FINITE MÖBIUS-PLANES ADMITTING A ZASSENHAUS GROUP AS GROUP OF AUTOMORPHISMS[1]

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We call an incidence structure $\mathfrak{M}$ consisting of points and circles and an incidence relation between points and circles a Möbius-plane (= inversive plane), if the following axioms are satisfied (see e.g. Benz [1]):

(1) *If P, Q, R are three different points of $\mathfrak{M}$, then there exists one and only one circle k in $\mathfrak{M}$ such that $P, Q, R \in k$.*

(2) *If k is a circle and P a point on k and if Q is a point not on k, then there exists one and only one circle l with $P, Q \in l$ and $k \cap l = \{P\}$.*

(3) *There are four points which do not all lie on the same circle, and every circle carries at least one point.*

σ is called an *automorphism* of $\mathfrak{M}$, if σ is a permutation of the points of $\mathfrak{M}$ which maps concyclic points on concyclic points. The full automorphism group of $\mathfrak{M}$ is called the *Möbius-group* of $\mathfrak{M}$.

If P is a point of $\mathfrak{M}$, then we derive an incidence structure $\mathfrak{A}(\mathfrak{M}, P)$ from $\mathfrak{M}$ and P in the following way:

(a) *The points of $\mathfrak{A}(\mathfrak{M}, P)$ are the points of $\mathfrak{M}$ which are different from P.*

(b) *The lines of $\mathfrak{A}(\mathfrak{M}, P)$ are the circles through P.*

(c) *A point Q and a line l of $\mathfrak{A}(\mathfrak{M}, P)$ are incident if and only if the corresponding point Q and the corresponding circle l are incident in $\mathfrak{M}$.*

It is a well known fact that $\mathfrak{A}(\mathfrak{M}, P)$ is an affine plane (Benz [1, Satz 1]).

If $\mathfrak{M}$ is a finite Möbius-plane, then it follows from the fact that $\mathfrak{A}(\mathfrak{M}, P)$ is an affine plane that the number of points of $\mathfrak{M}$ is $q^2 + 1$ and the number of points which lie on a circle is $q + 1$. It is easily seen that the number of circles is $q(q^2 + 1)$. We call q the *order* of $\mathfrak{M}$.

Let $\mathfrak{B}$ be a set of circles and P a point. We call $\mathfrak{B}$ a *tangent bundle* through P, if the following hold:

(i) $\mathfrak{B} \neq \emptyset$.

(ii) $k, l \in \mathfrak{B}$ and $k \neq l$ imply $k \cap l = \{P\}$.

(iii) $k \in \mathfrak{B}$ and $k \cap l = \{P\}$ imply $l \in \mathfrak{B}$.

Let Σ be a permutation group on the set $\mathfrak{P}$; then we call Σ *Zassenhaus transitive* on $\mathfrak{P}$, if Σ is doubly transitive on $\mathfrak{P}$ and if only the identity fixes three different elements of $\mathfrak{P}$.

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Now the only two known classes of finite Möbius-planes are the following ones:

1. The finite miquelian Möbius-planes: The Möbius-group of these planes contains a subgroup isomorphic to the $PSL_2(q^2)$, where q is the order of the plane. We shall show (Theorem 2) that this fact characterizes these planes. (For the definition and the properties of these planes see e.g. Benz [2, §2 and §4.6].)

2. The finite Möbius-planes of order $q = 2^{2r+1} > 2$ which are constructed with the Suzuki-group $S(2^{2r+1})$ (see Theorem 1): We shall show that a Möbius-plane of order $q = 2^{2r+1}$ on which the Suzuki-group $S(q)$ acts as an automorphism group is uniquely determined up to isomorphisms. We shall call these Möbius-planes Suzuki-planes.

The subgroup $PSL_2(q^2)$ of the Möbius-group of a miquelian Möbius-plane and the subgroup $S(q)$ of the Möbius-group of a Suzuki-plane act Zassenhaus-transitively on the points of these planes. We shall see that a Möbius-plane which admits an automorphism group being Zassenhaus transitive on the points is either miquelian or a Suzuki-plane.

Finally we shall prove the somewhat surprising theorem that the only finite Möbius-planes which admit an automorphism group which is transitive on the incident point-circle pairs and such that only the identity fixes three different points are the miquelian ones.

THEOREM 1. *If $q = 2^{2r+1} > 2$, then there is one, and up to isomorphism only one, Möbius-plane $\mathfrak{M}$ of order q which admits an automorphism group Σ isomorphic to the Suzuki-group $S(q)$.*

Proof. The existence part of this theorem was proved by Hughes in [8], so we have only to prove the uniqueness.

Let $\mathfrak{M}$ be a Möbius-plane with $q^2 + 1$ points ($q = 2^{2r+1} > 2$) and Σ an automorphism group of $\mathfrak{M}$ isomorphic to $S(q)$. If M is a 2-Sylow subgroup of Σ, then the normalizer $\mathfrak{N}$M of M in Σ is a subgroup of maximal order in Σ (Suzuki [11, Theorem 9]). Let $\mathfrak{Z}$ be a system of transitivity of points. We can assume that $\mathfrak{Z}$ contains at least two points. Now we have

$$(q^2 + 1)q^2(q - 1) = o(\Sigma) = |\mathfrak{Z}|\, o(\Sigma_P)$$

where P is a point of $\mathfrak{Z}$ and Σ_P the stabilizer of P. This implies

$$o(\Sigma) > o(\Sigma_P) \geqq q^2(q - 1) = o(\mathfrak{N}\mathrm{M}).$$

Hence $\Sigma_P = \mathfrak{N}\mathrm{M}$ for a suitable 2-Sylow subgroup M of Σ. It follows that $|\mathfrak{Z}| = q^2 + 1$, i.e. Σ is transitive on the points of $\mathfrak{M}$. The group of inner automorphisms of Σ is Zassenhaus transitive on the 2-Sylow subgroups of Σ. This and the fact that $\Sigma_P = \mathfrak{N}\mathrm{M}$ imply that Σ is Zassenhaus transitive on the points of $\mathfrak{M}$. We put $\mathrm{H} = \Sigma_P$. Then H is a Frobenius group and therefore $\mathrm{H} = \mathrm{MT}$ where M is a 2-Sylow subgroup of Σ and $\mathrm{M} \cap \mathrm{T} = 1$, $\mathrm{M} \lhd \mathrm{H}$ and $o(\mathrm{M}) = q^2$, $o(\mathrm{T}) = q - 1$. Finally M is transitive on the points different from P. The number of tangent bundles through P is $q + 1$. Therefore there exists a tangent bundle $\mathfrak{B}$ through P with $\mathfrak{B}^{\mathrm{M}} = \mathfrak{B}$. If $k \in \mathfrak{B}$, then $o(\mathrm{M}_k) = q$, since M is transitive and regular on the points different from P and since every point of $\mathfrak{M}$ is on a circle of $\mathfrak{B}$. This implies that $o(\mathrm{H}_k) = qt$ where t is a divisor of $q - 1$. Now we assume that Σ_k is different from H_k. Then there is a $\sigma \in \Sigma_k$ with $P^\sigma \neq P$. This implies that Σ_k is transitive on the points of k. Therefore $q + 1$ is a divisor of $o(\Sigma) = (q^2 + 1)q^2(q - 1)$, a contradiction proving $\Sigma_k = \mathrm{H}_k$. Let $\mathcal{K} = \{k^\sigma : \sigma \in \Sigma\}$. Then we have

$$(q^2 + 1)q^2(q - 1) = o(\Sigma) = |\mathcal{K}|\, o(\Sigma_k) = |\mathcal{K}|\, qt.$$

This implies, since t is a divisor of $q - 1$, that

$$q(q^2 + 1) \geqq |\mathcal{K}| = q(q^2 + 1)q^2(q - 1)t^{-1} \geqq q(q^2 + 1).$$

Hence $|\mathcal{K}| = q(q^2 + 1)$ and $t = q - 1$ so that H_k is sharply doubly transitive on $k - \{P\}$ and therefore M_k is an elementary abelian 2-group. This implies that $\mathrm{M}_k = \mathrm{ZM}$, the center of M (Suzuki [11, Theorem 6 and Lemma 1]). Let $\mathrm{H}_k = \Lambda$ and $\mathrm{ZM} = \mathrm{Z}$. Let $P = P_1, P_2, \cdots, P_{q+1}$ be all the points which are on k and H_i the stabilizer of P_i. Denote by $\mathcal{H}$ the set consisting of all the H_i. Finally we define $\Delta = \{\sigma \in \Sigma : \sigma^{-1}\mathrm{H}\sigma \in \mathcal{H}\}$. Now we can describe $\mathfrak{M}$ in Σ in the following way: We define the mappings

$$Q \to \mathrm{H}\sigma \quad \text{if and only if} \quad P^\sigma = Q,$$

$$l \to \Lambda\tau \quad \text{if and only if} \quad k^\tau = l.$$

These mappings are one-to-one and onto. We define incidence by $\mathrm{H}\sigma \,\mathrm{I}\, \Lambda\tau$ if and only if $P^\sigma \in k^\tau$. It is easily seen that $\mathrm{H}\sigma \,\mathrm{I}\, \Lambda\tau$ if and only if $\sigma\tau^{-1} \in \Delta$. If $\mathfrak{M}^*$ is a second Möbius-plane which satisfies the conditions of Theorem 1 and if Σ^*, H^*, M^* ,etc. have the same meaning as Σ, H, M, etc., then first of all Σ and Σ^* are isomorphic (Suzuki [11, Theorem 8]). Now there is exactly one 2-Sylow subgroup M_1 of Σ such that $\mathfrak{M}\mathrm{M} \cap \mathfrak{M}\mathrm{M}_1 = \mathrm{T}$ and exactly one 2-Sylow subgroup M_1^* of Σ^* such that $\mathfrak{M}\mathrm{M}^* \cap \mathfrak{M}\mathrm{M}_1^* = \mathrm{T}^*$. This implies, since the inner automorphisms of Σ are doubly transitive on the 2-Sylow subgroups of Σ, that there is an isomorphism α from Σ onto Σ^* such that $\mathrm{M}^\alpha = \mathrm{M}^*$ and $\mathrm{M}_1^\alpha = \mathrm{M}_1^*$. This implies that $\mathrm{T}^\alpha = \mathrm{T}^*, \mathrm{H}^\alpha = \mathrm{H}^*, \mathcal{H}^\alpha = \mathcal{H}^*$ and $\Delta^\alpha = \Delta^*$, Q.E.D.

The proof of Theorem 1 shows also the validity of the following:

COROLLARY 1. *Σ is transitive on the circles and Zassenhaus transitive on the points of $\mathfrak{M}$.*

COROLLARY 2. *In $\mathfrak{M}$ the bundle-theorem (Büschelsatz) is satisfied but not the theorem of Miquel. $\mathfrak{A}(\mathfrak{M}, P)$ is desarguesian for every point P of $\mathfrak{M}$.*

(For a statement of these two configuration theorems see e.g. Benz [2, §2].)

Proof. The Möbius-group of a finite miquelian Möbius-plane is isomorphic to the $P\Gamma L_2(q^2)$, if q is the order of the plane (see e.g. Benz [2, §4.6]). Since

$S(q)$ is not contained in the $P\Gamma L_2(q^2)$, we see that $\mathfrak{M}$ is a nonmiquelian Möbius-plane. Now let $\mathcal{O}$ be a Tits-ovaloid in the projective 3-space $\mathcal{S}$ over $GF(q)$ (see Tits [12, §§4.2 and 4.3]). If the points of $\mathcal{O}$ are the points of a geometry $\mathfrak{M}$ and the planar sections of $\mathcal{O}$ containing more than one point are the circles of $\mathfrak{M}$ and if incidence in $\mathfrak{M}$ is equivalent to incidence in $\mathcal{S}$, then it is easily seen that $\mathfrak{M}$ is a Suzuki-plane. It follows from this construction that $\mathfrak{M}$ satisfies the bundle-theorem and that $\mathfrak{A}(\mathfrak{M}, P)$ is desarguesian for all P of $\mathfrak{M}$. Corollary 2 follows now from Theorem 1.

THEOREM 2. *If $q = p^r$ (p a prime number), then there is one, and up to isomorphism only one, Möbius-plane $\mathfrak{M}$ of order q admitting an automorphism group Σ being isomorphic to the $PSL_2(q^2)$.*

Proof. The existence of such planes is well known (see e.g. Benz [2, §§2 and 4.6]), so we have only to prove their uniqueness. If $q \neq 3$, then it follows from Dickson [3, §263] that Σ is doubly transitive on the points of $\mathfrak{M}$. If $q = 3$ and Σ is not transitive on the points of $\mathfrak{M}$, then Σ has exactly four fixed points [3, §263] which is easily seen to be impossible. Therefore Σ is doubly transitive in either case. Let P be a point of $\mathfrak{M}$ and $\mathrm{H} = \Sigma_P$, the stabilizer of P. Then $o(\mathrm{H}) = a^{-1}q^2(q^2 - 1)$ with $a = 1$, if q is even, and $a = 2$, if q is odd. Furthermore $\mathrm{H} = \mathrm{MT}$, where M is a p-Sylow subgroup of Σ and $o(\mathrm{T}) = a^{-1}(q^2 - 1)$. Since the number of tangent bundles through P is $q + 1$, the p-Sylow subgroup M fixes some tangent bundle $\mathcal{B}$ through P. If $k \in B$, then $o(\mathrm{M}_k) = q$ and therefore $o(\mathrm{H}_k) = qt$ with t a divisor of $q - 1$. Now let $\mathcal{K}$ be the set $\{k^\sigma : \sigma \in \Sigma\}$. Then we have

$$q(q^2 + 1) \geqq |\mathcal{K}| = (q^2 + 1)q^2(q^2 - 1)(ao(\Sigma_k))^{-1}.$$

This implies $o(\Sigma_k) \geqq a^{-1}q(q^2 - 1)$. Therefore we have $\Sigma_k \neq \mathrm{H}_k$. Hence Σ_k is transitive on the points of k. Put $\Sigma_k = \Lambda$.

Case 1. $p = 2$. In this case $a = 1$ and $\Lambda \cong PSL_2(q)$ (this follows from the order of Λ and the list of subgroups of the $PSL_2(q^2)$ in [3, §260]). This implies $|\mathcal{K}| = q(q^2 + 1)$. Therefore Σ is transitive on the circles of $\mathfrak{M}$. Since Λ is transitive on the point of k, it follows that Σ is transitive on the incident point-circle pairs of $\mathfrak{M}$. Therefore we can describe $\mathfrak{M}$ within Σ in the following way: We define the mappings

$$Q \to \mathrm{H}\sigma \quad \text{if and only if} \quad P^\sigma = Q,$$

$$l \to \Lambda\tau \quad \text{if and only if} \quad k^\tau = l.$$

These mappings are one-to-one and onto. If we define incidence by $\mathrm{H}\sigma \mathrm{I} \Lambda\tau$ if and only if $P^\sigma \in k^\tau$, then it follows from Higman and McLaughlin [6, Proposition 1] that $\mathrm{H}\sigma \mathrm{I} \Lambda\tau$ if and only if $\mathrm{H}\sigma \cap \Lambda\tau \neq \emptyset$. A comparison with the miquelian plane of order q shows now that $\mathfrak{M}$ itself is miquelian.

Case 2. $p \neq 2$. In this case Λ contains a subgroup $\Lambda_0 \cong PSL_2(q)$ of index 1 or 2 (Dickson [3, §260]). If l is a circle with $l^{\Lambda_0} = l$, then we have

$l = k$; if $Q \in k$, then there is one and only one p-Sylow subgroup M_0 of Λ_0 such that $Q^{M_0} = Q$ and since two different p-Sylow subgroups of Σ intersect only in the identity, Q is the only fixed point of M_0. But M_0 must leave fixed a point on l, so Q is on l and $l = k$. Now Σ is isomorphic to the $PSL_2(q^2)$, so we have $\mathfrak{N}\Lambda_0 \cong PGL_2(q)$ (Dickson [3, §255]). Furthermore k is the only circle left fixed by Λ_0. This implies that $\Lambda = \mathfrak{N}\Lambda_0$. It follows that Σ splits the set $\mathfrak{C}$ of the circles of $\mathfrak{M}$ into two orbits $\mathfrak{C}_1$ and $\mathfrak{C}_2$. If $\mathfrak{M}_i$ $(i = 1, 2)$ is the incidence structure consisting of the points of $\mathfrak{M}$ and the circles of $\mathfrak{C}_i$, then Σ is transitive on the incident point-circle pairs of $\mathfrak{M}_i$. Let P be a point of $\mathfrak{M}$. There exist circles k_i $(i = 1, 2)$ such that $P \in k_i$ and $k_i \in \mathfrak{C}_i$. Let H be the stabilizer of P and Λ_i the stabilizer of k_i. Since Σ is transitive on the incident point-circle pairs of both $\mathfrak{M}_1$ and $\mathfrak{M}_2$, we can represent $\mathfrak{M}$ in the following way: We define the mappings

$$Q \rightarrow H\sigma \quad \text{if and only if} \quad P^\sigma = Q,$$

$$l \rightarrow \Lambda_i\tau \quad \text{if and only if} \quad l \in C_i \quad \text{and} \quad k_i^\tau = l \ (i = 1, 2).$$

These mappings are one-to-one and onto. It follows from Higman and McLaughlin [6] that $P^\sigma \in k_i^\tau$ if and only if $H\sigma \cap \Lambda_i\tau \neq \emptyset$. A comparison with the miquelian Möbius-plane of order q shows now that $\mathfrak{M}$ itself is miquelian, Q.E.D.

Lemma. *A finite Möbius-plane $\mathfrak{M}$ of order q admits an automorphism group which is sharply doubly transitive on the points of $\mathfrak{M}$, if and only if $q = 2$.*

Proof. If $\mathfrak{M}$ is the Möbius-plane of order 2, then $\mathfrak{M}$ has 5 points and every circle carries exactly 3 points. This implies that every set of three points is a circle. It follows that the Möbius-group of $\mathfrak{M}$ is the symmetric group of degree 5 which in fact contains a sharply doubly transitive subgroup.

To prove the converse we assume that $\mathfrak{M}$ is a Möbius-plane with $q^2 + 1$ points and that Σ is an automorphism group of $\mathfrak{M}$ which is sharply doubly transitive on the points of $\mathfrak{M}$. Since Σ is sharply doubly transitive the degree $q^2 + 1$ of Σ is a power of a prime p. Now V. A. Lebesgue [10] proved that $q^2 + 1 = p^r$ implies $r = 1$, so Σ_P is cyclic. Σ_P induces a collineation group in $\mathfrak{A}(\mathfrak{M}, P)$ which is cyclic and transitive on the points of $\mathfrak{A}(\mathfrak{M}, P)$. By Hoffman [7], $q = 2$, Q.E.D.

Theorem 3. *Let $\mathfrak{M}$ be a finite Möbius-plane of order q and Σ an automorphism group of $\mathfrak{M}$ which is Zassenhaus transitive on the points of $\mathfrak{M}$. Then, if q is odd, $\mathfrak{M}$ is miquelian and if q is even, then $\mathfrak{M}$ is either miquelian or a Suzuki-plane.*

Proof. If Σ contains a normal subgroup of order $q^2 + 1$, then it follows from Feit [4, Lemma 4.1] that Σ contains a subgroup which is sharply doubly transitive on the points of $\mathfrak{M}$. It follows from our lemma that $\mathfrak{M}$ is the miquelian plane of order 2. Therefore we can assume that Σ does not contain such a normal subgroup. This implies (Suzuki [11], Feit [4] and Ito [9]) that either $\Sigma \cong S(2^{2r+1})$ with $q = 2^{2r+1}$ or Σ contains a subgroup $\Sigma_0 \cong PSL_2(p^{2r})$ with $q = p^r$. Theorem 3 follows now from Theorems 1 and 2.

THEOREM 4. *A finite Möbius-plane $\mathfrak{M}$ is miquelian if and only if $\mathfrak{M}$ admits an automorphism group which is transitive on the incident point-circle pairs and such that only the identity leaves three distinct points fixed.*

Proof. If $\mathfrak{M}$ is a miquelian Möbius-plane, then $\mathfrak{M}$ has a sharply triply transitive automorphism group Σ (see e.g. Benz [2, §4.6]). It is obvious that Σ is transitive on the incident point-circle pairs and that only the identity leaves three distinct points fixed. To prove the converse assume that $\mathfrak{M}$ is a finite Möbius-plane and Σ an automorphism group of $\mathfrak{M}$ which satisfies the requirements of the theorem. Σ_P induces a collineation group of $\mathfrak{A}(\mathfrak{M}, P)$. Now $o(\Sigma_P) = q(q + 1)s$, since Σ_P is transitive on the circles through P. This implies that 2 is a divisor of $o(\Sigma_P)$. Therefore there is a nontrivial involution σ in Σ_P.

Case 1. q is even. In this case P is the only fixed point of σ, since only the identity fixes three different points. Therefore σ induces a translation in $\mathfrak{A}(\mathfrak{M}, P)$ Since Σ_P is transitive on the lines of $\mathfrak{A}(\mathfrak{M}, P)$ it follows from Gleason [5, Lemma 1.6] that $\mathfrak{A}(\mathfrak{M}, P)$ is a translation plane and that Σ_P contains the translation group of $\mathfrak{A}(\mathfrak{M}, P)$. This implies that Σ is Zassenhaus transitive on the points of $\mathfrak{M}$. It follows from Theorem 3 that $\mathfrak{M}$ is either miquelian or a Suzuki-plane. Since the translation group of $\mathfrak{A}(\mathfrak{M}, P)$ is a 2-Sylow subgroup of Σ and since it is elementary abelian, Σ cannot be the group $S(q)$ (Suzuki [11, Theorem 6 and Lemma 1]). Hence $\mathfrak{M}$ is miquelian.

Case 2. $q - 1$ is even. Then σ is a homology of $\mathfrak{A}(\mathfrak{M}, P)$. But it is obvious that σ is not the only involutory homology of $\mathfrak{A}(\mathfrak{M}, P)$ in Σ_P. This implies that there is a nontrivial translation of $\mathfrak{A}(\mathfrak{M}, P)$ in Σ_P and we deduce as above that $\mathfrak{A}(\mathfrak{M}, P)$ is a translation plane and that Σ_P contains the translation group of $\mathfrak{A}(\mathfrak{M}, P)$. It follows that Σ is Zassenhaus transitive on the points of $\mathfrak{M}$. Hence $\mathfrak{M}$ is miquelian by Theorem 3.

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