ON THE VANISHING OF TOR IN REGULAR LOCAL RINGS

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Introduction

The object of this paper is to provide a proof of the following conjecture of Auslander [1, p. 636]: Let C be a regular local ring, M and N C-modules of finite type. Then $\mathrm{Tor}_i^C(M, N) = 0$ implies $\mathrm{Tor}_j^C(M, N) = 0$ for $j \geq i$. This was previously known only when C is an unramified or equicharacteristic local ring. The proof uses two theorems of some independent interest, concerning the non-negativity of higher Euler characteristics.

The following notations and conventions are used throughout: If A is a ring, and M an A-module, $l(M)$ denotes the length of M. If M and N are two A-modules, we define

$$\chi_j^A(M, N) = \textstyle\sum_{i \geq j} (-1)^{i-j} l(\mathrm{Tor}_i^A(M, N)).$$

The use of these notations presupposes, in the first case, that M is an A-module of finite length, and in the second case, that $\mathrm{Tor}_i^A(M, N)$ is a module of finite length for $i \geq j$ and is 0 for i sufficiently large.

If A is a noetherian ring, and M an A-module of finite type, Supp M (the support of M) is the (closed) subset of Spec A consisting of all prime ideals p of A such that $M_p \neq 0$. The dimension of M (dim M) is the dimension of the noetherian topological space Supp M. We make the convention that dim $(0) = -1$. $\hat{\mathrm{T}}\mathrm{or}(M, N)$ denotes the "complete Tor." See [3 Chapter V] for details.

Statement and Proof of results

LEMMA 1. *Let A be a noetherian local ring with maximal ideal m. Let $x_1, x_2, \cdots, x_d$ be an A-sequence contained in m generating an ideal I. Let M be an A-module of finite type. Assume that M/IM is an A-module of finite length. Then $\mathrm{Tor}_i^A(A/I, M)$ is an A-module of finite length for $i \geq 1$ which is zero for i large, and $\chi_0^A(A/I, M) \geq 0$, with the equality holding iff* dim $M < d$.

Proof. Since Supp $(\mathrm{Tor}_i^A(A/I, M))$ is included in Supp $(M/IM) = \{m\}$, it is clear that the $\mathrm{Tor}_i^A(A/I, M)$ have finite length. The resolution of A/I by the Koszul complex with respect to $x_1, \cdots, x_d$ shows that A/I has finite homological dimension. The rest of the proof proceeds by induction on d. If $d = 0$, the statement is obvious. So assume $d \geq 1$, and let $B = A/x_1$, let $x = x_1$. Then we have the spectral sequence

$$\mathrm{Tor}_p^B(A/I, \mathrm{Tor}_q^A(B, M)) \Rightarrow \mathrm{Tor}_{p+q}^A(A/I, M).$$

Since B has homological dimension 1 as an A-module, the spectral sequence degenerates into an exact sequence:

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$$\cdots \to \operatorname{Tor}_{j-1}^{B}(A/I, \operatorname{Tor}_1^A(B, M)) \to \operatorname{Tor}_j^A(A/I, M)$$
$$\to \operatorname{Tor}_j^B(A/I, B \otimes_A M) \to \cdots \to A/I \otimes_B \operatorname{Tor}_1^A(B, M)$$
$$\to \operatorname{Tor}_1^A(A/I, M) \to \operatorname{Tor}_1^B(A/I, B \otimes_A M) \to 0.$$

It is easy to see that all the modules in this sequence are modules of finite length, and that

$$(*) \quad \chi_0^A(A/I, M) = \chi_0^B(A/I, B \otimes_A M) - \chi_0^B(A/I, \operatorname{Tor}_1^A(B, M)).$$

Now, $B \otimes_A M$ is just M/xM. Multiplication by x induces a surjective map

$$\varphi_r : x^r M/x^{r+1}M \to x^{r+1}M/x^{r+2}M$$

for all r. Let $K_r = \operatorname{Ker} \varphi_r$. Since M is a noetherian module, K_r is zero for r large. Since x is a non-zero-divisor in A, $\operatorname{Tor}_1^A(B, M)$ is just the set of elements of M killed by x. An immediate calculation then shows that

$$K_r = (\operatorname{Tor}_1^A(B, M) \cap x^r M)/\operatorname{Tor}_1^A(B, M) \cap x^{r+1}M.$$

Let $N = x^n M/x^{n+1}M$, where n is large enough so that φ_r is an isomorphism for $r \geq n$. Then one sees easily by the additivity of the Euler characteristic that

$$(**) \quad \chi_0^B(A/I, N) = \chi_0^B(A/I, B \otimes_A M) - \chi_0^B(A/I, \operatorname{Tor}_1^A(B, M)).$$

Hence, putting $(*)$ and $(**)$ together, we see that $\chi_0^A(A/I, M) \geq 0$ by the induction hypothesis.

Knowing this, we may reduce the last statement of the lemma to the case when $M = A/P$, P a prime ideal in A, by replacing M by a composition series for M whose factors are of the form A/P. Now we have two cases: either x is a non-zero-divisor on A/P, or x kills A/P. In the first case, $\operatorname{Tor}_1^A(B, M) = 0$ and $\dim M \otimes_A B = \dim M - 1$, so $\chi_0^A(A/I, M) = \chi_0^B(A/I, B \otimes_A M)$, and we are done by induction. In the second case, we have on the one hand $M \simeq B \otimes_A M \simeq \operatorname{Tor}_1^A(B, M)$ so $\chi_0^A(A/I, M) = 0$, and on the other hand $M/(x_2 \cdots x_d)M$ has finite length, so $\dim M \leq d - 1$.

THEOREM 1. *Let A be a noetherian local ring with maximal ideal m, M an A-module of finite type. Let $x_1 \cdots x_d$ be an A-sequence contained in m. Assume that there exists a j such that $\operatorname{Tor}_j^A(A/I, M)$ is an A-module of finite length, where $I = (x_1 \cdots x_d)$. Then $\operatorname{Tor}_k^A(A/I, M)$ is an A-module of finite length for $k \geq j$, and $\chi_j^A(A/I, M) \geq 0$. If $j \geq 1$ and $\chi_j = 0$, then $\operatorname{Tor}_j^A(A/I, M) = 0$.*

Proof. The case when $j = 0$ has been proved in Lemma 1, and by replacing M by a suitable module of syzygies, we may reduce the case $j \geq 1$ to the case $j = 1$. We prove this by induction on d. If $d = 0$, it is obvious. If $d \geq 1$, we let $x = x_1$, $B = A/x$, and consider the spectral sequence degenerating into an exact sequence which we used in Lemma 1. First we note that

$$l(\operatorname{Tor}_1^A(A/I, M)) < \infty$$
$$\Rightarrow \quad l(\operatorname{Tor}_1^B(A/I, B \otimes_A M)) < \infty$$

$\Rightarrow$ [by the induction hypothesis) $l(\mathrm{Tor}_k^B(A/I, B \otimes_A M)) < \infty$ for $k \geq 2$.

$\Rightarrow$ (by the exact sequence) $l(A/I \otimes_B \mathrm{Tor}_1^A(B, M)) < \infty$

$\Rightarrow$ (Lemma 1) $l(\mathrm{Tor}_k^B(A/I, \mathrm{Tor}_1^A(B, M))) < \infty$ for $k \geq 1$.

$\Rightarrow$ (by the exact sequence) $l(\mathrm{Tor}_k^A(A/I, M)) < \infty$ for $k \geq 1$.

Hence $\chi_1^A(A/I, M) = \chi_1^B(A/I, B \otimes_A M) + \chi_0^B(A/I, \mathrm{Tor}_1^A(B, M))$. So by induction and Lemma 1, $\chi_1^A(A/I, M) \geq 0$. If $\chi_1^A(A/I, M) = 0$, we must have

$$\chi_1^B(A/I, B \otimes_A M) = 0 \quad \text{and} \quad \chi_0^B(A/I, \mathrm{Tor}_1^A(B, M)) = 0,$$

hence $\mathrm{Tor}_1^B(A/I, B \otimes_A M) = 0$ and $\dim \mathrm{Tor}_1^A(B, M) \leq d - 2$. I claim that this implies that $\mathrm{Tor}_1^A(B, M) = 0$.

Let K be $\mathrm{Tor}_1^A(B, M)$, so that we have the exact sequence

$$0 \to K \to M \to xM \to M/xM \to 0.$$

Let $M_1 = M/K$. Let $K_1 = \mathrm{Tor}_1^A(B, M_1)$. Now M/xM has depth $\geq d - 1$, since

$$\mathrm{Tor}_1^B(B/(x_2 \cdots x_d), M/xM) = 0$$

and $x_2 \cdots x_d$ form a B-sequence. I claim that the hypotheses imply that M_1/xM_1 is isomorphic to M/xM and K_1 is isomorphic to K. We have the exact sequence

$$0 \to \mathrm{Tor}_1^A(B, K) \to \mathrm{Tor}_1^A(B, M) \to \mathrm{Tor}_1^A(B, M_1)$$
$$\to K/xK \to M/xM \to M_1/xM_1 \to 0$$

whose last four terms reduce to

$$0 \to K_1 \to K \to M/xM \to M_1/xM_1 \to 0.$$

Since depth $M/xM \geq d - 1$, all the associated primes of M/xM have dimension $\geq d - 1$. [3, IV, p. 14]. Since K has dimension $\leq d - 2$, the map of K to M/xM must be the zero map, so we are done. Letting

$$M_n = M_{n-1}/K_{n-1} \quad \text{and} \quad K_n = \mathrm{Tor}_1^A(B, M_n)$$

we find by induction that $K_n \simeq K$. Hence if $K \neq 0$, the M_n form an infinite strictly decreasing sequence of quotient modules of M, which is impossible since M is noetherian. So $\mathrm{Tor}_1^A(B, M) = 0$. It now follows that $\mathrm{Tor}_1^A(A/I, M) = 0$, and we have completed the proof.

THEOREM 2. *Let B be a complete unramified regular local ring and let M and N be B-modules of finite type.* (1) *If* $\mathrm{Tor}_i^B(M, N) = 0$ *then* $\mathrm{Tor}_j^B(M, N) = 0$ *for $j \geq i$.* (2) *If* $\mathrm{Tor}_i^B(M, N)$ *has finite length, then* $\mathrm{Tor}_j^B(M, N)$ *has finite length for $j \geq i$ and $\chi_i^B(M, N) \geq 0$.* (3) *If $i \geq 2$ and $\chi_i^B(M, N) = 0$ then* $\mathrm{Tor}_i^B(M, N) = 0$. (4) *If $\chi_1^B(M, N) = 0$ and either M or N is a torsion-free B-module, then* $\mathrm{Tor}_1^B(M, N) = 0$.

Proof. By the Cohen structure-theory for such rings, (see [2] for example). B is isomorphic to $R[[T_1 \cdots T_n]]$ where R is a discrete valuation ring or a field and the T_i's are indeterminates. Let $A = B \hat{\otimes}_R B$. Then Serre shows in [3], that there exists a spectral sequence

$$\mathrm{Tor}_p^A (B, \hat{\mathrm{Tor}}_q^R (M, N)) \Rightarrow \mathrm{Tor}_{p+q}^B (M, N).$$

Since the homological dimension of R is ≤ 1, the spectral sequence degenerates into an exact sequence

$$\cdots \to \mathrm{Tor}_{j-1}^A (B, \hat{\mathrm{Tor}}_1^R (M, N)) \to \mathrm{Tor}_j^B (M, N)$$
$$\to \mathrm{Tor}_j^A (B, M \hat{\otimes}_R N) \to \cdots \to B \otimes_A \hat{\mathrm{Tor}}_1^R (M, N)$$
$$\to \mathrm{Tor}_1^B (M, N) \to \mathrm{Tor}_1^A (B, M \hat{\otimes}_R N) \to 0.$$

Since A and B are regular, we see that the ideal I in A defining B is generated by an A-sequence of length n.

If $\mathrm{Tor}_1^B (M, N) = 0$, then $\mathrm{Tor}_1^A (B, M \hat{\otimes}_R N) = 0$, hence by Theorem 1, $\mathrm{Tor}_2^A (B, M \hat{\otimes}_R N) = 0$, hence $B \otimes_A \hat{\mathrm{Tor}}_1^R (M, N) = 0$, hence $\hat{\mathrm{Tor}}_1^R (M, N) = 0$, hence $\mathrm{Tor}_1^A (B, \hat{\mathrm{Tor}}_1^R (M, N)) = 0$, hence $\mathrm{Tor}_2^B (M, N) = 0$. The case of general i and j immediately reduces to $j = i + 1$, and then to $i = 1$, by replacing M by a suitable module of syzygies. So (1) is proved.

If $i = 0$, (2) was proved by Serre [3, V, p. 16]. So as above we may assume that $i = 1$. The result about finite lengths follows exactly as above, using Theorem 1. From the exact sequence, we obtain

$$\chi_1^B(M, N) = \chi_1^A(B, M \hat{\otimes}_R N) + \chi_0^A(B, \hat{\mathrm{Tor}}_1^R (M, N)) \geq 0.$$

By replacing M by a suitable module of syzygies, (3) follows from (4). However, if M is torsion-free, $\hat{\mathrm{Tor}}_1^R (M, N) = 0$ [3, V, p. 9], and

$$\mathrm{Tor}_i^B (M, N) \simeq \mathrm{Tor}_i^A (B, M \hat{\otimes}_R N),$$

and the result follows from Theorem 1.

THEOREM 3. *Let C be a local ring which is the quotient of an unramified (or equicharacteristic) regular local ring B by a non-zero element x. Let M and N be C-modules of finite type such that $\mathrm{Tor}_n^C (M, N) = 0$ for large n. Then if $\mathrm{Tor}_i^C (M, N) = 0$, $\mathrm{Tor}_j^C (M, N) = 0$ for $j \geq i$.*

Proof. We prove the theorem by induction on $\dim C$. (The proof of the induction step will give the result when $\dim C = 0$ as a special case.) So we assume that the theorem is true for rings C^1 with $\dim C^1 < \dim C$.

First, we have the spectral sequence:

$$\mathrm{Tor}_p^C (M, \mathrm{Tor}_q^B (N, C)) \Rightarrow \mathrm{Tor}_{p+q}^B (M, N).$$

Since $C \simeq B/xB$, x is a non-zero divisor, and N is a C-module, we have $\mathrm{Tor}_q^B (N, C) = 0$ for $q \geq 2$, and $\mathrm{Tor}_1^B (N, C) \simeq N \otimes_B C \simeq N$. Thus the spectral sequence degenerates into an exact sequence:

$$\cdots \to \mathrm{Tor}_{j-1}^{C}(M, N) \to \mathrm{Tor}_{j}^{B}(M, N) \to \mathrm{Tor}_{j}^{C}(M, N)$$
$$\to \cdots M \otimes_C N \to \mathrm{Tor}_{1}^{B}(M, N) \to \mathrm{Tor}_{1}^{C}(M, N) \to 0.$$

It is sufficient to prove that $\mathrm{Tor}_1^C(M, N) = 0$ implies $\mathrm{Tor}_2^C(M, N) = 0$, so assume $\mathrm{Tor}_1^C(M, N) = 0$. Let P be a prime ideal of $C \neq m =$ the maximal ideal of C. Then $C_P \simeq B_Q/xB_Q$, where Q is the inverse image of P in B and $\dim C_P < \dim C$. Since a ring of quotients of an unramified regular local ring with respect to a prime ideal is an unramified regular local ring [2, p. 99], the theorem is true for C_P. Let $C_P = D$. Then

$$\mathrm{Tor}_1^D(M_P, N_P) \simeq \mathrm{Tor}_1^C(M, N)_P = 0.$$

Hence by the induction hypothesis, $\mathrm{Tor}_j^D(M_P, N_P) = 0$ for $i \geq 1$ or $\mathrm{Tor}_j^C(M, N)_P = 0$ for $j \geq 1$. Hence $\mathrm{Tor}_j^C(M, N)$ has finite length for $j \leq 1$.

By the exact sequence, $\mathrm{Tor}_j^B(M, N)$ has finite length for $j \geq 2$. Let

$$\varphi : \mathrm{Tor}_2^C(M, N) \to M \otimes_C N.$$

From the exact sequence, we get $\chi_2^B(M, N) + l(\mathrm{Im}\,\varphi) = 0$. (Since $\mathrm{Tor}_1^C(M, N) = 0$.) But $\chi_2^B(M, N) = \chi_2^D(\hat{M}, \hat{N}) \geq 0$ by Theorem 2, so we have $\chi_2^B(M, N) = 0$ and φ is the zero map, i.e. the map of $\mathrm{Tor}_2^B(M, N)$ to $\mathrm{Tor}_2^C(M, N)$ is surjective. But by Theorem 2, $\mathrm{Tor}_2^D(\hat{M}, \hat{N}) = 0$, hence $\mathrm{Tor}_2^B(M, N) = 0$, hence $\mathrm{Tor}_2^C(M, N) = 0$.

Corollary 1. *Let C be an arbitrary regular local ring, M and N C-modules of finite type. Then if* $\mathrm{Tor}_i^C(M, N) = 0$, $\mathrm{Tor}_j^C(M, N) = 0$ *for $j \geq i$.*

Proof. We may clearly assume C complete. Since a regular local ring has finite homological dimension, and any complete regular local ring is the quotient of an unramified regular local ring by a non-zero element, C, M, N satisfy the hypotheses of Theorem 3.

We now obtain the following corollaries, all of which were proven in the case of unramified regular local rings by Auslander in [1], and which only depend on knowing that $\mathrm{Tor}_i^A(M, N) = 0 \Rightarrow \mathrm{Tor}_j^A(M, N) = 0$ for $j \geq i$.

Corollary 2. *Let A be a regular local ring, and let M and N be non-zero A-modules such that $M \otimes_A N$ is torsion-free. Then*

(a) *M and N are torsion free,*
(b) $\mathrm{Tor}_i^A(M, N) = 0$ *for all $i > 0$,*
(c) $hd(M) + hd(N) = hd(M \otimes_A N) < \dim A$.

Proof. See [1, pp. 636–637], especially the remark at the bottom of page 637.

Corollary 3. *Let A be a regular local ring of dimension $n > 0$. An A-module M is free iff the n-fold tensor product of M is torsion-free.*

COROLLARY 4. *Let A be a regular local ring, M an A-module; then we can have that*

(a) if $M \otimes M$ *is torsion-free, then M is reflexive*;
(b) *if $M \otimes M \otimes M^*$ is torsion-free and $M^* \neq 0$, then M is free.*

COROLLARY 5. *Let A be a regular local ring of dimension $n > 0$, and M an A-module satisfying the following conditions*:

(a) $hdM = hdM^*$.
(b) $M \otimes M^*$ *is torsion-free.*
(c) M_p *is A_p-free for each nonmaximal prime ideal p of A.*
Then $hd(M) = 0$ or $(n - 1)/2$.
Therefore if n is even, M must be free. If n is odd, then there are modules M satisfying (a), (b), *and* (c) *and such that $hd(M) = (n - 1)/2$.*

These corollaries are all consequences of Corollary 1. For proofs see [1, pp. 638–643].

COROLLARY 6. *Let A be a regular local ring, M an A-module of depth* 0, *and N an A-module. Then*

$$\operatorname{Tor}_i^A(M, N) = 0 \quad \Rightarrow \quad hd(N) < i.$$

Proof. It clearly suffices to prove that $\operatorname{Tor}_1^A(M, N) = 0 \Rightarrow N$ is free. Let d be the homological dimension of N, and assume $d > 0$. Then

$$\operatorname{Tor}_d^A(K, N) \neq 0 \quad \text{and} \quad \operatorname{Tor}_{d+1}^A(K, N) = 0,$$

where K is the residue field of A. Since M has depth zero, there is an exact sequence

$$0 \to K \to M \to Q \to 0.$$

Since $hd(N) = d$, we have

$$0 = \operatorname{Tor}_{d+1}^A(Q, N) \to \operatorname{Tor}_d^A(K, N) \to \operatorname{Tor}_d^A(M, N).$$

Since $d > 0$, $\operatorname{Tor}_d^A(M, N) = 0$ by Corollary 1, so $\operatorname{Tor}_d^A(K, N) = 0$, which is a contradiction. (This proof was told to me by M. Auslander.)

Open questions

The most important open question in this area is of course the "Serre Conjecture":

1. Let A be a ramified regular local ring, M and N A-modules such that $M \otimes_A N$ has finite length. Then $\chi_0^A(M, N) \geq 0$, with equality holding if and only if $\dim M + \dim N = \dim A$.

We then have the questions related to the higher Euler characteristics:

2. If A is a regular local ring, M and N are A-modules such that

$\operatorname{Tor}_j^A(M, N)$ has finite length, then $\chi_j^A(M, N) \geq 0$, with equality iff $\operatorname{Tor}_j^A(M, N) = 0$.

Theorem 2 answers this in the affirmative if A is unramified and either $j \geq 2$ or one of M and N is torsion-free, and it is well-known to be true if A is equicharacteristic. For ramified rings we know nothing.

3. For what local rings and what modules is Theorem 3 true?

If $A = K[[x, y]]/(xy, x^2)$, $M = A/xA$, $N = A/yA$, then $\operatorname{Tor}_1^A(M, N) = 0$ but $\operatorname{Tor}_j^A(M, N) \neq 0$ for $j \geq 2$, so the answer is not "all rings and all modules." But we know of no counter-example where $\operatorname{Tor}_j^A(M, N)$ is zero for large j. There are obviously many possible conjectures.

4. For a given local ring A, which modules M have the property that $\operatorname{Tor}_1^A(M, N) = 0$ implies that N is free?

Corollary 6 shows that for A regular, this is true if and only if depth $M = 0$. If A is not regular, then M must in addition (at least) not have finite homological dimension.

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