THE LIE ALGEBRAS WITH A NONDEGENERATE TRACE FORM

BY

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1. Introduction

A trace form on a Lie algebra L is a bilinear form f on L for which there is a representation Δ of L, of finite degree, such that

$$f(a, b) = \mathrm{tr}(\Delta a \Delta b) \qquad (a, b \in L).$$

In this paper we shall show that if L is a Lie algebra over an algebraically closed field F of characteristic $p > 3$, and if L has a nondegenerate trace form, then L is a direct sum of Lie algebras which are either one-dimensional, isomorphic to a total matrix algebra $M_n(F)$ with n a multiple of p, or simple of classical type but not of type PA. The simple algebras of classical type were classified by Mills and Seligman in [3]; they are the analogues (as described, for example, in [4]) over F of the complex simple algebras (including the five exceptional algebras). Included among the simple algebras of classical type are the algebras of *type* PA, a Lie algebra L over F being said to be of type PA if for some multiple n of p, $L \cong PSM_n(F)$, the Lie algebra of all $n \times n$ matrices of trace 0, modulo scalar matrices.

Conversely, it is known that all of the direct summands mentioned above do have a nondegenerate trace form, with the possible exception of the algebra of type E_8 when $p = 5$.

It is to be hoped that the results of this paper will be applicable to the theory of finite groups. For the relationship to that subject, see [6]. Actually, for this application, one is concerned with the case in which the base field F is finite. Theorem 5.1 below gives the structure of L over finite fields; the actual classification of the algebras in this case is, however, a quite different topic.

2. Preliminaries

If f is a trace form on a Lie algebra L, we shall denote by $L^\perp$ the radical of f, that is, the set of all a in L such that $f(a, b) = 0$ for all b in L. Now $L^\perp$ is an ideal of L and f induces a bilinear form $\bar{f}$ on the quotient algebra $\bar{L} = L/L^\perp$. By a *quotient trace form* on a Lie algebra $\bar{L}$ is meant any bilinear form $\bar{f}$ arising in this way from a trace form f on an algebra L such that $\bar{L} = L/L^\perp$. Thus a quotient trace form is in particular a nondegenerate symmetric invariant form.

It has been shown by Block [1] that if L is a simple Lie algebra over an algebraically closed field F of characteristic $p > 3$, and if L has a quotient trace form, then L is of classical type. The algebras of type PA have a

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quotient trace form, but it was shown in [1] that they have no nondegenerate trace form. The other simple algebras of classical type over F are known to have a nondegenerate trace form, except that information is lacking about the algebra of type E_8 when $p = 5$.

In the Structure Theorem of Zassenhaus [5] it is proved that a Lie algebra L with a quotient trace form is a direct sum of mutually orthogonal ideals each of which is orthogonally indecomposable and has a quotient trace form, and the center zL of L has the same dimension as L/L^2; moreover, if the characteristic is neither 2 nor 3 and the algebra L with a quotient trace form is orthogonally indecomposable but is neither one-dimensional nor simple, then $0 \subset zL \subset L^2$ and L^2 is the sum of mutually orthogonal perfect ideals $L_1, \cdots, L_m$ of L such that there is the decomposition

$$L^2/zL = \sum_{j=1}^{m}(L_j + zL)/zL$$

of the factor algebra L^2/zL into the direct sum of the m ideals $(L_j + zL)/zL$, each of which is simple.

It is also stated in [5] that if L is a Lie algebra of characteristic $p \neq 2, 3$, and if L has a quotient trace form and is orthogonally indecomposable, then L^2 and L are indecomposable. There is a gap in [5] in the proof of the indecomposability of L^2 (and thus in the proof of the indecomposability of L, which in [5] is made to follow from that of L^2). We wish to use the fact that L^2 is indecomposable; we therefore close the gap in [5] here with the following proofs.

LEMMA 2.1. *Let L be a finite-dimensional Lie algebra over a field F of characteristic $\chi \neq 2$, with a nondegenerate symmetric invariant bilinear form f. Then L is a direct sum of mutually orthogonal ideals $L_1, \cdots, L_n$, each of which is indecomposable.*

Proof. If L is abelian, then, since $\chi \neq 2$, there is an element x in L such that $f(x, x) \neq 0$, so that L is orthogonally decomposable into the direct sum of the ideal Fx and the orthogonal complement $(Fx)^\perp$ of Fx; applying induction to $(Fx)^\perp$, we see that the lemma holds in this case. Now suppose that L is not abelian; then there is a decomposition of L into the direct sum of the ideals L_1 and R, where L_1 is nonabelian and indecomposable. Now $f(L_1^2, R) = f(L_1, L_1R) = 0$, so that $L_1^2 \cap L_1^\perp \subseteq (L_1 + R)^\perp = 0$. Since

$$L_1 \cap L_1^\perp \cap L_1^2 = L_1^\perp \cap L_1^2 = 0,$$

there is a linear subspace X of L_1 containing L_1^2 and complementary to $L_1 \cap L_1^\perp$ in L_1. Since $L_1^2 \subseteq X$, X is an ideal of L_1. Therefore L_1 is the direct sum of the ideals X and $L_1 \cap L_1^\perp$. Since L_1 is nonabelian, $X \neq 0$; since L_1 is indecomposable, it follows that $L_1 \cap L_1^\perp = 0$. Hence L is the direct sum of L_1 and $L_1^\perp$, and the lemma follows by induction on the dimension.

With the result of Lemma 2.1, the proof given in [5, p. 71] of the following statement is now essentially correct but needs clarification, which we provide here.

LEMMA 2.2. *Let L be an indecomposable Lie algebra, over a field F of characteristic distinct from 2 and 3, with a quotient trace form f. Then L^2 is indecomposable.*

Proof. We may assume that F is infinite by using the reduction[2] to this case given in [5]. Let H be a Cartan subalgebra of L. Then H is abelian by Lemma 4 of [5] (see Lemma 3.1 of [1]). Suppose that there is a decomposition of L^2 into the direct sum of two proper ideals A and B of L^2. Then $L^2 \neq L$; also L^2 is perfect and $zL \subseteq L^2$ (see [5]). Since L^2 is perfect, A and B are perfect and hence are orthogonal ideals of L. The adjoint mapping induces a representation of L on A; let

$$A = A_0 + \sum_{\alpha \neq 0} A_\alpha$$

be the decomposition into weight spaces (with respect to H) for this representation. Each weight α is a root of L and $A_\alpha \subseteq L_\alpha$; hence $f(A_\alpha, H) = 0$ for all nonzero α. Now

$$A + zL = (A_0 + zL) + \sum_{\alpha \neq 0} A_\alpha,$$

where $A_0 + zL \subseteq H$ and $\sum_{\alpha \neq 0} A_\alpha \subseteq H^\perp$. Since f induces a nondegenerate bilinear form on H, we have $(A_0 + zL) \cap H^\perp = 0$,

$$(A + zL) \cap H^\perp \subseteq \sum_{\alpha \neq 0} A_\alpha \subseteq A.$$

But $zL = (L^2)^\perp$, so that

$$(A^\perp \cap L^2)^\perp \cap H^\perp = (A + zL) \cap H^\perp \subseteq A,$$

$$(A^\perp \cap L^2) + H \supseteq A^\perp,$$

and similarly with A replaced by B. Hence there are linear subspaces H_1 and H_2 of H such that, as linear spaces, $A^\perp$ and $B^\perp$ have the decompositions

$$A^\perp = H_1 \dot{+} (A^\perp \cap L^2), \qquad B^\perp = H_2 \dot{+} (B^\perp \cap L^2).$$

Since $A \cap B = 0$, we have $A^\perp + B^\perp = L$. But $L^2 = A + B$, so that

$$L = H_1 + H_2 + L^2 = H_1 + B + H_2 + A.$$

Also $A^\perp \cap B^\perp = zL = (A^\perp \cap L^2) \cap (B^\perp \cap L^2)$ and $(A^\perp \cap L^2) + (B^\perp \cap L^2) = L^2$, so that, writing d for dimension, we have

$$\mathrm{d}H_1 + \mathrm{d}H_2 = \mathrm{d}A^\perp + \mathrm{d}B^\perp - \mathrm{d}(A^\perp \cap L^2) - \mathrm{d}(B^\perp \cap L^2) = \mathrm{d}L - \mathrm{d}(L^2).$$

It follows that L, as a linear space, is the direct sum $L = H_1 \dot{+} B \dot{+} H_2 \dot{+} A$. Also $f(AA^\perp, L) \subseteq f(A, A^\perp) = 0$, so that $AA^\perp = 0$ and $AH_1 = 0$; similarly $BH_2 = 0$. Since H is abelian, it now follows that L is the direct sum of the

[2] This is needed (if one is interested in the nonalgebraically closed case) because it is not known whether Lie algebras over a finite field necessarily possess a Cartan subalgebra. There are two misprints in the argument in [5, pp. 71–72]: in the two places where it occurs, the index s should be replaced by t.

ideals $H_1 + B$ and $H_2 + A$. This contradicts the indecomposability of L, and the lemma is proved.

3. The irreducible representations of a direct sum of Lie algebras

LEMMA 3.1. *Let L_1 and L_2 be Lie algebras over an algebraically closed field F, and let Γ be an irreducible representation of the direct sum $L = L_1 \dotplus L_2$. Then Γ is equivalent to the Lie-Kronecker sum $\Gamma_1 \otimes \Gamma_2$ of two irreducible representations Γ_1 and Γ_2 of L such that*

$$\Gamma_1(L_2) = \Gamma_2(L_1) = 0. \tag{3.1}$$

Proof. Let $\mathfrak{A}_i$ be the enveloping associative algebra (including the identity I) of $\Gamma(L_i)$ $(i = 1, 2)$. Then $\mathfrak{A}_1$ commutes elementwise with $\mathfrak{A}_2$, so that an irreducible representation Φ of the Kronecker product algebra $\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2$ over F is obtained by setting $\Phi(A_1 \otimes A_2) = A_1 A_2$ $(A_i \in \mathfrak{A}_i, i = 1, 2)$. It then follows that there are irreducible representations Φ_1 of $\mathfrak{A}_1$ and Φ_2 of $\mathfrak{A}_2$ such that Φ is the Kronecker product $\Phi_1 \otimes \Phi_2$—indeed, since $\mathfrak{A}$ has a unit and Φ is irreducible, Φ is equivalent to a constituent of the regular representation of $\mathfrak{A}$ and hence to a constituent of the Kronecker product of the regular representations of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. But upon taking Kronecker products of the terms in reductions of representations of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, one obtains a reduction of the Kronecker product representation. Hence Φ is equivalent to a constituent of $\Theta = \Theta_1 \otimes \Theta_2$, where Θ_i is an irreducible constituent of the regular representation of $\mathfrak{A}_i$ $(i = 1, 2)$; since F is algebraically closed, by a theorem of Burnside Θ is irreducible and hence equivalent to Φ. Now Φ_1 and Φ_2 induce irreducible representations Γ_1 and Γ_2 of L satisfying (3.1), and Γ is their Lie-Kronecker sum, since for a_1 in L_1 and a_2 in L_2,

$$\begin{aligned}\Gamma(a_1 + a_2) &= (\Gamma a_1)I + I(\Gamma a_2) = \Phi(\Gamma a_1 \otimes I + I \otimes \Gamma a_2) \\ &= \Phi_1(\Gamma a_1) \otimes \Phi_2 I + \Phi_1 I \otimes \Phi_2(\Gamma a_2) = (\Gamma_1 a_1) \otimes I_2 + I_1 \otimes (\Gamma_2 a_2).\end{aligned}$$

This proves the lemma.

If the representation Γ is assumed to be centrally irreducible, the above result holds over any field F (see [7, I, p. 173; II, p. 67]); however in proving our structure results we shall need to use it only in the algebraically closed case.

4. Structure of the derived algebra

The following key lemma is given in more generality than is needed in this paper, in order that it may be of use also in the determination of the algebras with a quotient trace form.

LEMMA 4.1. *Let L be a Lie algebra of characteristic distinct from 2 and 3, with a trace form f such that $L^\perp \subseteq zL$. Denote the quotient algebra $L/L^\perp$ by $\bar{L}$, and suppose that $\bar{L}^2 = \bar{L}_1 + \bar{L}_2$, where $\bar{L}_1$ and $\bar{L}_2$ are orthogonal perfect* *ideals of* $\bar{L}$. *Let* L_i *be the inverse image of* $\bar{L}_i$ *under the natural mapping of* L *onto* $\bar{L}$ $(i = 1, 2)$. *Then* $L_1 L_2 = 0$ *and* $L_1^2 \cap L_2^2 \subseteq L^{\perp}$.

Proof. By extending the base field, we may assume without loss of generality that it is algebraically closed. Since $\bar{L}_i$ is perfect,

$$L_i = L_i^2 + L^{\perp} \qquad (i = 1, 2).$$

It follows that L_i^2 is perfect $(i = 1, 2)$; indeed, since $L^{\perp} \subseteq zL$, we have $L_i^2 = (L_i^2 + L^{\perp})^2 = (L_i^2)^2$. Since $\bar{L}_1$ and $\bar{L}_2$ are orthogonal, $f(L_1 L_2, L) \subseteq f(L_1, L_2) = 0$, that is, $L_1 L_2 \subseteq L^{\perp} \subseteq zL$. Therefore by the Jacobi identity,

$$0 = (L_1 L_1)L_2 = (L_1^2 + L^{\perp})L_2 = L_1 L_2,$$

which proves the first statement of the conclusion.

Now suppose that $L_1^2 \cap L_2^2 \not\subseteq L^{\perp}$. Then we may take elements c and d in L such that

$$c \in L_1^2 \cap L_2^2, \qquad f(c, d) \neq 0.$$

Then for some irreducible constituent, say Δ, of the representation of L of which f is the trace form, we have

$$\operatorname{tr}(\Delta c \Delta d) \neq 0. \tag{4.1}$$

Since $L_1^2 + L_2^2$ is an ideal of L, by Lemma 1 of [5] the irreducible constituents of the restriction of Δ to $L_1^2 + L_2^2$ are all equivalent, say to Θ. Now form the direct sum of L_1^2 and L_2^2, and denote it by M. Since $L_1 L_2 = 0$, a representation Γ of M is obtained by setting $\Gamma(a_1, a_2) = \Theta a_1 + \Theta a_2$ (where (a_1, a_2) denotes the element of M with component a_i in L_i^2). Clearly Γ is irreducible. Therefore, by Lemma 3.1, Γ is equivalent to $\Gamma_1 \otimes \Gamma_2$ for some irreducible representations Γ_1 and Γ_2 of M such that $\Gamma_1(0, a_2) = \Gamma_2(a_1, 0) = 0$ for all a_i in L_i^2 $(i = 1, 2)$. Denote the degree of Γ_i by r_i.

Now suppose that p does not divide r_1. Note that $L_1 \cap L_2 \subseteq z(L_1 + L_2)$ since $L_1 L_2 = 0$. If $e \in L_1^2 \cap L_2^2$ then $(e, 0) \in zM$, and since M is perfect, $\Gamma_1(e, 0)$ is a scalar matrix of trace 0, and hence $\Gamma_1(e, 0) = 0$. Therefore

$$\Theta e = \Gamma(e, 0) = \Gamma_1(e, 0) \otimes I_{r_2} + I_{r_1} \otimes \Gamma_2(e, 0) = 0,$$

so that Δe is nilpotent. Thus the restriction of the irreducible representation Δ to the ideal $L_1^2 \cap L_2^2$ is a nilrepresentation, so that by Lemma 2 of [5], it is a null representation of $L_1^2 \cap L_2^2$ and therefore $\Delta c = 0$. This contradicts (4.1).

Therefore $p \mid r_1$. Similarly $p \mid r_2$. Note that for any a and b in M, since M is perfect, $\operatorname{tr}(\Gamma_1 a)\operatorname{tr}(\Gamma_2 b) = 0$. Thus for any a and b in M,

$$\begin{aligned}\operatorname{tr}(\Gamma a \Gamma b) &= \operatorname{tr}((\Gamma_1 a \otimes I_{r_2} + I_{r_1} \otimes \Gamma_2 a)(\Gamma_1 b \otimes I_{r_2} + I_{r_1} \otimes \Gamma_2 b)) \\ &= r_2 \operatorname{tr}(\Gamma_1 a \Gamma_1 b) + \operatorname{tr}\Gamma_1 a \operatorname{tr}\Gamma_2 b + \operatorname{tr}\Gamma_1 b \operatorname{tr}\Gamma_2 a + r_1 \operatorname{tr}(\Gamma_2 a \Gamma_2 b) = 0.\end{aligned}$$

It follows that the trace form f_Θ *of* Θ vanishes identically. Therefore f_Δ vanishes identically on $L_1^2 + L_2^2$. Since $L_1^2 + L_2^2$ is a perfect ideal of L, we have

$$f_\Delta(L_1^2 + L_2^2, L) = f_\Delta(L_1^2 + L_2^2, (L_1^2 + L_2^2)L) = 0.$$

This contradicts (4.1), and the lemma is proved.

5. The determination of L

THEOREM 5.1. *Let L be a Lie algebra of characteristic distinct from 2 and 3, with a nondegenerate trace form. Then L is the direct sum of indecomposable mutually orthogonal ideals $L_1, \cdots, L_r$, where each L_i is either one-dimensional, simple, or such that $0 \subset zL_i \subset L_i^2$, $zL_i \cong L_i/L_i^2$, L_i^2 is perfect, and L_i^2/zL_i is simple.*

Proof. By Lemma 2.1, L is the direct sum of indecomposable mutually orthogonal ideals, each with a nondegenerate trace form, so that we may assume without loss of generality that L itself is indecomposable. We may also suppose that L is neither abelian nor simple. Now by Theorem 3 of [5], we have $0 \subset zL \subset L^2$, $zL \cong L/L^2$, and L^2 is the sum of mutually orthogonal perfect ideals $L_1, \cdots, L_m$ of L such that $(L_j + zL)/zL$ is simple for $j = 1, \cdots, m$. The only thing remaining to be proved is that $m = 1$. If $m > 1$ then we may take L_1 and $L_2 + \cdots + L_m$ in the roles of $\bar{L}_1$ and $\bar{L}_2$, respectively, in Lemma 4.1. We conclude that

$$L_1 \cap \left(\textstyle\sum_{j=2}^m L_j\right) = L_1^2 \cap \left(\textstyle\sum_{j=2}^m L_j\right)^2 \subseteq L^\perp = 0,$$

that is, L_1 is a direct summand of L^2. But by Lemma 2.2, L^2 is indecomposable, a contradiction. Hence $m = 1$, and the theorem is proved.

THEOREM 5.2. *Let L be a Lie algebra over an algebraically closed field F of characteristic $p > 3$, with a nondegenerate trace form. Then L is the direct sum of mutually orthogonal ideals which are either one-dimensional, simple of classical type but not of type PA, or isomorphic to the total matrix algebra $M_n(F)$ for some multiple n of p.*

Proof. We may assume that L itself is one of the indecomposable direct summands described in Theorem 5.1, and that L is not one-dimensional. If L is simple then by [1], L is of classical type but not of type PA. Now we may suppose that $0 \subset zL \subset L^2$, $zL \cong L/L^2$, L^2 is perfect and L^2/zL is simple. Since $(L^2)^\perp = zL$, we see that L^2/zL has a quotient trace form and thus by [1] is simple of classical type.

It is proved in [2] (and also in [8]) that if $\pi : M \to \bar{M}$ is a homomorphism of a perfect Lie algebra M over F onto a simple Lie algebra $\bar{M}$ of classical type, with kernel K contained in zM, then either $K = 0$ or else for some multiple n of p, $\bar{M} \cong PSM_n(F)$ and $M \cong SM_n(F)$, the algebra of all $n \times n$ matrices of trace 0. In the present case, with $M = L^2$ and $K = zL$, it follows that $L^2 \cong SM_n(F)$, where $p \mid n$. Now since L/L^2 is one-dimensional, L is spanned by L^2 and some element x. Note that ad x does not induce an inner derivation of L^2, since otherwise L^2 would be a direct summand of L. But every derivation of $SM_n(F)$ is induced by an element of $M_n(F)$. It follows that $L \cong M_n(F)$, and the theorem is proved.

References

1. R. Block, *Trace forms on Lie algebras*, Canad. J. Math., vol. 14 (1962), pp. 553–564.
2. ———, *On the extensions of Lie algebras*, to appear.
3. W. H. Mills and G. B. Seligman, *Lie algebras of classical type*, J. Math. Mech., vol. 6 (1957), pp. 519–548.
4. R. Steinberg, *Automorphisms of classical Lie algebras*, Pacific J. Math., vol. 11 (1961), pp. 1119–1131.
5. H. Zassenhaus, *On trace bilinear forms on Lie algebras*, Proc. Glascow Math. Assoc., vol. 14 (1959), pp. 62–72.
6. ———, *On an application of the theory of Lie algebras to group theory*, Proceedings of Symposia in Pure Mathematics, vol. 1, Amer. Math. Soc., 1959, pp. 105–108.
7. ———, *Representation theory*, mimeographed notes, California Institute of Technology, 1959, I, 203 pp.; II, 103 pp.
8. R. Steinberg, *Générateurs, relations et revêtements des groupes algébriques*, Colloque sur la Theorie des Groupes Algébriques, tenu à Bruxelles, Juin 1962, Centre Belge de Recherches Math., pp. 113–127.

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