$. In particular, we may select $(z, \eta) \in M$ so that $\eta \leqq \frac{1}{2}$; hence, there exists $k \in D$ so that

$$\mu_n = \lambda_n/(1 + \lambda_n) \leqq \eta \leqq \tfrac{1}{2}$$

(i.e., $\lambda_n \leqq 1$) for all $n > k$. Therefore, we have $(0,0) \leqq (x_n, \lambda_n) \leqq (0, 2)$ for all

$n > k$, which means that net $\{(x_n, \lambda_n), n \in D\}$ is eventually bounded. Hence, o-convergence and uo-convergence coincide in $E(p)$.

We now wish to prove the last assertion of the lemma. Assume first that $\lim p(x_n) = 0$. Let us define $\mu_n = \min \{p(x_n), 1\}$ and then define

$$\lambda_n = \sup \{\mu_m : m > n \in D\}.$$

Then set $M = \{(0, \lambda_n) : n \in D\}$. It is easily verified that M is directed to $(0, 0)$. Now if we select any $(0, \lambda_i) \in M$, then there exists $k \in D$ such that $k > i$ and $\lambda_k < 1$. Therefore, $-(0, \lambda_i) \leqq (x_n, 0) \leqq (0, \lambda_i)$ for all $n > k$. Hence, the net $\{(x_n, 0), n \in D\}$ o-converges to $(0, 0)$. To prove the converse, let us assume that the net $\{(x_n, 0), n \in D\}$ o-converges to $(0, 0)$. Hence, there exists non-empty $M \subset E(p)$ which is directed to $(0, 0)$ and is such that for any $(z, \eta) \in M$ there exists $k \in D$ such that

$$-(z, \eta) \leqq (x_n, 0) \leqq (z, \eta)$$

for all $n > k$. Using the latter inequalities, we have

$$2p(x_n) \leqq p(x_n + z) + p(x_n - z) \leqq (0 + \eta) + (\eta - 0) = 2\eta;$$

hence, $p(x_n) \leqq \eta$ for all $n > k$. Since $\inf \{\eta : (z, \eta) \in M\} = 0$, we have $\lim p(x_n) = 0$, Q.E.D.

Note. If $(0, 0) \leqq (z, \eta) \in E(p)$, then $(z, \eta) \leqq (0, 2\eta)$. Hence, in discussing o-convergence of nets in $E(p)$ it is sufficient to consider only one subset M which is directed to $(0, 0)$, namely, $M = \{(0, \eta) : \eta > 0\} \subset E(p)$.

THEOREM 1. *Let E be a l.c.l.t.s. (the topology is Hausdorff). Then there exists a p.o.l.s. X and a one-to-one linear mapping $i : E \to X$ (into) such that a net $\{x_n, n \in D\}$ of elements from E converges to 0 with respect to the topology for E if and only if the net $\{i(x_n), n \in D\}$ of elements from X uo-converges to 0.*

Proof. Let P be the family of semi-norms which define the topology for E. For each $p \in P$ let $E(p)$ be the p.o.l.s. constructed as in the above lemma. We now define X as the direct product of all $E(p)$, where $p \in P$. If X is partially ordered coordinatewise, then X becomes a p.o.l.s. The mapping $i : E \to X$ is defined as follows: if $x \in E$, then $i(x)$ is the element in the direct product X in which each component has the form $(x, 0)$. The mapping i is linear and one-to-one since E is a Hausdorff space. Hence, if $x \in E$ and $x \neq 0$, then there exists $p \in P$ such that $p(x) \neq 0$ so that $(x, 0) \in E(p)$ and $\{x, 0) \neq (0, 0)$.

Now let $\{x_n, n \in D\}$ be a net of elements from E which converges to 0. This means, of course, that $\lim p(x_n) = 0$ for all $p \in P$. For each $n \in D$ and $p \in P$ define $\lambda_n(p) = p(x_n)$ and then define $z_n \in X$ as follows: the component of z_n in $E(p)$ is $(0, \lambda_n(p))$. It is clear that $-z_n \leqq i(x_n) \leqq z_n$ for all $n \in D$. We will now use the fact that $\lim \lambda_n(p) = 0$ for all $p \in P$ to show that the net $\{z_n, n \in D\}$ uo-converges to $0 \in X$. To do this we must show that if $\{y_n, n \in D\}$ is any bounded net such that $0 \leqq y_n \leqq z_n$, then it o-converges to

$0 \in X$. If we write $(y_n(p), \mu_n(p))$ as the component of y_n in $E(p)$, then we can set $\beta_n(p) = \sup \{\mu_m(p) : m > n \in D\}$, where $\beta_n(p) < \infty$ because the net $\{y_n, n \in D\}$ is bounded and, hence, so is the net of components in each $E(p)$. Since $0 \leqq \mu_n(p) \leqq \lambda_n(p)$ and $\lim \lambda_n(p) = 0$, we must have $\lim \beta_n(p) = 0$ for all $p \in P$. Now if we define $u_n \in X$ as that element having component $(0, 2\beta_n(p))$ in $E(p)$ and define $M = \{u_n : n \in D\}$, then it is readily seen that M is directed to $0 \in X$. Using the fact that $\mu_n(p) \leqq \beta_n(p)$, we see that

$$(0, 0) \leqq (y_n(p), \mu_n(p)) \leqq (0, 2\beta_n(p))$$

for all $n \in D$ and $p \in P$; hence, $0 \leqq y_n \leqq u_n$ for all $n \in D$. Therefore, the net $\{y_n, n \in D\}$ o-converges to 0. This in turn proves that the net $\{z_n, n \in D\}$ uo-converges to 0; hence, the net $\{i(x_n), n \in D\}$ uo-converges to 0.

Now let $\{x_n, n \in D\}$ be a net of elements from E such that the net $\{i(x_n), n \in D\}$ uo-converges to $0 \in X$. This in turn means that for each $p \in P$ the net $\{(x_n, 0), n \in D\}$ of elements from $E(p)$ uo-converges to $(0, 0) \in E(p)$. By the above lemma, this means that $\lim p(x_n) = 0$ for all $p \in P$; hence, the net $\{x_n, n \in D\}$ converges to $0 \in E$ with respect to the topology for E, Q.E.D.

We now wish to show that continuous linear operators on a l.c.l.t.s. E can be represented by positive linear operators on X (where E and X are related as in Theorem 1).

THEOREM 2. *Let E and F be l.c.l.t. spaces and let P and Q be the families of all semi-norms which determine the topologies for E and F, respectively. Let X and Y be the p.o.l. spaces which are constructed as in the proof of Theorem 1. Accordingly we obtain linear one-to-one mappings $i : E \to X$ and $j : F \to Y$. Then if $H : E \to F$ is a continuous linear operator, there exists a positive linear operator $T : X \to Y$ such that for each $x \in E$, $T(i(x)) = j(H(x))$.*

Proof. For each $q \in Q$ let us select $p \in P$ so that $q(H(x)) \leqq p(x)$ for all $x \in E$. This is possible since H is continuous. We will regard this selection as determining a correspondence $c : Q \to P$.

Now if $z \in X$, we will define $T(z) \in Y$ as follows: since Y is the direct product of all $F(q)$, $q \in Q$, we will define the component of $T(z)$ in $F(q)$ to be $(H(x), \lambda)$, where (x, λ) is the component of z in $E(c(q))$.

It follows immediately that T is linear. Referring to the proof of Theorem 1 to see how the partial ordering is defined in X and Y and using the definition of $c(q)$, it is easily seen that T is positive. From the definition of the mappings i and j it follows that $T(i(x)) = j(H(x))$, Q.E.D.

If in Theorem 2 we take F as the real line and define $Y = F$ with j as the identity mapping, then the proof of Theorem 2 can be modified so that it applies to linear functionals; i.e., if H is a continuous linear functional on E, then there exists a positive linear functional T on X such that $T(i(x)) = H(x)$.

The following question is open: using the notation of Theorem 2, under what conditions is it possible to find a continuous linear operator $H : E \to F$ such that $T(i(x)) = j(H(x))$ for all $x \in E$, where T is a given positive linear operator mapping X into Y? The following theorem gives a partial answer to this question in the case where linear functionals are considered.

THEOREM 3. *Let E be a l.c.l.t.s. such that the topology for E is determined by a countable family $P = \{p_1, p_2, \cdots\}$ of seminorms. Let X be the p.o.l.s. constructed from E and P as in the proof of Theorem* 1. *Then if f is any positive linear functional defined on X, $f(i(\cdot))$ is a continuous linear functional defined on E.*

Proof. Referring to the construction of X as given in Theorem 1, we see that X is the space of sequences of elements $(x_n, \alpha_n) \in E(p_n)$. If we consider the subspace X_0 of sequences with components of the form $(0, \alpha_n) \in E(p_n)$, then f must be a positive linear functional on X_0. This is possible only if

$$f(z) = \beta_1 \alpha_1 + \cdots + \beta_k \alpha_k,$$

where $z = \{(0, \alpha_1), (0, \alpha_2), \cdots\} \in X_0$ and $\beta_1, \cdots, \beta_k$ are non-negative real numbers determined by f.

Now if $x \in E$, $\lambda_n = p_n(x)$, and $z = \{(0, \lambda_1), (0, \lambda_2), \cdots\} \in X_0$, then $f(z) + f(i(x)) \geqq 0$ and $f(z) - f(i(x)) \geqq 0$, where the inequalities follow from the fact that $\pm i(x) \leqq z$. Therefore,

$$|f(i(x))| \leqq \beta_1 p_1(x) + \cdots + \beta_k p_k(x) \qquad \text{for all } x \in E;$$

hence, $f(i(\cdot))$ is continuous, Q.E.D.

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