RELATIVE HOMOLOGICAL ALGEBRA AND ABELIAN GROUPS

BY
CAROL PEERCY WALKER[1]

Introduction

Relative homological algebra has risen from two principal sources: Hochschild's study of exact sequences of R-modules which split as S-modules, where S is a subring of R [10], and Harrison's exploitation of the homological aspects of pure exact extensions of Abelian groups [8]. In each case one deals with a special class of exact sequences in an Abelian category and with the corresponding Ext functors. Possible axioms for these special classes of exact sequences have been put forth by Heller [9], Buchsbaum [2] and Yoneda [17]. The axioms formulated by Buchsbaum will be taken as a basis for relative homological algebra considerations in this paper with some of the terminology borrowed from [16]. In particular, if $\mathcal{E}$ is a class of short exact sequences of an Abelian category $\mathcal{C}$, then $\mathcal{E}$ is a *proper class* if and only if the class of monomorphisms

$$A \xrightarrow{f} B$$

for which the exact sequence

$$0 \to A \xrightarrow{f} B \to \operatorname{Cok} f \to 0$$

belongs to $\mathcal{E}$ form an h.f. class [2].

This paper offers two simple methods (dual to one another) for obtaining proper classes and studies the relative homological algebras that arise. Particular attention is given to the category of Abelian groups. These methods include as special cases many (though not all) of the generalizations of purity in Abelian group theory. In particular the standard notion of purity can be obtained by these methods.

Recall that a group[2] A is a *pure subgroup* of B if $A \cap nB = nA$ for all positive integers n. An equivalent statement is that A is a summand of every subgroup C of B such that $A \subset C$ and C/A is finite. This notion was generalized to arbitrary infinite cardinals $\mathfrak{m}$ by Gacsályi [7] who defined $\mathfrak{m}$-pure subgroups in terms of systems of equations, in such a way that $\aleph_0$-pure coincides with pure. It has been shown (Łoś [15]) that a subgroup A of a group B is $\mathfrak{m}$-pure in B if and only if A is a summand of every subgroup C of B such that $A \subset C$ and C/A is generated by a subset of cardinal less than $\mathfrak{m}$. The concept of a neat subgroup (due to Honda [11]) can be defined in an analogous fashion. Namely, A is a neat subgroup of B if and only if A is a summand of every sub-

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[2] The word *group* in this paper means *Abelian group*.

group C of B such that $A \subset C$ and C/A is elementary (i.e. a torsion group in which every element has square-free order). These examples among others lead to the following generalization of purity. Let $\mathcal{I}$ be a class of objects of an Abelian category $\mathcal{C}$, and let A be a subobject of an object B. Call A *$\mathcal{I}$-pure* in B if A is a summand of every subobject C of B such that $A \subset C$ and C/A belongs to $\mathcal{I}$. Assuming that $\mathcal{I}$ is closed under quotient objects,[3] the monomorphisms

$$A \xrightarrow{f} B$$

with $\operatorname{Im} f$ $\mathcal{I}$-pure in B form an h.f. class (Theorem 2.1). Consequently, many theorems about pure subgroups are carried over to $\mathcal{I}$-pure subobjects, including homological properties of $\operatorname{Pext}_{\mathcal{I}}(C, A)$, the collection of equivalence classes of $\mathcal{I}$-pure extensions of A by C analogous to those of the functor Pext (C, A). The general properties of $\mathcal{I}$-purity are discussed in Section II.

The properties of pure subgroups which hold in the more setting of $\mathcal{I}$-purity are those which depend on the nature of the quotient with respect to an $\mathcal{I}$-pure subobject and not on properties of the $\mathcal{I}$-pure subobject itself. For example the theorem that a pure bounded subgroup is a direct summand has no analogue for $\mathcal{I}$-purity. To obtain theorems of this nature a dual definition can be made. Let $\mathcal{S}$ be a class of objects of $\mathcal{C}$ and let A be a subobject of an object B. Call A an *$\mathcal{S}$-copure* subobject of B if S a subobject of A with A/S in $\mathcal{S}$ implies A/S is a summand of B/S. If $\mathcal{S}$ is closed under subobjects, the monomorphisms

$$A \xrightarrow{f} B$$

with $\operatorname{Im} f$ $\mathcal{S}$-copure in B form an h.f. class. Ordinary purity for Abelian groups is self-dual in this sense, for pure and copure with respect to the class of finite groups both yield the ordinary concept of a pure subgroup. It is not generally true however, when $\mathcal{S}$ is a class of objects closed under both quotient objects and subobjects, that $\mathcal{S}$-pure and $\mathcal{S}$-copure coincide. The properties of $\mathcal{S}$-copurity are dual to those of $\mathcal{I}$-purity, so are not stated in detail.

In Sections III and IV, purity and copurity with respect to some special classes are investigated. In particular, in Section IV the notion of $\mathfrak{m}$-purity for infinite cardinals $\mathfrak{m}$, is extended to modules. In characterizing $\mathfrak{m}$-pure projectives a result that is perhaps of interest in itself is proven, namely that a direct summand of a direct sum of modules each generated by a subset of cardinal $\leq \mathfrak{m}$ is again a direct sum of such modules. (In [13] Kaplansky proved this for $\mathfrak{m} = \aleph_0$).

I. Preliminaries

The following notations and conventions will be followed. Throughout this paper $\mathcal{C}$ denotes an Abelian category. In addition to the standard axioms for an Abelian category the following two axioms are assumed.

[3] This condition is actually more restrictive than necessary, as explained in the remark following Theorem 2.7.

(0) For each object A there is a set of maps, each monic with range A, which contains a representative of every subobject of A and dually, for quotient objects of A.

(00) For each pair of objects B, A and each $n \geq 1$ there is a set of n-fold exact sequences from A to B containing a representative of every equivalence class of such sequences.

(The reader is referred to [16] for a complete exposition of the axioms for an Abelian category, including (0) and (00) above, as well as an explanation of the equivalence relation on n-fold extensions.)

A subobject of an object A is an equivalence class of monomorphisms

$$S \xrightarrow{f} A.$$

We will abuse the language and refer to the domain of a given monomorphism as a subobject of A, writing $S \subset A$. The symbol $S \subset A$ assumes implicitly a fixed monomorphism

$$S \xrightarrow{f} A,$$

and this map (called the inclusion map) will be specified when there is any cause for confusion. If $\mathcal{J}$ is a class of objects of $\mathcal{C}$ closed under isomorphic images, axiom (0) allows us to speak of the set $\{A_\alpha\}_{\alpha \epsilon J}$ of subobjects of an object A which belong to $\mathcal{J}$, meaning of course that

$$\{A_\alpha \xrightarrow{f_\alpha} A\}_{\alpha \epsilon J}$$

is a set of monomorphisms containing a representative from each relevant equivalence class. When it is assumed that $\mathcal{C}$ has infinite direct sums, it will suffice that each set of objects $\{A_\alpha\}_{\alpha \epsilon K}$ which is a subset of the set of subobjects of some object A have a direct sum.

If A and C are subobjects of B, with inclusion maps

$$A \xrightarrow{f} B \quad \text{and} \quad C \xrightarrow{g} B,$$

B/A denotes a cokernel of f, $A \cap C$ a kernel of the map

$$B \xrightarrow{\text{Cok } f \times \text{Cok } g} B/A \oplus B/C,$$

and $A + C$ an image of the map

$$A \oplus C \xrightarrow{f + g} B.$$

If S is a subobject of B/A, one can write $S = T/A$ where T is a kernel of the composition

$$B \to B/A \to (B/A)/S.$$

When Ext (B, A) appears without a subscript or superscript, $\text{Ext}^1_{\mathcal{C}}(B, A)$ is always implied. Similarly $\text{Pext}_{\mathcal{J}}(B, A) = \text{Pext}^1_{\mathcal{J}}(B, A)$ and $\text{Copext}_{\mathcal{S}}(B, A) = \text{Copext}^1_{\mathcal{S}}(B, A)$. Also Hom (B, A) is used for $\text{Map}_{\mathcal{C}}(B, A)$.

A pertinent foundational question has not been resolved, namely whether

$\mathrm{Pext}^n_{\mathcal{J}}(B, A)$ is a set for $n > 1$. In lieu of an answer one may follow the precedent set by others (e.g. [2], [3] and [17]) and set the question aside, or restrict consideration by one means or another to situations in which $\mathrm{Pext}^n_{\mathcal{J}}(B, A)$ is a set. In most of this paper it is assumed that the category $\mathcal{C}$ has enough projectives. It turns out then that $\mathrm{Pext}_{\mathcal{J}}$ also has enough projectives (Theorem 2.4), so in this case $\mathrm{Pext}^n_{\mathcal{J}}(B, A)$ is a set. Similar remarks apply to the other relative Ext functors which occur.

II. $\mathcal{J}$-pure extensions

Throughout this section $\mathcal{J}$ is a class of objects in $\mathcal{C}$ which is closed under quotient objects (i.e.

$$I \xrightarrow{f} J$$

an epimorphism and $I \in \mathcal{J}$ implies $J \in \mathcal{J}$). The duals of all statements in this section are valid with "$\mathcal{J}$-pure" replaced by "$\mathcal{S}$-copure" where $\mathcal{S}$ is a class of objects which is closed under subobjects.

Definition. A subobject A of an object B is *$\mathcal{J}$-pure* in B if A is a direct summand of every subobject C of B such that $A \subset C$ and $C/A \in \mathcal{J}$. An exact sequence

$$\cdots \longrightarrow E_{k+1} \xrightarrow{f_k} E_k \xrightarrow{f_{k-1}} E_{k-1} \longrightarrow \cdots$$

is $\mathcal{J}$-pure if $\operatorname{Ker} f_{i-1}$ is $\mathcal{J}$-pure in E_i for each i.

The first theorem shows that the class of $\mathcal{J}$-pure short exact sequences satisfies the axioms of a proper class.

2.1 Theorem. *Let $A \subset B \subset C$ be objects of $\mathcal{C}$. The following hold.*

(i) *If the sequence $0 \to A \to B \to B/A \to 0$ is $\mathcal{J}$-pure then any equivalent sequence is $\mathcal{J}$-pure.*

(ii) *If A is a direct summand of B then A is $\mathcal{J}$-pure in B.*

(iii) *If A is $\mathcal{J}$-pure in C then A is $\mathcal{J}$-pure in B.*

(iv) *If B is $\mathcal{J}$-pure in C, then B/A is $\mathcal{J}$-pure in C/A.*

(v) *If A is $\mathcal{J}$-pure in B and B is $\mathcal{J}$-pure in C, then A is $\mathcal{J}$-pure in C.*

(vi) *If A is $\mathcal{J}$-pure in C and B/A is $\mathcal{J}$-pure in C/A then B is $\mathcal{J}$-pure in C.*

Proof. Statements (i), (ii) and (iii) are immediate. To prove (iv), assume B is $\mathcal{J}$-pure in C, and suppose $B/A \subset I \subset C/A$ with $I/(B/A)$ in $\mathcal{J}$. Then $I = J/A$ with $B \subset J \subset C$, and $J/B \approx (J/A)/(B/A) = I/(B/A)$ in $\mathcal{J}$ implies J/B is in $\mathcal{J}$. Thus B is a summand of J, say $J = B \oplus R$, and $I = (B/A) \oplus ((R \oplus A)/A)$. Hence B/A is $\mathcal{J}$-pure in C/A.

Assume A is $\mathcal{J}$-pure in B and B is $\mathcal{J}$-pure in C, and suppose $A \subset S \subset C$ with $S/A \in \mathcal{J}$. The natural epimorphism $S/A \to (S + A)/B$ implies $(S + B)/B$ is in $\mathcal{J}$, so B is a summand of $S + B$, say $S + B = B \oplus R$. Let

$$S \xrightarrow{i} B \oplus R \quad \text{and} \quad B \oplus R \xrightarrow{p} B$$

be the inclusion and projection maps, respectively, and let E be an image of *pi*. The epimorphism $S/A \to E/A$ induced by *pi* implies E/A is in $\mathcal{J}$, and hence $E = A \oplus T$ for some $T \subset E$. Now

$$S \subset E \oplus R = A \oplus T \oplus R \quad \text{implies} \quad S = A \oplus (S \cap (T \oplus R)),$$

and A is a summand of S. Hence A is $\mathcal{J}$-pure in C.

Assume A is $\mathcal{J}$-pure in C and B/A is $\mathcal{J}$-pure in C/A, and suppose $B \subset S \subset C$ with S/B in $\mathcal{J}$. Then $(S/A)/(B/A)$ is in $\mathcal{J}$ and hence B/A is a summand of S/A, say $S/A = (B/A) \oplus (R/A)$. Now $R/A \approx S/B$ implies R/A is in $\mathcal{J}$ and hence A is a summand of R, say $R = A \oplus T$. Then

$$S/A = (B/A) \oplus ((A \oplus T)/A) \quad \text{implies} \quad S = B \oplus T.$$

Thus B is $\mathcal{J}$-pure in C.

The theorem above has important homological implications. For A, B in $\mathcal{C}$ let $\mathrm{Pext}_{\mathcal{J}}^{n}(B, A)$ denote the set of equivalence classes of $\mathcal{J}$-pure n-fold extensions of A by B. The reader is referred to [16] for a proof of the following theorem for proper classes.

2.2 Theorem. *For each $n \geq 1$, $\mathrm{Pext}_{\mathcal{J}}^{n}(B, A)$ is a bifunctor on $\mathcal{C}$ to Abelian groups. If*

$$0 \to A \to B \to C \to 0$$

is an $\mathcal{J}$-pure exact sequence and D is any object of $\mathcal{C}$, there are exact sequences

$$\begin{aligned} 0 &\to \mathrm{Hom}(D, A) \to \mathrm{Hom}(D, B) \to \mathrm{Hom}(D, C) \\ &\to \mathrm{Pext}_{\mathcal{J}}^{1}(D, A) \to \mathrm{Pext}_{\mathcal{J}}^{1}(D, B) \to \cdots \to \mathrm{Pext}_{\mathcal{J}}^{n}(D, C) \\ &\to \mathrm{Pext}_{\mathcal{J}}^{n+1}(D, A) \to \mathrm{Pext}_{\mathcal{J}}^{n+1}(D, B) \to \mathrm{Pext}_{\mathcal{J}}^{n+1}(D, C) \to \cdots \end{aligned}$$

and

$$\begin{aligned} 0 &\to \mathrm{Hom}(C, D) \to \mathrm{Hom}(B, D) \to \mathrm{Hom}(A, D) \\ &\to \mathrm{Pext}_{\mathcal{J}}^{1}(C, D) \to \mathrm{Pext}_{\mathcal{J}}^{1}(B, D) \to \cdots \to \mathrm{Pext}_{\mathcal{J}}^{n}(A, D) \\ &\to \mathrm{Pext}_{\mathcal{J}}^{n+1}(C, D) \to \mathrm{Pext}_{\mathcal{J}}^{n+1}(B, D) \to \mathrm{Pext}_{\mathcal{J}}^{n+1}(A, D) \to \cdots \end{aligned}$$

with the maps given by composition.

In the remainder of this section it is assumed that the category $\mathcal{C}$ has infinite direct sums and enough projectives (i.e. for each $A \in \mathcal{C}$ there exists an epimorphism $F \to A$ with F projective).

Definition. An object P is *$\mathcal{J}$-pure-projective* if for every $\mathcal{J}$-pure exact sequence

$$0 \to A \to B \to C \to 0$$

the induced sequence

$$0 \to \mathrm{Hom}(P, A) \to \mathrm{Hom}(P, B) \to \mathrm{Hom}(P, C) \to 0$$

is exact, or equivalently, if $\mathrm{Pext}_{\mathcal{J}}(P, K) = 0$ for all objects K. An $\mathcal{J}$-pure-injective is defined dually. Let Cl $\mathcal{G}$ denote the class of $\mathcal{G}$-pure-projectives. The class Cl $\mathcal{G}$ is the *projective closure* of $\mathcal{G}$.

The following proposition describes a basic class of $\mathcal{G}$-pure projectives.

2.3 PROPOSITION. *If $P = \sum_\alpha S_\alpha$ with each S_α either projective or a member of $\mathcal{G}$, then P is $\mathcal{G}$-pure-projective.*

Proof. Let

$$0 \to A \to B \xrightarrow{f} C \to 0$$

be an $\mathcal{G}$-pure exact sequence. This sequence induces a commutative diagram

$$\begin{array}{ccc} \text{Hom}\,(\sum_\alpha S_\alpha\,, B) & \to & \text{Hom}\,(\sum_\alpha S_\alpha\,, C) \\ \downarrow & & \downarrow \\ \prod_\alpha \text{Hom}\,(S_\alpha\,, B) & \to & \prod_\alpha \text{Hom}\,(S_\alpha\,, C) \end{array}$$

with the vertical maps isomorphisms, and the lower horizontal map the product of the maps

$$\text{Hom}\,(S_\alpha\,, B) \xrightarrow{f_\alpha} \text{Hom}\,(S_\alpha\,, C)$$

where $f_\alpha = \text{Hom}\,(S_\alpha\,, f)$. Thus it suffices to show that the maps f_α are each epimorphisms. If S_α is projective this is clear. Suppose $S_\alpha \in \mathcal{G}$ and let $g \in \text{Hom}\,(S_\alpha\,,\ C)$. Let B_α be the inverse image in B of $C_\alpha = g(S_\alpha)$, i.e. $B = \text{Ker}\,(B \to C \to C/C_\alpha)$. Then the sequence

$$0 \to A \to B_\alpha \xrightarrow{f} {}'C_\alpha \to 0$$

is $\mathcal{G}$-pure exact and $C_\alpha = \text{Im}\, g$ is in $\mathcal{G}$, since $\mathcal{G}$ is closed under homomorphic images. Thus there exists a map $h : C_\alpha \to B_\alpha$ such that $fh = 1_C$. Now $fhg = g = f_\alpha(hg)$. Thus f_α is an epimorphism.

2.4 THEOREM. *If A is any object of $\mathcal{C}$, there exists an epimorphism $P \to A$ with P $\mathcal{G}$-pure-projective and* Ker f *$\mathcal{G}$-pure in P. The $\mathcal{G}$-pure projective P may be chosen to be a direct sum $F \oplus S$, and F projective and S a direct sum of members of $\mathcal{G}$.*

Proof. Let

$$S_\alpha \xrightarrow{g_\alpha} A \quad (\alpha \in I)$$

be the set of subobjects of A which belong to $\mathcal{G}$, and let

$$F \xrightarrow{f} A$$

be an epimorphism with F projective. These objects and maps yield an epimorphism

$$F \oplus \textstyle\sum_{\alpha \in I} S_\alpha \xrightarrow{f + \Sigma g_\alpha} A$$

with $P = F \oplus \sum_{\alpha \in I} S_\alpha$ $\mathcal{G}$-pure-projective. Let

$$K \xrightarrow{k} P$$

be a kernel of $h = f + \sum g_\alpha$, and suppose S is a subobject of P with S/K in $\mathcal{G}$.

Let

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d, "k'"'] & P \\ S \arrow[ur, "g"'] & \end{tikzcd}$$

be the diagram of monomorphisms. Now $h(S) \approx S/K$ in $\mathcal{S}$ implies $h(S)$ is equivalent, as a subobject of A, to S_β for some $\beta \in I$, and there is a map

$$S \xrightarrow{h'} S_\beta$$

with the diagram

$$\begin{tikzcd} S \arrow[r, "h'"] \arrow[d, "g"] & S_\beta \arrow[d, "g_\beta"] \\ P \arrow[r, "h"] & A \end{tikzcd}$$

commutative. Let

$$S_\beta \xrightarrow{i_\beta} F \oplus \sum S_\alpha = P$$

be the injection map. Then $g - i_\beta h'$ is a map from S to P, and

$$h(g - i_\beta h') = hg - hi_\beta h' = hg - g_\beta h' = 0.$$

Hence since K is a kernel of h, there is a map

$$S \xrightarrow{t} K$$

such that $kt = g - i_\beta h'$. Now

$$ktk' = (g - i_\beta h')k' = gk' - i_\beta h'k' = k - i_\beta 0 = k,$$

and since k is monic, this implies $tk' = 1_K$ and K is a summand of S. Hence K is $\mathcal{S}$-pure in P.

Let $\mathcal{P}$ denote the class of projective objects of $\mathcal{C}$, and let $\mathcal{P}(\mathcal{S})$ (resp. $\mathcal{S}_\Sigma$) denote the class of objects of $\mathcal{C}$ which can be expressed as a direct summand of a direct sum of members of $\mathcal{S} \cup \mathcal{P}$ resp. $\mathcal{S}$). The classes $\mathcal{S}_\Sigma$ and $\mathcal{P}(\mathcal{S})$ are clearly closed under direct sums and summands.

2.5 THEOREM. *An object P is $\mathcal{S}$-pure-projective if and only if P belongs to* $\mathcal{P}(\mathcal{S})$.

Proof. Suppose P is $\mathcal{S}$-pure-projective. By Theorem 2.4, there exists an epimorphism

$$F \oplus S \xrightarrow{f} P$$

with $F \in \mathcal{P}$, $S \in \mathcal{S}_\Sigma$ and $\operatorname{Ker} f$ $\mathcal{S}$-pure in $F \oplus S$. Since P is $\mathcal{S}$-pure-projective, this implies P is isomorphic to a summand of $F \oplus S$, and hence P belongs to $\mathcal{P}(\mathcal{S})$. The converse follows from Proposition 2.3.

If $\mathcal{C}$ has global dimension 1, the $\mathcal{S}$-pure-projectives can be described in terms of $\mathcal{P}$ and $\mathcal{S}_\Sigma$.

2.6 THEOREM. *Suppose $\mathcal{C}$ has a global dimension* 1. *Then P is $\mathcal{S}$-pure-projective if and only if $P = F \oplus S$ with $F \in \mathcal{P}$ and $S \in \mathcal{S}_\Sigma$.*

Proof. From Theorem 2.5, one may assume that $P \oplus Q = F' \oplus S'$ with $F' \in \mathcal{P}$ and $S' \in \mathcal{S}_\Sigma$. The composition

$$P \xrightarrow{i} F' \oplus S' \xrightarrow{p} F',$$

where i is the inclusion map and p the projection, has kernel $S = P \cap S'$ and image $P' \subset F'$. Since $\mathcal{C}$ has dimension 1, P' is projective so that $P = S \oplus F$ with $F \approx P'$ in $\mathcal{P}$. Now

$$F' \oplus S' = P \oplus Q = S \oplus F \oplus Q \quad \text{implies} \quad S' = S \oplus (S' \cap (F \oplus Q)),$$

so that S is in $\mathcal{S}_\Sigma$. The converse follows from Proposition 2.3.

The converse of Theorem 2.5 is also true, i.e. the $\mathcal{S}$-pure short exact sequences are the largest class of short exact sequences for which $\mathcal{P}(\mathcal{S})$ is a projective class, or in the terminology of [9], the class of $\mathcal{S}$-pure short exact sequences is projectively closed.

2.7 THEOREM. *A short exact sequence*

$$0 \to A \to B \to C \to 0$$

is $\mathcal{S}$-pure if and only if the sequence

$$0 \to \operatorname{Hom}(P, A) \to \operatorname{Hom}(P, B) \to \operatorname{Hom}(P, C) \to 0$$

is exact for all P in $\mathcal{P}(\mathcal{S})$.

Proof. This follows from Theorems 2.4 and 2.5 of this paper and Proposition 5.7 of [9].

The results of this section can be interpreted from a somewhat different point of view, as a result of Theorems 2.5 and 2.7. Given a class $\mathcal{J}$ of objects of $\mathcal{C}$, let $\mathcal{E}(\mathcal{J})$ denote the class of short exact sequences

$$E : 0 \to A \to B \to C \to 0$$

for which the induced sequence

$$(*) \qquad 0 \to \operatorname{Hom}(J, A) \to \operatorname{Hom}(J, B) \to \operatorname{Hom}(J, C) \to 0$$

is exact for all J in $\mathcal{J}$. It is generally known that $\mathcal{E}(\mathcal{J})$ is a proper class. Recall that the projective closure $\operatorname{Cl} \mathcal{J}$ of $\mathcal{J}$ is the class of objects J for which $(*)$ is exact for all E in $\mathcal{E}(\mathcal{J})$. There is no reasonable characterization of the classes $\mathcal{J}$ which are projectively closed. The preceding theorems provide a sufficient condition, for categories $\mathcal{C}$ which have enough projectives and infinite direct sums. Namely, if $\mathcal{J}$ contains a subclass $\mathcal{S}$ which is closed under quotient objects and $\mathcal{J} = \mathcal{P}(\mathcal{S})$ then $\mathcal{J}$ is projectively closed (i.e. $\mathcal{J} = \operatorname{Cl} \mathcal{J}$), and moreover, $\mathcal{J}$ provides enough relative projectives for the proper class $\mathcal{E}(\mathcal{J})$. It is clear that the results of this section are valid with $\mathcal{S}$ replaced by any class $\mathcal{J}$ with $\mathcal{S} \subset \mathcal{J} \subset \mathcal{P}(\mathcal{S})$.

The group $\operatorname{Pext}_{\mathcal{S}}(B, A)$ may be described as a subgroup of $\operatorname{Ext}(B, A)$ in the following sense.

2.8 THEOREM. *Let A and B be objects of $\mathfrak{C}$ and let $\{B_\alpha\}_{\alpha\in I}$ be the set of subobjects of B which belong to $\mathcal{J}$. Let*

$$\operatorname{Ext}(B/B_\alpha, A) \xrightarrow{f_\alpha} \operatorname{Ext}(B, A)$$

be the homomorphism induced by the projection $B \to B/B_\alpha$. Then

$$\operatorname{Pext}_{\mathcal{J}}(B, A) = \textstyle\bigcap_{\alpha\in I} \operatorname{Im} f_\alpha .$$

Proof. For each $\alpha \in I$ the exact sequence

$$0 \to B_\alpha \to B \to B/B_\alpha \to 0$$

induces an exact sequence

$$\operatorname{Ext}(B/B_\alpha, A) \xrightarrow{f_\alpha} \operatorname{Ext}(B, A) \xrightarrow{g_\alpha} \operatorname{Ext}(B_\alpha, A).$$

Suppose an exact sequence

$$(1) \qquad 0 \to A \to G \to B \to 0$$

represents an element of $\operatorname{Ext}(B, A)$. Let S be a subobject of G with $A \subset S$ and $S/A \in \mathcal{J}$. Then $S/A \approx B_\alpha$ in a natural way for some $\alpha \in I$. Under g_α, the element represented by (1) maps onto the element represented by

$$(2) \qquad 0 \to A \to S \to B_\alpha \to 0$$

which splits if and only if (1) is in the image of f_α. But (1) represents an element of $\operatorname{Pext}_{\mathcal{J}}(B, A)$ if and only if for every $S \subset G$ such that $A \subset S$ and $S/A \in \mathcal{J}$, the sequence

$$(3) \qquad 0 \to A \to S \to S/A \to 0$$

splits. But (3) is equivalent to (2). Thus (1) represents an element of $\operatorname{Pext}_{\mathcal{J}}(B, A)$ if and only if (1) represents an element of $\bigcap_\alpha \operatorname{Im} f_\alpha$, and hence $\operatorname{Pext}_{\mathcal{J}}(B, A) = \bigcap_\alpha \operatorname{Im} f_\alpha$.

2.9 COROLLARY. *Let $\{B_\alpha\}_{\alpha\in I}$ be the set of subobjects of B which belong to $\mathcal{J}$, let $B_{\mathcal{J}} \to B$ be an image of the map $\sum_{\alpha\in I} B_\alpha \to B$, and let C be any subobject f B such that $B_{\mathcal{J}} \subset C$. Then for any object A the image of the map*

$$\operatorname{Ext}(B/C, A) \to \operatorname{Ext}(B, A)$$

induced by $B \to B/C$, is a subgroup of $\operatorname{Pext}_{\mathcal{J}}(B, A)$. *If $B_{\mathcal{J}} \in \mathcal{J}$ then* $\operatorname{Pext}_{\mathcal{J}}(B, A)$ *is the image of the map*

$$\operatorname{Ext}(B/B_{\mathcal{J}}, A) \to \operatorname{Ext}(B, A)$$

induced by $B \to B/B_{\mathcal{J}}$.

Proof. If $B_\alpha \subset B$ with $B_\alpha \in \mathcal{J}$ then the inclusion $B_\alpha \to C$ induces a map $B/B_\alpha \to B/C$. The corresponding diagram

$$\begin{tikzcd}
\operatorname{Ext}(B/B_\alpha, A) \arrow[r, "f_\alpha"] & \operatorname{Ext}(B, A) \\
\operatorname{Ext}(B/C, A) \arrow[u] \arrow[ur, "f"] &
\end{tikzcd}$$

is commutative for all α, implying that $\operatorname{Im} f_\alpha \subset \bigcap_{\alpha \in I} \operatorname{Im} f = \operatorname{Pext}_{\mathcal{G}}(B, A)$. If $B_{\mathcal{G}}$ is in $\mathcal{G}$, then letting $C = B_{\mathcal{G}}$ in the diagram above, $f = f_\alpha$ for some $\alpha \in I$ and $\operatorname{Im} f = \bigcap_{\alpha \in I} \operatorname{Im} f_\alpha = \operatorname{Pext}_{\mathcal{G}}(B, A)$.

The Ext functor corresponding to the dual concept of $\mathcal{S}$-copurity will be denoted by $\operatorname{Copext}^n_{\mathcal{S}}(B, A)$. The statements of 2.8 and 2.9 dualize as follows.

2.8′ THEOREM. *Let $\{A_\alpha\}_{\alpha \in J}$ be the set of subobjects of an object A such that A/A_α is in $\mathcal{S}$. Let*

$$\operatorname{Ext}(B, A_\alpha) \xrightarrow{g_\alpha} \operatorname{Ext}(B, A)$$

be the map induced by the inclusion $A_\alpha \to A$. Then

$$\operatorname{Copext}_{\mathcal{S}}(B, A) = \bigcap_{\alpha \in J} \operatorname{Im} g_\alpha .$$

2.9′ COROLLARY. *Let $\{A_\alpha\}_{\alpha \in J}$ be the set of subobjects of an object A such that A/A_α is in $\mathcal{S}$, and let $A^{\mathcal{S}} = \bigcap_{\alpha \in J} A_\alpha (= \operatorname{Ker}(A \to \prod_{\alpha \in J} A/A_\alpha))$. Let $C \subset A^{\mathcal{S}}$. Then for any object B the image of the map*

$$\operatorname{Ext}(B, C) \to \operatorname{Ext}(B, A)$$

is a subgroup of $\operatorname{Copext}_{\mathcal{S}}(B, A)$. *If $A^{\mathcal{S}}$ belongs to $\mathcal{S}$ then* $\operatorname{Copext}_{\mathcal{S}}(B, A)$ *is the image of the map*

$$\operatorname{Ext}(B, A^{\mathcal{S}}) \to \operatorname{Ext}(B, A).$$

III. Pure and copure relative to torsion theories

Throughout this section it is assumed that $\mathcal{C}$ is a complete Abelian category, i.e. has arbitrary infinite sums and products.

DEFINITION. A *torsion theory* for the category $\mathcal{C}$ is a pair of subclasses $\mathfrak{J}$, $\mathfrak{F}$ *of* $\mathcal{C}$ such that $\mathfrak{J} \cap \mathfrak{F} = 0$ (the zero object of $\mathcal{C}$) and satisfying

(i) $\mathfrak{J}$ is closed under quotient objects.
(ii) $\mathfrak{F}$ is closed under subobjects.
(iii) For each object A there exists an exact sequence

$$0 \to T \to A \to F \to 0$$

with $T \in \mathfrak{J}$ and $F \in \mathfrak{F}$.

The properties of $\operatorname{Pext}_{\mathfrak{J}}$ and $\operatorname{Copext}_{\mathfrak{F}}$ are investigated in this section. The definition of a torsion theory and the results in the following paragraph are due to S. Dickson [4].

Let $\mathfrak{J}$, $\mathfrak{F}$ be a torsion theory for $\mathcal{C}$. The objects in $\mathfrak{J}$ will be called torsion and those of $\mathfrak{F}$ will be called torsion free. For each object A the extension (iii) is unique, and hence A has a maximum torsion subobject A_t and A/A_t is torsion free. In fact

$$A_t = \bigcap \{S \mid S \subset A, A/S \in \mathfrak{F}\}.$$

An object T belongs to $\mathfrak{J}$ if and only if $\operatorname{Hom}(T, F) = 0$ for all $F \in \mathfrak{F}$, and conversely an object F belongs to $\mathfrak{F}$ if and only if $\operatorname{Hom}(T, F) = 0$ for all $T \in \mathfrak{J}$. The class $\mathfrak{J}$ is closed under arbitrary sums and $\mathfrak{F}$ is closed under arbitrary products. For proofs of these statements and further results the reader is referred to [4].

3.1 THEOREM. *For any objects A and B the sequence*

$$0 \to \mathrm{Hom}\,(B/B_t\,,\,A) \to \mathrm{Hom}\,(B,\,A) \to \mathrm{Hom}\,(B_t\,,\,A)$$
$$\to \mathrm{Ext}\,(B/B_t\,,\,A) \to \mathrm{Pext}_{\mathfrak{I}}\,(B,\,A) \to 0$$

is exact.

Proof. This is the statement of 2.9, since B_t contains all torsion subobjects of B and B_t belongs to $\mathfrak{I}$.

3.2 PROPOSITION. *Let $F \in \mathfrak{F}$ and let A be any object. Then*

$\mathrm{Pext}_{\mathfrak{I}}\,(F,\,A) = \mathrm{Ext}\,(F,\,A)$ *and* $\mathrm{Pext}_{\mathfrak{I}}\,(A,\,F) \approx \mathrm{Ext}\,(A/A_t\,,\,F)$.

Proof. The first statement follows from the fact that F has no non-zero torsion subobjects. The second statement is obtained from the exact sequence

$$0 = \mathrm{Hom}\,(A_t\,,\,F) \to \mathrm{Ext}\,(A/A_t\,,\,F) \to \mathrm{Pext}_{\mathfrak{I}}\,(A,\,F) \to 0$$

of Theorem 3.1.

The $\mathfrak{I}$-pure-injectives are determined by $\mathfrak{F}$ as follows.

3.3 THEOREM. *An object I is $\mathfrak{I}$-pure-injective if and only if* $\mathrm{Ext}\,(F, I) = 0$ *for all $F \in \mathfrak{F}$.*

Proof. From 3.2, if I is $\mathfrak{I}$-pure-injective and $F \in \mathfrak{F}$, then $\mathrm{Ext}\,(F,\,I) = \mathrm{Pext}_{\mathfrak{I}}\,(F,\,I) = 0$. Conversely if $\mathrm{Ext}\,(F,\,I) = 0$ for all $F \in \mathfrak{F}$ and A is any object then the exact sequence

$$0 = \mathrm{Ext}\,(A/A_t\,,\,I) \to \mathrm{Pext}_{\mathfrak{I}}\,(A,\,I) \to 0$$

of 3.1 implies $\mathrm{Pext}_{\mathfrak{I}}\,(A,\,I) = 0$ so that I is $\mathfrak{I}$-pure-injective.

In the case of Abelian groups with the standard torsion theory the $\mathfrak{I}$-pure-injectives are those groups which are the direct sum of a cotorsion[4] group with a divisible group and are of considerable interest [8]. In this case $\mathrm{Pext}_{\mathfrak{I}}\,(B,\,A)$ is the divisible subgroup $\mathrm{Dext}\,(B,\,A)$ of $\mathrm{Ext}\,(B,\,A)$, investigated in [12].

It is not necessary in a torsion theory for the class $\mathfrak{I}$ of torsion objects to be closed under subobjects. In particular if A is a subobject of B then it may not be true that $A_t = A \cap B_t$ but only that $A_t \subset A \cap B_t$. However if A is $\mathfrak{I}$-pure in B this equality must hold as is revealed in the proof of the following theorem.

[4] A group G is *cotorsion* if and only if both $\mathrm{Hom}\,(Q, G) = 0$ and $\mathrm{Ext}\,(Q, G) = 0$, where Q denotes the additive group of rational numbers.

3.4 THEOREM. *An exact sequence*

$$(1) \qquad 0 \to A \to B \to C \to 0$$

is $\mathfrak{J}$*-pure if and only if the sequence of subobjects*

$$(2) \qquad 0 \to A_t \to B_t \to C_t \to 0$$

is splitting exact.

Proof. Suppose (1) is $\mathfrak{J}$-pure exact, and let $(B/A)_t = S/A$. Then $S/A \in \mathfrak{J}$ implies A is a summand of S, say $S = A \oplus T$, and $T \approx S/A$ in $\mathfrak{J}$ implies $T \subset B_t$. Since $\mathfrak{J}$ is closed under quotient objects,

$$B_t/(A \cap B_t) \approx (B_t + A)/A$$

is torsion. Hence

$$(B_t + A)/A \subset (B/A)_t = (T \oplus A)/A \subset (B_t + A)/A$$

implies

$$(B/A)_t = (B_t + A)/A.$$

Also $B_t + A = A \oplus T$ with $T \subset B_t$ implies $B_t = (B_t \cap A) \oplus T$, and then $B_t \cap A$ a quotient object of B_t implies $B_t \cap A$ is torsion and hence $B_t \cap A = A_t$. Exactness of (2) follows. Now since $T \approx C_t$ under the map $B \to C$, the sequence (2) is splitting exact.

Assume (2) is splitting exact, say $B_t = A_t \oplus T$ with $T \approx C_t$. Since the map $B_t \to C_t$ contains $A \cap B_t$ in its kernel, the exactness of (2) implies $A \cap B_t \subset A_t$ and hence $A \cap B_t = A_t$. Then $B_t + A = A \oplus T$ since

$$A \cap T = A \cap (B_t \cap T) = (A \cap B_t) \cap T = A_t \cap T = 0.$$

Also $(B/A)_t = (B_t + A)/A$. Suppose E is a subobject of B with $A \subset E$ and $E/A \in \mathfrak{J}$. Then $E/A \subset (B/A)_t = (B_t + A)/A$ so that $E \subset B_t + A = A \oplus T$, and $E = A \oplus (E \cap T)$. Thus (1) is $\mathfrak{J}$-pure.

3.5 COROLLARY. *If A is* $\mathfrak{J}$*-pure in B then*
(i) $A_t = A \cap B_t$, *and*
(ii) $(B/A)_t = (B_t + A)/A$.

Theorems 3.6 to 3.10 follow by duality and proofs are omitted.

3.6 THEOREM. *For any objects A and B the sequence*

$$0 \to \operatorname{Hom}(B, A_t) \to \operatorname{Hom}(B, A) \to \operatorname{Hom}(B, A/A_t)$$

$$\to \operatorname{Ext}(B, A_t) \to \operatorname{Copext}_{\mathfrak{F}}(B, A) \to 0$$

is exact.

3.7 PROPOSITION. *Let $T \in \mathfrak{J}$ and let B be any object. Then*

$$\operatorname{Copext}_{\mathfrak{F}}(B, T) = \operatorname{Ext}(B, T) \quad \textit{and} \quad \operatorname{Copext}_{\mathfrak{F}}(T, B) \approx \operatorname{Ext}(T, B_t).$$

3.8 THEOREM. *An object P is $\mathfrak{F}$-copure-projective if and only if* Ext $(P, T) = 0$ *for all* $T \in \mathfrak{J}$.

In the theory of Abelian groups, with $\mathfrak{J}$ the standard torsion class, the characterization of the $\mathfrak{F}$-copure-projectives is a classical problem.

The $\mathfrak{F}$-copure short exact sequences are characterized as follows.

3.9 THEOREM. *An exact sequence*

$$(1) \qquad 0 \to A \to B \to C \to 0$$

is $\mathfrak{F}$-copure if and only if the sequence

$$(2) \qquad 0 \to A/A_t \to B/B_t \to C/C_t \to 0$$

is splitting exact.

3.10 COROLLARY. *If A is $\mathfrak{F}$-copure in B then*
(i) $A \cap B_t = A_t$, *and*
(ii) $(B/A)_t = (B_t + A)/A$.

In the remainder of this section it is assumed that $\mathfrak{C}$ has enough projectives and enough injectives. Then there are also enough $\mathfrak{J}$-pure-projectives and enough $\mathfrak{F}$-copure-injectives. Every $\mathfrak{J}$-pure-projective is a summand of a direct sum $T \oplus P$ with $T \in \mathfrak{J}_\Sigma = \mathfrak{J}$ and P projective. Every $\mathfrak{F}$-copure-injective is a summand of a direct sum $F \oplus I$ with $F \in \mathfrak{F}_\pi = \mathfrak{F}$ and I injective. Since each object has a maximum torsion subobject, resolutions can be obtained in a simpler fashion than in the general case.

3.11 THEOREM. *Let A be any object and let*

$$P \xrightarrow{f} A$$

be an epimorphism with P projective. Then $P \oplus A_t$ is $\mathfrak{J}$-pure-projective and the sequence

$$0 \to K \to P \oplus A_t \xrightarrow{f+i} A \to 0$$

is $\mathfrak{J}$-pure exact, where

$$A_t \xrightarrow{i} A$$

is the inclusion map and $K =$ Ker $(f + i)$. *Let B be any object and let*

$$B \xrightarrow{g} I$$

be a monomorphism with I injective. Then $I \oplus (B/B_t)$ is $\mathfrak{F}$-copure-injective and the sequence

$$0 \to B \xrightarrow{g \times j} I \oplus (B/B_t) \to C \to 0$$

is $\mathfrak{F}$-copure exact, where

$$B \xrightarrow{j} B/B_t$$

is the quotient map and $C =$ Cok $(g \times j)$.

Proof. The proof is similar to the proof of Theorem 2.4 and will be omitted.

The next theorem shows that for $\mathfrak{I}$-purity the relative homological dimension of an object A does not exceed the homological dimension of A/A_t. Hence the relative (to $\mathfrak{I}$-purity) global dimension does not exceed the global dimension of $\mathcal{C}$.

3.12 THEOREM. *Let A be an object of $\mathcal{C}$. If A/A_t has a projective resolution of length n, then A has a $\mathfrak{I}$-pure projective resolution of length n.*

Proof. If A/A_t has a projective resolution of length 0 then A/A_t is projective. Thus $A \approx A_t \oplus (A/A_t)$ is $\mathfrak{I}$-pure-projective and has a $\mathfrak{I}$-pure-projective resolution of length 0. Assume $n > 0$ and that the theorem holds for objects having a projective resolution of length $< n$. Suppose

$$0 \to P_{n-1} \xrightarrow{f_{n-1}} \cdots \to P_0 \xrightarrow{f_0} A/A_t \to 0$$

is exact with P_i projective, $i = 0, \cdots, n-1$. Since P_0 is projective and

$$A \xrightarrow{p} A/A_t$$

is an epimorphism (the natural map), there exists a map

$$P_0 \xrightarrow{g} A$$

such that $pg = f_0$. Let

$$A_t \xrightarrow{i} A$$

be the inclusion map. The sequence

$$0 \to K \xrightarrow{k} P_0 \oplus A_t \xrightarrow{g+i} A \to 0$$

may be checked to be $\mathfrak{I}$-pure exact, where

$$K \xrightarrow{k} P_0 \oplus A_t$$

is a kernel for $g + i$. Let

$$J \xrightarrow{j} P_0$$

be a kernel for f_0. Then $pgj = f_0 j = 0$ so there exists a map

$$J \xrightarrow{g'} A_t$$

such that $ig' = gj$. Now $(g + i)(j \times (-g')) = gj - ig' = 0$ so there exists a map

$$J \xrightarrow{h} K$$

such that $kh = j \times (-g')$. Let

$$P_0 \xrightarrow{i_1} P_0 \oplus A_t \xrightarrow{p_1} P_0$$

be the injection and projection maps. Now $p(g + i) = pg + pi = pg + 0 = pgp_1$, so that $0 = p(g + i)k = pgp_1k = f_0p_1k$. Hence there exists a map

$$K \xrightarrow{h'} J$$

such that $jh' = p_1 k$. Then $jh'h = p_1 kh = p_1(j \times (-g')) = j$ and j monic implies $h'h = 1_J$. Now $gp_1 k + ip_2 k = (g + i)k = 0$ so that

$ig'h' = gjh' = gp_1 k = -ip_2 k$ implies $-g'h' = p_2 k$. Then $khh' = (j \times (-g'))h' = jh' \times (-g')h' = p_1 k \times p_2 k = k$ and k monic implies $hh' = 1_K$. Thus h is an isomorphism and K has a projective resolution of length $n - 1$. By the induction hypothesis K has a $\mathfrak{Z}$-pure-projective resolution of length $n - 1$. Connecting the two sequences yields a $\mathfrak{Z}$-pure-projective resolution of A of length n.

A dual procedure yields the following.

3.13 THEOREM. *Let A be an object of $\mathcal{C}$. If A_t has an injective resolution of length n then A has an $\mathfrak{F}$-copure-injective resolution of length n.*

The statement of Theorem 3.12 is actually valid in a more general setting. Namely, if $\mathcal{I}$ is a class of objects closed under quotient objects and it happens that every object A has a maximum subobject $A_{\mathcal{I}} \in \mathcal{I}$ then the theorem will hold for $\mathcal{I}$-pure-projective resolutions. This is the case whenever $\mathcal{I} = \mathcal{I}_\Sigma$. Similarly, Theorem 3.13 is valid for $\mathcal{S}$-copure-injective resolutions when $\mathcal{S}$ is a class of objects closed under subobjects such that $\mathcal{S} = \mathcal{S}_\pi$.

The class of exact sequences which are both $\mathfrak{Z}$-pure and $\mathfrak{F}$-copure is of some interest. Since the intersection of two proper classes is a proper class, one obtains for $n \geq 1$, a bifunctor $\text{Bipext}^n_{\mathfrak{Z},\mathfrak{F}}(B, A)$ to Abelian groups (with $\text{Bipext}^1_{\mathfrak{Z},\mathfrak{F}}(B, A) = \text{Pext}^1_{\mathfrak{Z}}(B, A) \cap \text{Copext}^1_{\mathfrak{F}}(B, A))$, together with the appropriate long exact sequences. A subobject A of an object B will be called $(\mathfrak{Z}, \mathfrak{F})$-bipure in B if it is both $\mathfrak{Z}$-pure and $\mathfrak{F}$-copure in B. From Theorems 3.4 and 3.9 one obtains the following.

3.14 PROPOSITION. *A short exact sequence*

$$0 \to A \to B \to C \to 0$$

is $(\mathfrak{Z}, \mathfrak{F})$-bipure if and only if the sequences

$$0 \to A_t \to B_t \to C_t \to 0$$

$$0 \to A/A_t \to B/B_t \to C/C_t \to 0$$

are both splitting exact.

For any object A, A_t is $(\mathfrak{Z}, \mathfrak{F})$-bipure in A, leading to the following exact sequences.

3.15 THEOREM. *Let A, B be objects of $\mathcal{C}$. The following sequences are exact.*

$$0 \to \text{Hom}\,(B, A_t) \to \text{Hom}\,(B, A) \to \text{Hom}\,(B, A/A_t)$$
$$\to \text{Pext}_{\mathfrak{Z}}\,(B, A_t) \to \text{Bipexts}_{\mathfrak{Z},\mathfrak{F}}\,(B, A) \to 0$$

$$0 \to \text{Hom}\,(B/B_t, A) \to \text{Hom}\,(B, A) \to \text{Hom}\,(B_t, A)$$
$$\to \text{Copext}_{\mathfrak{F}}\,(B/B_t, A) \to \text{Bipext}_{\mathfrak{Z},\mathfrak{F}}\,(B, A) \to 0.$$

Proof. These sequences are obtained from the standard long exact sequence for $\mathrm{Bipext}_{\mathfrak{I},\mathfrak{F}}$ together with the facts that

$$\mathrm{Bipext}_{\mathfrak{I},\mathfrak{F}}(B, A_t) = \mathrm{Pext}_{\mathfrak{I}}(B, A_t),$$

$$\mathrm{Bipext}_{\mathfrak{I},\mathfrak{F}}(B/B_t, A) = \mathrm{Copext}_{\mathfrak{F}}(B/B_t, A)$$

and

$$\mathrm{Bipext}_{\mathfrak{I},\mathfrak{F}}(B, A/A_t) = 0 = \mathrm{Bipext}_{\mathfrak{I},\mathfrak{F}}(B\ , A).$$

The projectives and injectives for $\mathrm{Bipext}_{\mathfrak{I},\mathfrak{F}}$ may be described in the following sense.

3.16 Proposition. *An object P is $(\mathfrak{I}, \mathfrak{F})$-bipure-projective if and only if* $\mathrm{Pext}_{\mathfrak{I}}(P, T) = 0$ *for all $T \in \mathfrak{I}$. An object I is $(\mathfrak{I}, \mathfrak{F})$-bipure-injective if and only if* $\mathrm{Copext}_{\mathfrak{F}}(F, I) = 0$ *for all $F \in \mathfrak{F}$.*

Proof. These statements follow easily from the exact sequences of Theorem 3.15.

The following theorems show that if $\mathrm{Pext}_{\mathfrak{I}}$ has enough injectives then so does $\mathrm{Bipext}_{\mathfrak{I},\mathfrak{F}}$ and if $\mathrm{Copext}_{\mathfrak{F}}$ has enough projectives then so does $\mathrm{Bipext}_{\mathfrak{I},\mathfrak{F}}$. In the case of Abelian groups with the standard torsion theory, $\mathrm{Pext}_{\mathfrak{I}}$ has enough injectives, namely the class of groups of the form $C \oplus D$ with C cotorsion and D divisible. Applying Theorem 3.17, it can be shown that $\mathrm{Bipext}_{\mathfrak{I},\mathfrak{F}}$ has enough injectives and they are precisely the class of groups of the form $C \oplus F \oplus D$ with C cotorsion, F torsion free and D divisible. Also in this case, $\mathrm{Bipext}^n_{\mathfrak{I},\mathfrak{F}} = 0$ for $n > 1$ by 3.17.

3.17 Theorem. *For any object A, if A_t has a $\mathfrak{I}$-pure-injective resolution of length n, then A has a $(\mathfrak{I}, \mathfrak{F})$-bipure-injective resolution of length n. If* $\mathrm{Pext}_{\mathfrak{I}}$ *has enough injectives for $\mathfrak{I}$ then the following hold.*

(i) $\mathrm{Bipext}_{\mathfrak{I},\mathfrak{F}}$ *has enough injectives.*

(ii) *An object is $(\mathfrak{I}, \mathfrak{F})$-bipure-injective if and only if it is a direct summand of a sum $I \oplus F$ with I $\mathfrak{I}$-pure-injective and $F \in \mathfrak{F}$.*

Proof. Suppose

$$A_t \xrightarrow{f} I$$

is a monomorphism with A_t $\mathfrak{I}$-pure in I and I $\mathfrak{I}$-pure injective. Since A_t is $\mathfrak{I}$-pure in A there exists a map

$$A \xrightarrow{g} I$$

such that $gi = f$ (where

$$A_t \xrightarrow{i} A$$

is the inclusion map). Let

$$A \xrightarrow{p} A/A_t \quad \text{and} \quad I \xrightarrow{q} I/A_t$$

be cokernels for i and f respectively. Since $A_t \subset \mathrm{Ker}\ qg = \mathrm{Ker}\,(-qg)$, there exists a map

$$A/A_t \xrightarrow{\bar{g}} I/A_t$$

with $\bar{g}p + qg = 0$. Then the map

$$I \oplus A/A_t \xrightarrow{q+\bar{g}} I/A_t$$

is an epimorphism. Let

$$I \oplus A/A_t \xrightarrow{h} C$$

be a cokernel for $g \times p$. Now $(q + \bar{g})(g \times p) = qg + \bar{g}p = 0$, so there exists a map

$$C \xrightarrow{s} I/A_t$$

such that $sh = q + \bar{g}$. Let

$$I \xrightarrow{i_1} I \oplus A/A_t \quad \text{and} \quad A/A_t \xrightarrow{i_2} I \oplus A/A_t$$

be the injection maps. Then $0 = h(g \times p)i = h(gi \times pi) = h(f \times 0) = hi_1 f$ so there exists a map

$$I/A_t \xrightarrow{t} C$$

such that $tq = hi_1$. Now $stq = shi_1 = (q + \bar{g})i_1 = q$, with q epic, implies $st = 1_C$. Also $0 = h(g \times p) = h(i_1 g + i_2 p) = hi_1 g + hi_2 p$, so that $t\bar{g}p = -tqg = -hi_1 g = hi_2 p$, implying $t\bar{g} = hi_2$. Then $tsh = t(q + \bar{g}) = tg + t\bar{g} = hi_1 + hi_2 = h$ implies $ts = 1_{I/A_t}$. In particular s is an isomorphism and there is an exact sequence

$$0 \to A \xrightarrow{g\times p} I \oplus A/A_t \xrightarrow{q+\bar{g}} I/A_t \to 0.$$

The diagram

$$\begin{array}{ccccccc}
0 \to A_t & \xrightarrow{(g\times p)i} & (I \oplus A/A_t)_t & \xrightarrow{q+\bar{g}} & (I/A_t)_t \to 0 \\
\downarrow 1 & & \downarrow p_1 & & \downarrow 1 \\
0 \to A_t & \xrightarrow{f} & I_t & \longrightarrow & (I/A_t)_t \to 0
\end{array}$$

commutes, in fact is an equivalence. Thus A_t $\mathfrak{J}$-pure in I implies the top row is splitting exact. The diagram

$$\begin{array}{ccccc}
A/A_t & \xrightarrow{k} & (I \oplus A/A_t)/(I \oplus A/A_t)_t & \xrightarrow{k'} & (I/A_t)/(I/A_t)_t \\
\downarrow 1 & & \downarrow & & \downarrow \\
0 \to A/A_t & \xrightarrow{-\bar{g}\times 1} & I/I_t \oplus A/A_t & \xrightarrow{1+\bar{g}} & I/I_t \to 0
\end{array}$$

with k and k' the maps induced by $g \times p$ and $q + \bar{g}$ respectively and the vertical arrows the natural maps, is commutative. It is also an equivalence, since the vertical maps are isomorphisms. The bottom row splits, for the composition

$$I/I_t \xrightarrow{i_1} I/I_t \oplus A/A_t \xrightarrow{1+\bar{g}} I/I_t$$

is the identity. Thus the top row is splitting exact (with k a monomorphism and k' an epimorphism), and therefore

$$A \xrightarrow{g\times p} I \oplus A/A_t$$

is $(\mathfrak{J}, \mathfrak{F})$-bipure.

If $\mathrm{Pext}_{\mathfrak{I}}$ has enough injectives for $\mathfrak{I}$, then (i) and (ii) follow immediately from the paragraph above.

Assume A_t has a $\mathfrak{I}$-pure-injective resolution of length n. If $n = 0$, A_t is $\mathfrak{I}$-pure-injective so is a summand of A. Then $A \approx A_t \oplus A/A_t$ is $(\mathfrak{I}, \mathfrak{F})$-bipure-injective and has a resolution of length 0. Suppose $n > 0$ and the statement holds for $<n$. Let

$$0 \to A_t \xrightarrow{f_0} I_0 \to \cdots \xrightarrow{f_{n-1}} I_{n-1} \to 0$$

be a $\mathfrak{I}$-pure-injective resolution of A_t. Then f_0 can be extended to a map

$$A \xrightarrow{g} I_0,$$

and

$$A \xrightarrow{g \times p} I_0 \oplus A/A_t$$

is a $(\mathfrak{I}, \mathfrak{F})$-bipure monomorphism with $\mathrm{Cok}\,(g \times p) \approx \mathrm{Cok}\, f_0 = \mathrm{Im}\, f_1$ by the previous paragraph. But $\mathrm{Im}\, f_1$ has a $\mathfrak{I}$-pure-injective resolution of length $n - 1$ so by the assumption has a $(\mathfrak{I}, \mathfrak{F})$-bipure-injective resolution of length $n - 1$. Composing the two sequences yields a $(\mathfrak{I}, \mathfrak{F})$-bipure-injective resolution of A of length n.

A dual procedure yields the following.

3.18 THEOREM. *For any object A, if A/A_t has an $\mathfrak{F}$-copure-projective resolution of length n, then A has a $(\mathfrak{I}, \mathfrak{F})$-bipure-projective resolution of length n. If* $\mathrm{Copext}_{\mathfrak{F}}$ *has enough projectives for $\mathfrak{F}$ then the following hold.*

(i) $\mathrm{Bipext}_{\mathfrak{I},\mathfrak{F}}$ *has enough projectives.*

(ii) *An object is $(\mathfrak{I}, \mathfrak{F})$-bipure-projective if and only if it is a direct summand of a sum $P \oplus T$ with P $\mathfrak{F}$-copure-projective and $T \in \mathfrak{I}$.*

IV. Generalized purity in a category of modules

Throughout this section Λ denotes a ring with identity, and the word module implies unitary Λ-module. Let $\mathfrak{m}$ denote an infinite cardinal. Let $\mathcal{G}_{\mathfrak{m}}$ be the class of Λ-modules A such that $g(A) < \mathfrak{m}$ and $\mathcal{S}_{\mathfrak{m}}$ the class of Λ-modules A such that $|A| < \mathfrak{m}$ (where $g(A)$ denotes the minimal cardinal of a set of generators of A and $|A|$ the cardinal of A). Since $\mathcal{G}_{\mathfrak{m}}$ is closed under homomorphic images, and $\mathcal{S}_{\mathfrak{m}}$ is closed under both submodules and homomorphic images, the theorems of Section II apply to the notions of $\mathcal{G}_{\mathfrak{m}}$-pure, $\mathcal{S}_{\mathfrak{m}}$-pure and $\mathcal{S}_{\mathfrak{m}}$-copure. In particular one obtains decreasing chains of subfunctors $\mathrm{Pext}_{\mathcal{G}_{\mathfrak{m}}}(B, A)$, $\mathrm{Pext}_{\mathcal{S}_{\mathfrak{m}}}(B, A)$ and $\mathrm{Copext}_{\mathcal{S}_{\mathfrak{m}}}(B, A)$ of $\mathrm{Ext}^1_{\Lambda}(B, A)$. Other classes may be obtained from a torsion theory $\mathfrak{I}$, $\mathfrak{F}$ by taking $\mathfrak{I}_{\mathfrak{m}}$ to be the class of torsion modules T with $g(T) < \mathfrak{m}$, $\mathfrak{I}'_{\mathfrak{m}}$ the class of torsion modules with $|T| < \mathfrak{m}$ and $\mathfrak{F}_{\mathfrak{m}}$ the class of torsion free modules F with $|F| < \mathfrak{m}$.

The following theorem is a generalization of a theorem of Kaplansky [13] who proved the theorem for the countable case. This theorem is valid in a general algebraic setting as the proof involves only universal properties of direct summands. It will be applied here to obtain a description of the projectives for some of the types of purity mentioned above.

4.1 LEMMA. *Let $\mathfrak{m}$ be an infinite cardinal, and let $M = \sum_{i \in I} M_i$ with each M_i a Λ-module generated by a subset of cardinal $\leq \mathfrak{m}$. Suppose $M = P \oplus Q$ and let $x \in M$. Then there is a submodule M_x of M such that*

(i) $x \in M_x$,
(ii) $M_x = \sum_{i \in I_x} M_i$ *where* $|I_x| \leq \mathfrak{m}$, *and*
(iii) $M_x = (P \cap M_x) \oplus (Q \cap M_x)$.

Proof. Let τ be an ordinal of cardinal $\mathfrak{m}$, and let

$$\mathfrak{A} = \{(\alpha_1, \cdots, \alpha_k) \mid 0 < \alpha_i < \tau, 0 < k < \omega\} \cup \{(0)\}.$$

For notational convenience, when $k = 1$ the symbol $(\alpha_1, \cdots, \alpha_{k-1})$ will be used to denote (0). The set $\mathfrak{A}$ is well ordered as follows. For $(\alpha_1, \cdots, \alpha_k)$ and $(\beta_1, \cdots, \beta_j) \in \mathfrak{A}$,

$$(0) \leq (\alpha_1, \cdots, \alpha_k) \quad \text{and} \quad (\alpha_1, \cdots, \alpha_k) \leq (\beta_1, \cdots, \beta_j)$$

if and only if either $k < j$ or $k = j$ and $\alpha_i = \beta_i$ for $i < t \leq k$ implies $\alpha_t \leq \beta_t$.

With each $\mathfrak{A} \in \mathfrak{A}$ will be associated a set $R(\mathfrak{A}) = \{x_{\mathfrak{A},\beta}\}_{0<\beta<\tau}$ of elements of M such that

(1) $\langle R(\mathfrak{A})\rangle = \sum_{i \in J(\mathfrak{A})} M_i$ with $J(\mathfrak{A}) \subset I$ and $|J(\mathfrak{A})| < \aleph_0$, and
(2) the element x is factored into its P and Q components in $\langle R((0))\rangle$ and for $k \geq 1$, $x_{(\alpha_1,\cdots,\alpha_{k-1}),\alpha_k}$ is factored into its P and Q components in $\langle R((\alpha_1, \cdots, \alpha_k))\rangle$.

The sets $R(\mathfrak{A})$ will be defined inductively. Write $x = p + q$ with $p \in P$ and $q \in Q$. There is a finite set $J_0 \subset I$ with both p and $q \in \sum_{i \in J_0} M_i$. Assume $\mathfrak{A} = (\alpha_1, \cdots, \alpha_k) > (0)$ and the sets $R(\mathfrak{A}')$ have been chosen for $\mathfrak{A}' < \mathfrak{A}$. Then $R((\alpha_1, \cdots, \alpha_{k-1}))$ and in particular $x_{(\alpha_1,\cdots,\alpha_{k-1}),\alpha_k}$ have been determined. Write $x_{(\alpha_1,\cdots,\alpha_{k-1}),\alpha_k} = p' + q'$ with $p' \in P$ and $q' \in Q$. There is a finite set $J(\mathfrak{A}) \subset I$ with both p' and q' in $\sum_{i \in J(\mathfrak{A})} M_i$. Let $R(\mathfrak{A}) = \{x_{\mathfrak{A},\beta}\}_{0<\beta<\tau}$ be a set of generators for $\sum_{i \in J(\mathfrak{A})} M_i$, then $R(\mathfrak{A})$ satisfies (1) and (2).

Let $I_x = \bigcup_{\mathfrak{A} \in \mathfrak{A}} J(\mathfrak{A})$. Then $M_x = \sum_{i \in I_x} M_i$ satisfies the conditions (i), (ii) and (iii).

4.2 THEOREM. *Let M be a direct sum of (any number of) Λ-modules M_i with each M_i generated by a subset of cardinal $\leq \mathfrak{m}$ (where $\mathfrak{m}$ is an infinite cardinal). Then any summand of M is also a direct sum of Λ-modules, each generated by a subset of cardinal $\leq \mathfrak{m}$.*

Proof. Let $M = \sum_{i \in I} M_i$ with $g(M_i) \leq \mathfrak{m}$ $(i \in I)$, and suppose $M = P \oplus Q$. An increasing chain of submodules S_α $(\alpha \in J)$ of M such that $M = \bigcup_{\alpha \in J} S_\alpha$ will be constructed having the following properties:

(i) if α is a limit ordinal, $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$.
(ii) $S_\alpha = \sum_{i \in I_\alpha} M_i$ for some $I_\alpha \subset I$.
(iii) $g(S_{\alpha+1}/S_\alpha) \leq \mathfrak{m}$.
(iv) $S_\alpha = P_\alpha \oplus Q_\alpha$ with $P_\alpha = P \cap S_\alpha$ and $Q_\alpha = Q \cap S_\alpha$.

The chain is constructed inductively. Let $S_0 = 0$. Assume $\alpha > 0$ and for $\beta < \alpha$ there is a chain of submodules S_β of M having properties (i)–(iv). If α is a limit ordinal let $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$ and S_α clearly satisfies (i)–(iv). Suppose $\alpha = \beta + 1$. Choose $x \in M$ such that $x \notin S_\beta$ and let $S_\alpha = \langle S_\beta , M_x\rangle$ where M_x is the submodule of Lemma 4.1. Then S_α clearly satisfies (i)–(iv).

If $S_\alpha \neq M$ then $S_\alpha \neq S_{\alpha+1}$. Thus if $| \alpha | > | M |$, it must be true that $S_\alpha = M$, so one has $M = \bigcup_{|\alpha| \leqq |M|} S_\alpha$. Now $P_\alpha = P \cap S_\alpha$ is a direct summand of S_α by (iv) so P_α is a direct summand of M and hence a summand of $P_{\alpha+1}$. Writing $P_{\alpha+1} = P_\alpha \oplus N_\alpha$ for $| \alpha | \leq | M |$, one obtains $P = \bigcup_\alpha P_\alpha = \sum_\alpha N_\alpha$, where the sum is direct. Then

$$S_{\alpha+1}/S_\alpha \approx (P_{\alpha+1}/P_\alpha) \oplus (Q_{\alpha+1}/Q_\alpha)$$

and $g(S_{\alpha+1}/S_\alpha) \leq \mathfrak{m}$ imply $g(P_{\alpha+1}/P_\alpha) \leq \mathfrak{m}$. Then since $N_\alpha \approx P_{\alpha+1}/P_\alpha$ it is true that $g(N_\alpha) \leq \mathfrak{m}$ for $| \alpha | \leq | M |$.

With a slight change in the proof of the previous theorem one obtains the following.

4.3 Theorem. *Let M be a direct sum of Λ-modules M_i $(i \in I)$, with $| M_i | \leq \mathfrak{m}$ for each $i \in I$ and a fixed infinite cardinal $\mathfrak{m}$. Then any summand of M is again a direct sum of Λ-modules each of cardinal $\leq \mathfrak{m}$.*

Proof. In the proof of Lemma 4.1, substitute a listing $\{x_{\mathfrak{A},\beta}\}_{0<\beta<\tau}$ of the elements of $\sum_{i \in J(\mathfrak{A})} M_i$ for the listing of a set of generators, and substitute

(iii′) $| S_{\alpha+1}/S_\alpha | \leq \mathfrak{m}$

for (iii) in the proof of the theorem.

It follows from Theorems 2.5 and 4.2 that the $\mathcal{G}_\mathfrak{m}$-pure projectives are simply direct sums of members of $\mathcal{G}_\mathfrak{m}$, when the cardinal $\mathfrak{m}$ has a predecessor. If Λ has global dimension 1, it follows from Theorems 2.6 and 4.3 that the $\mathcal{S}_\mathfrak{m}$-pure-projectives are of the form $(\sum_{i \in I} P_i) \oplus P$ where $| P_i | < \mathfrak{m}$ and P is Λ-projective, again whenever the cardinal $\mathfrak{m}$ has a predecessor. If $| \Lambda | < \mathfrak{m}$ then of course $\mathcal{G}_\mathfrak{m}$ and $\mathcal{S}_\mathfrak{m}$ are the same and the first remark applies also to $\mathcal{S}_\mathfrak{m}$-pure-projectives. Similar statements are true for $\mathfrak{J}_\mathfrak{m}$ and $\mathfrak{J}'_\mathfrak{m}$ when Λ has global dimension 1.

The types of purity mentioned above are all direct generalizations of ordinary purity for Abelian groups, for the concepts of $\mathcal{G}_{\aleph_0}$-pure, $\mathcal{S}_{\aleph_0}$-pure $\mathfrak{J}_{\aleph_0}$-pure, $\mathfrak{J}'_{\aleph_0}$-pure as well as $\mathcal{S}_{\aleph_0}$-copure and $\mathfrak{J}'_{\aleph_0}$-copure all coincide with ordinary purity for Abelian groups (with $\mathfrak{J}$ the standard class of torsion groups).

From now on $\mathfrak{m}$-pure will be used to denote $\mathcal{G}_\mathfrak{m}$-pure. The notion of $\mathfrak{m}$-pure subgroups of Abelian groups for arbitrary cardinals $\mathfrak{m}$ was first proposed by Gacsályi [7], who defined them in terms of systems of equations. This generalization of purity was investigated further by Loś [15] and Fuchs [6] and [5] (pp. 87–91), who showed that Gacsályi's definition of $\mathfrak{m}$-pure was equivalent to the definition that appears here, in the case of Abelian groups. The next theorem shows that this result may be extended to unitary modules.

DEFINITION. *A (compatible) system of equations* over a Λ-module A is a triple $[M, F, f]$ where M is a submodule of a free Λ-module and f is a Λ-homomorphism from M into A.

This corresponds to the usual concept of a system of equations as follows. Let $\Lambda(\mathfrak{m}) = \sum_{\beta \in J} \Lambda x_\beta$ be the free Λ-module of rank $\mathfrak{m} = |J|$. A submodule M of $\Lambda(\mathfrak{m})$ consists of a set of linear forms $f_\alpha = \lambda_{\alpha 1} x_{\beta_1} + \cdots + \lambda_{\alpha k} x_{\beta_k}$ $(\lambda_{\alpha i} \in \Lambda, \alpha \in I)$, and the homomorphism f yields a system of equations $f_\alpha = a_\alpha$ $(\alpha \in I)$ over A. The set $\{x_\beta\}_{\beta \in J}$ is called the unknowns of the system. A solution of this system in A is a set of elements a_β $(\beta \in J)$ of A such that

$$\lambda_{\alpha 1} a_{\beta_1} + \cdots + \lambda_{\alpha k} a_{\beta_k} = a_\alpha \qquad (\alpha \in I).$$

The following definition is shown by Kertesz [14] to be equivalent to the usual definition for systems of equations over a ring with a (right) unit element, and carries over immediately to systems over a unitary module.

DEFINITION. A system of equations $[M, F, f]$ over a Λ-module A is *solvable* in A if and only if the homomorphism f may be extended to a Λ-homomorphism $\bar{f}$ from F to A. The number of unknowns of a system of equations $[M, F, f]$ over A is the rank of the free module F.

4.4 THEOREM. *Let A be a submodule of B. Then A is $\mathfrak{m}$-pure in B if and only if every system of equations over A with less than $\mathfrak{m}$ unknowns which is solvable in B is also solvable in A.*

Proof. Assume A is $\mathfrak{m}$-pure in B and let $[M, F, f]$ be a system of equations over A with less than $\mathfrak{m}$ unknowns which is solvable in B. Since $[M, F, f]$ is solvable in B the map

$$M \xrightarrow{f} A \subset B$$

can be extended to a homomorphism

$$F \xrightarrow{h} B.$$

Let $C = \langle h(F), A\rangle$. Then $g(C/A) \leq g(h(F)) \leq g(F) < \mathfrak{m}$ so A is a summand of C. Let

$$C \xrightarrow{p} A$$

be a projection onto A. Then

$$F \xrightarrow{ph} A$$

is a solution in A of the system $[M, F, f]$.

Conversely, assume the second statement holds and let $A \subset C \subset B$ with $g(C/A) < \mathfrak{m}$. Let

$$F \xrightarrow{h} C/A$$

be an epimorphism with F a free Λ-module and $g(F) = g(C/A)$. Suppose $S \subset F$ and

$$S \xrightarrow{k} A$$

is a homomorphism which can be extended to a homomorphism $F \to C$. Since $[S, F, k]$ is solvable in C it is solvable in B and hence in A, i.e. there exists an extension $F \to A$ of k. This implies A is a summand of C, and hence A is $\mathfrak{m}$-pure in B.

Assume Λ is a commutative ring. Let A be a Λ-module and I an ideal of Λ. Define $I^{\alpha}A$ for finite and transfinite ordinals α inductively as follows: $I^0A = A$, $I^{\alpha+1}A = I(I^{\alpha}A)$ and for limit ordinals α, $I^{\alpha}A = \bigcap_{\beta<\alpha} I^{\beta}A$. In [5] Fuchs has shown that if G is an Abelian p-group and S is an $\mathfrak{m}$-pure subgroup of G then $S \cap p^{\alpha}G = p^{\alpha}S$ for $|\alpha| < \mathfrak{m}$. A similar statement may be made for unitary modules over a commutative ring Λ for principal ideals I of Λ, as well as a corresponding statement for the quotient G/S. The following lemma is needed.

4.5 LEMMA. *Assume Λ is commutative. Let A be a Λ-module and let $I = \Lambda\lambda$ be a principal ideal of Λ. Then $a \in I^{\alpha}A$ is equivalent to the solvability in A of a system of equations over $\langle a \rangle$ with not more than $|\alpha|$ unknowns.*

Proof. If $\alpha = 1$, $a \in I^{\alpha}A$ is equivalent to the solvability in A of the equation $\lambda x = a$. Assume the statement of the lemma holds for ordinals $\beta < \alpha$. If α is a limit ordinal there is no problem. If $\alpha = \beta + 1$ for some β then $a \in I^{\alpha}A = I(I^{\beta}A)$ is equivalent to the condition $a = \lambda b$ with $b \in I^{\beta}A$. By the induction assumption $b \in I^{\beta}A$ is equivalent to the solvability in A of a system of equations $f_{\tau} = \lambda_{\tau}\, b$ $(\tau \in J)$ with not more than $|\beta|$ unknowns. Then $a \in I^{\alpha}A$ is equivalent to the solvability in A of the set of equations $\lambda x_0 = a$, $f_{\tau} - \lambda_{\tau}\, x_0 = 0$ $(\tau \in J)$ which has $\leq |\alpha|$ unknowns.

4.6 THEOREM. *Let Λ be a commutative ring (with identity), I a principal ideal of Λ and A a submodule of a Λ-module B. If A is $\mathfrak{m}$-pure in B then for $|\alpha| < \mathfrak{m}$,*

(i) $A \cap I^{\alpha}B = I^{\alpha}A$

(ii) $I^{\alpha}(B/A) = (I^{\alpha}B + A)/A$.

Proof. Assume A is $\mathfrak{m}$-pure in B. Statement (i) and the inclusion

$$(I^{\alpha}B + A)/A \subset I^{\alpha}(B/A)$$

follow immediately from Lemma 4.5 and Theorem 4.4. Let $b + A \in I^{\alpha}(B/A)$. This is equivalent to the solvability in B/A of a system of equations $[M, F, f]$ over $\langle b + A \rangle$ involving $\leq |\alpha|$ unknowns. Let $F = \Lambda(|\alpha|)$ and let

$$F \xrightarrow{\bar{f}} B/A$$

be a solution to $[M, F, f]$. Let S/A be the image of $\bar{f}$. Then $g(S/A) \leq g(F) = |\alpha| < \mathfrak{m}$ and A $\mathfrak{m}$-pure in B imply A is a summand of S, say $S = A \oplus C$. Let

$$S/A \xrightarrow{h} C$$

be the natural isomorphism. Then the system $[M, F, hf]$ of equations with the solution $[F, h\bar{f}]$ imply $h(b + A) \in I^{\alpha}C$. Then

$$b + A = h(b + A) + A \in (I^\alpha C + A)/A \subset (I^\alpha B + A)/A,$$

completing the proof.

The conditions above are not sufficient, even for Abelian groups, to imply A is $\mathfrak{m}$-pure in B. For example, let $T = (\prod_{n=1}^{\infty} C(p^n))_t$. The group T has no elements of infinite height and is not a direct sum of cyclic groups. Let B be the direct sum of the finite cyclic groups $\langle t \rangle$ $(t \in T)$. There is a natural homomorphism f of B onto T and the kernel A of f is a pure ($\aleph_0$-pure) subgroup of B (see [8]). Then since every countable subgroup of T is a direct sum of cyclic groups, A is $\aleph_1$-pure in B. Also A is not a summand of B since $B/A \approx T$ is not a direct sum of cyclic groups, so A is not $\aleph_2$-pure in B. However conditions (i) and (ii) of the previous theorem hold for all ideals of the ring of integers and all ordinals α. Note that this is also an example of a subgroup which is $\mathfrak{J}_{\aleph_1}$-pure but not $\mathfrak{J}_{\aleph_2}$-pure.

It is interesting to note that in general $\mathrm{Pext}_{\mathfrak{g}_\mathfrak{m}}(B, A)$ is not a natural subgroup of the group $\mathrm{Ext}^1_\Lambda(B, A)$. For Abelian groups, $\mathrm{Pext}_{\mathfrak{g}(\aleph_0)}(B, A) = \mathrm{Pext}(B, A)$, where $\mathfrak{g}(\aleph_0)$ is written for $\mathfrak{g}_{\aleph_0}$, is the subgroup of elements of infinite height of $\mathrm{Ext}(B, A)$ and hence depends only on the group $\mathrm{Ext}(B, A)$ and not on the particular choice of B and A. But for $\mathrm{Pext}_{\mathfrak{g}(\aleph_1)}$ this does not hold. The example above shows there exists a torsion group T and a group A with $\mathrm{Pext}_{\mathfrak{g}(\aleph_1)}(T, A) \neq 0$. Let Q denote the additive group of rational numbers and Z the group of integers. Then $T \approx \mathrm{Tor}(Q/Z, T)$ and

$$\mathrm{Ext}(T, A) \approx \mathrm{Ext}(\mathrm{Tor}(Q/Z, T), A) \approx \mathrm{Ext}(Q/Z, \mathrm{Ext}(T\ A)).$$

However since Q/Z is countable $\mathrm{Pext}_{\mathfrak{g}(\aleph_1)}(Q/Z, \mathrm{Ext}(T, A)) = 0$.

The following proposition is used to give an example of a non-trivial $\aleph_1$-copure subgroup.

4.7 Proposition. *If C is a countable cotorsion Abelian group then C is bounded.*

Proof. If L is a torsion free cotorsion group then

$$L \approx \mathrm{Hom}(Q/Z, Q/Z \otimes L)$$

is uncountable [8], so C has no torsion free summand, i.e. C is adjusted. Thus

$$C \approx \mathrm{Ext}(Q/Z, C_t) \qquad [8].$$

Let B be a basic subgroup of C_t. There is an epimorphism $C_t \to B \to 0$ and the exact sequence

$$C \approx \mathrm{Ext}(Q/Z, C_t) \to \mathrm{Ext}(Q/Z, B) \to 0$$

implies

$$|C| \geq |\mathrm{Ext}(Q/Z, B)|.$$

Write $B = \sum B_\alpha$ with B_α a cyclic prime power group. Suppose B is unbounded. Then $(\prod B_\alpha)/(\sum B_\alpha)$ has a non-zero divisible subgroup. This gives an exact sequence

$$0 \to \mathrm{Hom}\,(Q/Z,\ (\prod B_\alpha)/(\sum B_\alpha)) \to \mathrm{Ext}\,(Q/Z,\ B)$$

so that $\aleph_1 \leq |\ \mathrm{Ext}\,(Q/Z,\ B)| \leq |\ C\ |$. Thus B must be bounded, so that C_t and hence $C \approx \mathrm{Ext}\,(Q/Z,\ C_t)$ are bounded.

Let P be the group of p-adic integers, and let A be a pure subgroup of P such that $|\ P/A\ | = \aleph_0$. Let S be a subgroup of A and suppose $|\ A/S\ | \leq \aleph_0$. Then also $|\ P/S\ | \leq \aleph_0$. Now P is cotorsion and hence P/S is cortorsion plus divisible. By the proposition, P/S must be the direct sum of a bounded group and a divisible group. Then since A is pure in P, A/S is pure in P/S. It follows that A/S is a summand of P/S and hence A is $\aleph_1$-pure in P. But A is not a summand of P since the p-adics are indecomposable [1], and $|\ A\ | = \aleph_1$, so A is not $\aleph_2$-copure in P.

Bibliography

1. R. Baer, *Abelian groups without elements of finite order*, Duke Math. J., vol. 3 (1937), pp. 68–122.
2. D. Buchsbaum, *A note on homology in categories*, Ann. of Math. (2), vol. 69 (1959), pp. 66–74.
3. M. C. R. Butler and G. Horrocks, *Classes of extensions and resolutions*, Philos. Trans. Roy. Soc. London, vol. 254 (1961), pp. 155–222.
4. S. Dickson, *Torsion theories for Abelian categories*, Doctoral Dissertation, New Mexico State Univ., 1963.
5. L. Fuchs, *Abelian groups*, Budapest, Hungarian Academy of Sciences, 1958.
6. ———, *On generalized pure subgroups of Abelian groups*, Ann. Univ. Sci. Budapest, vol. 1 (1958), pp. 41–47.
7. S. Gacsályi, *On pure subgroups and direct summands of Abelian groups*, Publ. Math. Debrecen, vol. 4 (1955), pp. 89–92.
8. D. K. Harrison, *Infinite Abelian groups and homological methods*, Ann. of Math. (2), vol. 69 (1959), pp. 366–391.
9. A. Heller, *Homological algebra in Abelian categories*, Ann. of Math. (2), vol. 68 (1958), pp. 484–525.
10. G. Hochschild, *Relative homological algebra*, Trans. Amer. Math. Soc., vol. 82 (1956), pp. 246–269.
11. K. Honda, *Realism in the theory of Abelian groups*, I, Comm. Math. Univ. St. Pauli, vol. 5 (1956), pp. 37–75.
12. J. Irwin, C. Peercy and E. Walker, *On p^α-pure sequences of Abelian groups*, Topics in Abelian groups, Chicago, Scott Foresman, 1963.
13. I. Kaplansky, *Projective modules*, Ann. of Math. (2), vol. 68 (1958), pp. 372–377.
14. A. Kertész, *The general theory of linear equation systems over semisimple rings*. Publ. Math. Debrecen, vol. 4 (1955), pp. 79–86.
15. J. Loś, *Linear equations and pure subgroups*, Bull. Acad. Polon. Sci., vol. 7 (1959), pp. 13–18.
16. S. MacLane, *Homology*, Berlin, Springer-Verlag, 1963.
17. N. Yoneda, *On Ext and exact sequences*, J. Fac. Univ. Sci. Tokyo, vol. 8 (1960), pp. 507–526.

New Mexico State University
University Park, New Mexico
Institute for Advanced Study
Princeton, New Jersey