LOCAL TIME AS A DERIVATIVE OF OCCUPATION TIMES

BY

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1. Introduction

In a joint paper [1] Blumenthal and Getoor obtained local times for a large class of Markov processes by considering local time as an additive functional of a Markov process. Abstract representation theorems insure the existence of continuous additive functionals with prescribed potentials. By prescribing a certain potential Blumenthal and Getoor were thus able to obtain a continuous additive functional that they called local time. The connection between local time and occupation times was then made under Hunt's hypothesis (F), [5, III]. Thus this method of obtaining local time is an indirect one.

It is of interest whether local time can be constructed for processes satisfying hypothesis (F) and certain regularity conditions by more direct and intuitive methods than those employed by Blumenthal and Getoor. In this paper local time is constructed as the limit (in some sense) of natural approximating densities.

2. Preliminaries

We refer the reader to Getoor's expository paper [3] for notation, definitions and results used below concerning Hunt processes and additive functionals.

Let $X = \{X_t, t \geqq 0\}$ be a Hunt process on a state space E. E is assumed to be a locally compact separable metric space with a point Δ adjoined to E as the point at infinity if E is not compact or an isolated point if E is compact. By convention all extended real valued functions on E are defined on $E \cup \{\Delta\}$ by $f(\Delta) = 0$. We denote the λ-potential operator of the process for $\lambda \geqq 0$ by U^λ, i.e.,

$$U^\lambda f(x) = E_x \int_0^\infty e^{-\lambda t} f(X_t)\, dt$$

where f is a bounded real-valued universally measurable function on E. Recall that if a Hunt process satisfies hypothesis (F) [5, III, p. 154] then there exists a measure ξ on E and point kernels $U^\lambda(x, y)$ defined on $E \times E$ for $\lambda \geqq 0$ such that

$$U^\lambda f(x) = \int U^\lambda(x, y) f(y)\, d\xi(y). \tag{2.1}$$

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We will need the following facts implied by hypothesis (F):

(i) The measure ξ is strictly positive on non-empty open sets and finite on compact sets [5, III, p. 154].

(ii) Given a measure μ on E one can define the λ-potential, U_μ^λ, of μ as follows:

$$U_\mu^\lambda(x) = \int U^\lambda(x, y)\, d\mu(y).$$

U_μ^λ is positive lower semi-continuous and it is λ-excessive when finite almost everywhere with respect to ξ [5, III, p. 169].

3. Regularity conditions

Local time will be constructed as an additive functional of X under the regularity conditions given below. First we need the following definition:

A point $x \in E$ is *regular for itself* if $P_x\{T_x = 0\} = 1$, where $T_x = \inf\{t > 0 : X_t = x\}$ is the hitting time for x.

In the sequel we assume X is a Hunt process that satisfies hypothesis (F). We assume that X also satisfies the following conditions:

(1) Each point of E is regular for itself.

(2) For each $\lambda > 0$, $\lim_{y \to x_0} U^\lambda(x, y) = U^\lambda(x, x_0)$ uniformly in x, for all x, y, x_0 in E.

We remark that condition (1) implies that for fixed y, $U^\lambda(x, y)$ as a function of x belongs to $C_0(E)$ if $\lambda > 0$, where $C_0(E)$ is the class of functions continuous on E that vanish at infinity. This is true since then Hunt's hypothesis (I) is true; see [5, III, pp. 196 and 200]. Also condition (1) implies that $U^\lambda(x, y) \leqq U^\lambda(x, x) < \infty$ if $\lambda > 0$, for all x, y in E; see [5, iii, p. 200].

4. The construction

In this section we fix an arbitrary point x_0 in E and $\lambda > 0$. We wish to construct the local time at x_0. The additive functional that is constructed first depends on λ. The dependence on λ will be removed later.

Define a new process X^λ as follows:

$$\begin{aligned} X_t^\lambda &= X_t \quad \text{if} \quad t < S^\lambda \\ &= \Delta \quad \text{if} \quad t \geqq S^\lambda \end{aligned}$$

where S^λ is an exponentially distributed random variable with parameter λ completely independent of X, i.e., $P_x\{S^\lambda > t\} = e^{-\lambda t}$ for all x. X^λ is a Hunt process satisfying hypothesis (F) with respect to the same reference measure ξ.

Let $\{B_n\}$ be a decreasing sequence of open sets with compact closures each containing the fixed point $x_0 \in E$ and $x_0 = \bigcap_n \bar{B}_n$.

We define continuous additive functionals of X^λ for our fixed λ as follows:

Let $\quad A_n^\lambda(t) = \int_0^{t \wedge S^\lambda} h_n(X_s), \quad \text{where} \quad h_n(x) = I_{B_n}(x)/\xi(A_n)$

($a \wedge b$ denotes the minimum of a and b). Note that $0 < \xi(B_n) < \infty$ by comment (i) in Section 2. For each n, $A_n^\lambda = \{A_n^\lambda(t), t \geqq 0\}$ is a continuous additive functional of X^λ and $A_n^\lambda(t) = A_n^\lambda(S^\lambda)$ for $t \geqq S^\lambda$.

For a fixed time $t > 0$ and fixed sample point ω we can define a measure $\mu(\cdot, t, \omega)$ on E as follows: for a Borel set B in E let

$$\mu(B, t, \omega) = \int_0^t I_B(X_s(\omega))\, ds,$$

where I_B is the indicator function of B. One may call $\mu(\cdot, t, \omega)$ the "occupation measure" up to time t for the path ω, for $\mu(B, t, \omega)$ gives the amount of time spent by the path in B up to time t. Note that

$$A_n^\lambda(t, \omega) = \mu(B_n, t \wedge S^\lambda, \omega)/\xi(B_n).$$

These latter quotients are of the kind that arise in differentiating one measure with respect to another, so that the $A_n(t)$ can be interpreted as approximate densities of the occupation measure $\mu(\cdot, t, \omega)$ with respect to the measure ξ.

Let f_n^λ be the potential (of parameter zero) of A_n^λ. That is,

$$\begin{aligned}
f_n^\lambda(x) &= E_x A_n^\lambda(\infty) = E_x A_n^\lambda(S^\lambda) = E_x \int_0^{S^\lambda} g_n(X_s)\, ds \\
&= E_x \int_0^\infty \lambda e^{-\lambda u} \int_0^u g_n(X_s)\, ds\, du \\
&= E_x\left\{ -e^{-\lambda u} \int_0^u g_n(X_s)\, ds \right]_0^\infty \\
&\qquad\qquad + \left. \int_0^\infty e^{-\lambda u} g_n(X_u)\, du \right\} \qquad \text{(by integration by parts)} \\
&= E_x \int_0^\infty e^{-\lambda u} g_n(X_u)\, du \\
&= \int U^\lambda(x, y) g_n(y)\, d\xi(y) \qquad \text{(by equation (2.1) in Section 2)} \\
&= \int U^\lambda(x, y) I_{B_n}(y)/\xi(B_n)\, d\xi(y) \\
&= \frac{1}{\xi(B_n)} \cdot \int_{B_n} U^\lambda(x, y)\, d\xi(y).
\end{aligned}$$

Define measures ν_n on E as follows: if D is a Borel set we define

$$\nu_n(D) = \int_D g_n(y)\, d\xi(y) = \frac{\xi(D \cap B_n)}{\xi(B_n)}.$$

Then $f_n^\lambda(x) = U_{\nu_n}^\lambda(x) = \int U^\lambda(x, y)\, d\nu_n(y)$. Note also that $\| \nu_n \| = \nu_n(B_n) = 1$, for each n.

Now,

$$f_n^\lambda(x) = \frac{1}{\xi(B_n)} \int_{B_n} U^\lambda(x, y)\, d\xi(y),$$

Since $\{\bar{B}_n\} \downarrow x_0$ and $U^\lambda(x, y) \to U^\lambda(x, x_0)$ uniformly in x as $y \to x_0$ by regularity assumption (2), it is clear that the sequence $\{f_n^\lambda\}$ converges uniformly to f^λ, where $f^\lambda(x) = U^\lambda(x, x_0)$. This observation is of basic importance in what follows.

We will show the existence of an additive functional A^λ of X^λ such that the potential of A^λ if f^λ and A is the limit in an appropriate manner of the A_n.

The following remarks will be needed for the lemma that follows: One way of stating the strong Markov property is to require that if h is a bounded $\mathfrak{F}$-measurable function (see [3] for the definition of $\mathfrak{F}$ and $\mathfrak{F}_T$) defined on the sample space then

$$E_x\{h(\theta_T\,\omega) \mid \mathfrak{F}_T\} = E_{X_T}\, h, \qquad \text{a.e.} \quad P_x,$$

for any stopping time T, where $\theta_T\,\omega$ is the sample point defined by the equation $X_u(\theta_T\,\omega) = X_{u+T(\omega)}(\omega)$. We write $X_u \circ \theta_T = X_{u+T}$. We also note that if T is a stopping time then

$$S^\lambda(\theta_T\,\omega) = S^\lambda(\omega) - T(\omega), \qquad \text{a.e.} \quad P_x,$$

on $\{T < S^\lambda\}$ for all x; see [4, p. 24].

Lemma. *Let $A_n^\lambda = \{A_n^\lambda(t)\}$ and f_n^λ be defined as before. Then,*

$$E_n\{A_n^\lambda(\infty) \mid \mathfrak{F}_{t\wedge S^\lambda}\} = A_n^\lambda(t) + f_n^\lambda(X_t^\lambda), \qquad \text{a.e.} \qquad P_x.$$

Proof. For notational conveneince let $T = t \wedge S^\lambda$ and

$$h(\omega) = \int_0^{S^\lambda(\omega)} g_n(X_u(\omega))\, du.$$

Then,

$$\begin{aligned} E_x\{A_n^\lambda(\infty) \mid \mathfrak{F}_T\} &= E_x\left\{\int_0^T g_n(X_u)\, du \mid \mathfrak{F}_T\right\} + E_x\left\{\int_T^{S^\lambda} g_n(X_u)\, du \mid \mathfrak{F}_T\right\} \\ &= A_n^\lambda(t) + E_x\left\{\int_0^{S^\lambda - T} g_n(X_u \circ \theta_T)\, du \mid \mathfrak{F}_T\right\} \\ &= A_n^\lambda(t) + E_x\{I_{\{t<S^\lambda\}} \cdot h(\theta_T\,\omega) \mid \mathfrak{F}_T\} \\ &= A_n^\lambda(t) + I_{\{t<S^\lambda\}} E_{X_T} h \quad \text{(by the strong Markov property)} \\ &= A_n^\lambda(t) + I_{\{t<S^\lambda\}}\, f_n^\lambda(X_T) \\ &= A_n^\lambda(t) + f_n^\lambda(X_t^\lambda), \qquad \text{a. e. } P_x, \text{ Q. E. D.} \end{aligned}$$

By this lemma, if we let

$$M_n^\lambda(t) = A_n^\lambda(t) + f_n^\lambda(X_t^\lambda)$$

then $\{M_n^\lambda(t), \mathfrak{F}_{t\wedge S^\lambda}, P_x\}$ is a martingale for each n and x. This martingale is separable because $f_n^\lambda(X_t)$ is almost surely right continuous in t by a theorem of Hunt [5, I] since f_n^λ is λ-excessive and $A_n^\lambda(t)$ is clearly almost surely continuous in t. The phrase "almost surely" is used to denote a.e. P_x for all x in E.

By a standard inequality for separable martingales (see [3, p. 353]) for $\delta > 0$,

$$
\begin{aligned}
&P_x\{\sup_t | M_n^\lambda(t) - M_m^\lambda(t)| \geqq \delta\} \\
&\leqq \delta^{-2}E_x\{(M_n^\lambda(\infty) - M_m^\lambda(\infty))^2\} \\
&= \delta^{-2}E_x\{(A_n^\lambda(\infty) - A_m^\lambda(\infty))^2\} \qquad (\text{since } f_n^\lambda(X_\infty) = f_n^\lambda(\Delta) = 0) \\
&= \delta^{-2}E_x\{(A_n^\lambda(S^\lambda) - A_m^\lambda(S^\lambda))^2\}.
\end{aligned} \tag{4.1}
$$

Now,

$$
\begin{aligned}
A_n^\lambda(S^\lambda)^2 &= \int_0^{S^\lambda} (A_n^\lambda(S^\lambda) - A_n^\lambda(t))\, dA_n^\lambda(t) \\
&= 2\int_0^{S^\lambda} (A_n^\lambda(S^\lambda) - A_n^\lambda(t))g_n(X_t)\, dt \\
&= 2\int_0^{S^\lambda} g_n(X_t) \int_t^{S^\lambda} g_n(X_u)\, du\, dt.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E_x\{A_n^\lambda(S^\lambda)^2\} &= 2E_x \int_0^{S^\lambda} g_n(X_t) \int_t^{S^\lambda} g_n(X_u)\, du\, dt \\
&= 2E_x \int_0^{S^\lambda} g_n(X_t)E_{X_t} \int_0^{S^\lambda} g_n(X_u)\, du\, dt \\
&= 2E_x \int_0^{S^\lambda} g_n(X_t)E_{X_t}\, A_n^\lambda(S^\lambda)\, dt \\
&= 2E_x \int_0^{S^\lambda} g_n(X_t)f_n^\lambda(X_t)\, dt \\
&= 2E_x \int_0^{\infty} e^{-\lambda u}g_n(X_u)f_n^\lambda(X_u)\, du\ ;
\end{aligned}
$$

the latter equality is obtained by using the fact that S^λ is exponentially distributed with parameter λ.

But applying equation (2.1) in Section 2 to the latter expression we obtain

$$E_x\{A_n^\lambda(S^\lambda)^2\} = 2\int U^\lambda(x, y)g_n(y)f_n^\lambda(y)\, d\xi(y).$$

By a similar computation,

$E_x\{(A_n^\lambda(S^\lambda) - A_m^\lambda(S^\lambda))^2\}$

$$= 2 \int U^\lambda(x, y)(g_n(y) - g_m(y))(f_n^\lambda(y) - f_m^\lambda(y))\, d\xi(y)$$

$$= 2 \int U^\lambda(x, y)(f_n^\lambda(y) - f_m^\lambda(y))(d\nu_n(y) - d\nu_m(y))$$

$$\leqq 4U^\lambda(x, x)\| f_n^\lambda - f_m^\lambda \|.$$

The latter inequality follows from the fact that

$$U^\lambda(x, y) \leqq U^\lambda(x, x) < \infty \quad \text{and} \quad \| \nu_n \| = 1.$$

Since the sequence $\{f_n^\lambda\}$ converges uniformly to f^λ,

$$\lim_{n,m\to\infty} E_x\{(A_n^\lambda(S^\lambda) - A_m^\lambda(S^\lambda))^2\} = 0.$$

Hence, by expression (4.1) we obtain

$$(4.2) \qquad P_x\{\sup_t | M_n^\lambda(t) - M_m^\lambda(t)| \geqq \delta\} \to 0 \quad \text{as} \quad n, m \to \infty$$

for all $\delta > 0$.

But since $M_n^\lambda(t) = A_n^\lambda(t) + f_n^\lambda(X_t^\lambda)$,

$$P_x\{\sup_t | A_n^\lambda(t) - A_m^\lambda(t)| \geqq \delta\}$$
$$\leqq P_x\{\sup_t | M_n^\lambda(t) - M_m^\lambda(t)| \geqq \delta/2\} + P_x\{\sup_t |f_n^\lambda(X_t^\lambda) - f_m^\lambda(X_t^\lambda)| \geqq \delta/2\}.$$

Recalling that $f_n^\lambda \to f^\lambda$ uniformly and applying (4.2) we have

$$(4.3) \qquad P_x\{\sup_t | A_n^\lambda(t) - A_m^\lambda(t)| \geqq \delta\} \to 0 \quad \text{as} \quad n, m \to \infty$$

for all $\delta > 0$.

Thus the sequence $\{A_n^\lambda(t)\}$ converges in P_x-probability uniformly in t for all x. Hence, for each x there is a subsequence $\{A_{n_k}^\lambda(t)\}$ of $\{A_n^\lambda(t)\}$ which converges almost everywhere (P_x) uniformly in t. The subsequence $\{A_{n_k}^\lambda(t)\}$ in general depends on x. Let $B_x^\lambda(t, \omega) = \lim_{k\to\infty} A_{n_k}^\lambda(t, \omega)$. The subscript x is to denote the dependence of the limit on x. $B_x^\lambda(t, \omega)$ is almost everywhere (P_x) continuous in t since it is the uniform limit in t of the $A_{n_k}^\lambda(t)$ which are almost surely continuous in t. We now define $A^\lambda(t, \omega) = B_x^\lambda(t, \omega)$ on the set $\{\omega : X_0^\lambda(\omega) = x\}$. It is easy to check that $A^\lambda = \{A^\lambda(t); t \geqq 0\}$ is an additive functional of X^λ. Also $t \to A^\lambda(t, \omega)$ is almost surely continuous since $t \to B_x^\lambda(t, \omega)$ is almost everywhere (P_x) continuous. Thus A^λ is a *continuous* additive functional of X^λ.

The following argument shows that the potential of A^λ is f^λ. As before

$$E_x(A_n^\lambda(\infty)^2) = 2 \int U^\lambda(x, y) g_n(y) f_n^\lambda(y)\, d\xi(y)$$

$$= 2 \int U^\lambda(x, y) f_n^\lambda(y)\, d\nu_n(y)$$

$$\leqq 2U^\lambda(x, x)\, \| f_n^\lambda \|\, \| \nu_n \|.$$

Recall that $f_n^\lambda \to f^\lambda$ uniformly and

$$0 \leqq f^\lambda(x) = U^\lambda(x, x_0) \leqq U^\lambda(x_0, x_0) < \infty$$

and also $\| \nu_n \| = 1$ for all n. Therefore, $E_x(A_n^\lambda(\infty)^2) \leqq K < \infty$ for large enough n, where K is a constant depending on x.

For large n the random variables $A_n^\lambda(\infty)$ are thus uniformly integrable, so

$$\begin{aligned} E_x A^\lambda(\infty) &= E_x \lim_k A_{n_k}^\lambda(\infty) = \lim_k E_x A_{n_k}^\lambda(\infty) \\ &= \lim_k f_{n_k}^\lambda(x) = f^\lambda(x) = U^\lambda(x, x_0). \end{aligned}$$

Thus, $A^\lambda = \{A^\lambda(t)\}$ is a continuous additive functional of X^λ with potential f^λ.

We wish to obtain an additive functional of X (independent of λ) with λ-potential f^λ. The following argument is standard (see [1, p. 53] or [4, p. 49]).

Let $\lambda, \mu > 0$ and S^λ, S^μ be independent random variables exponentially distributed with parameters λ and μ respectively and independent of X.

By the above there exist additive functionals A^λ of X^λ and A^μ of X^μ such that $E_x A^\lambda(S^\lambda) = f^\lambda(x)$ and $E_x A^\mu(S^\mu) = f^\mu(x)$.

Now,

$$\begin{aligned} E_x\{A^\mu(S^\lambda \wedge S^\mu)\} &= E_x A^\mu(S^\mu) - E_x\{A^\mu(S^\mu) - A^\mu(S^\lambda), S^\lambda < S^\mu\} \\ &= f^\mu(x) - E_x\{A^\mu(S^\mu) - A^\mu(S^\lambda), S^\lambda < S^\mu\}. \end{aligned}$$

Using the fact that S^λ and S^μ are exponentially distributed and applying Fubini's theorem we obtain

$$\begin{aligned} E_x\{A^\mu(S^\mu) - A^\mu(S^\lambda); S^\lambda < S^\mu\} &= \lambda \int_0^\infty E_x\{A^\mu(S^\mu) - A^\mu(t); t < S^\mu\}e^{-\lambda t}\, dt \\ &= \lambda \int_0^\infty E_x\{E_{X_t}(A^\mu(S^\mu)); t < S^\mu\}e^{-\lambda t}\, dt \\ &= \lambda \int_0^\infty E_x\{f^\mu(X_t), t < S^\mu\}e^{-\lambda t}\, dt \\ &= \lambda\mu \int_0^\infty \int_t^\infty E_x\{f^\mu(X_t)\}e^{-\mu s}e^{-\lambda t}\, ds\, dt \\ &= \lambda \int_0^\infty E_x f^\mu(X_t)e^{-(\mu+\lambda)t}\, dt \\ &= \lambda E_x \int_0^\infty e^{-(\mu+\lambda)t} f^\mu(X_t)\, dt \\ &= \lambda U^{\mu+\lambda} f^\mu(x). \end{aligned}$$

Thus $E_x A^\mu(S^\lambda \wedge S^\mu) = f^\mu(x) - \lambda U^{\lambda+\mu} f^\mu(x)$. But by the resolvent equation

$$U^\mu f(x) - \lambda U^{\lambda+\mu} U^\mu f(x) = U^{\lambda+\mu} f(x)$$

for all bounded measurable functions f and x in E. Since $U^\lambda(x, y)$ is the kernel for the operator U^λ, this implies

$$U^\mu(x, y) - \lambda U^{\lambda+\mu} U^\mu(\cdot, y)(x) = U^{\lambda+\mu}(x, y)$$

for all x, y in E. However, $f^\lambda(x) = U^\lambda(x, x_0)$, so

$$f^\mu - \lambda U^{\lambda+\mu} f^\mu = f^{\lambda+\mu}.$$

$f^{\lambda+\mu}$ is symmetric in λ and μ, so by Meyer's uniqueness theorem [3, p. 416] $A^\lambda(t) = A^\mu(t)$ a.s. on $\{t < S^\lambda \wedge S^\mu\}$. It now follows from a standard construction that there exists an additive functional A of the process X such that for each $\lambda > 0$,

$A(t) = A^\lambda(t)$ if $t < S^\lambda$, and $A(S^\lambda) = A^\lambda(t)$ if $t \geqq S^\lambda$. See [4, p. 50].

Hence the λ-potential of A is f^λ for each $\lambda > 0$:

$$U_A^\lambda(x) = E_x \int_0^\infty e^{-\lambda t}\, dA(t)$$

$$= \lambda \int_0^\infty E_x(A(t)) e^{-\lambda t}\, dt \qquad \text{(by integration by parts and Fubini's theorem)}$$

$$= E_x A(S^\lambda) = E_x A^\lambda(\infty) = f^\lambda(x).$$

We thus obtain the following theorem.

THEOREM 4.1. *Assume the process X satisfies hypothesis* (F) *and regularity conditions* (1) *and* (2). *Fix x_0 in E. Let $f^\lambda(x) = U^\lambda(x, x_0)$ for $\lambda > 0$. Then there exists a unique (up to equivalence) continuous additive functional $A = \{A(t)\}$ of X such that the λ-potential of A is f^λ for each $\lambda > 0$, and if $\lambda > 0$ then the sequence of additive functionals*

$$A_n^\lambda(t) = \frac{1}{\xi(B_n)} \int_0^{t \wedge S^\lambda} I_{B_n}(X_u)\, du$$

converges in P_x-probability for each x, uniformly in t to $A(t)$ on $\{t \leqq S\}$. Also for each x there is a subsequence $\{A_{n_k}^\lambda(t)\}$ of $\{A_n^\lambda(t)\}$ such that $A_{n_k}^\lambda(t)$ converges to $A(t)$ almost everywhere (P_x) uniformly in t on $\{t \leqq S^\lambda\}$.

DEFINITION. The additive functional, A, of Theorem 4.1 is called the local time of the process X at the point x_0.

The local time constructed here coincides with that obtained by Blumenthal and Getoor since the two local times have the same λ-potentials.

5. An alternative condition

In this section we give a useful condition that together with hypothesis (F) and regularity condition (1) implies condition (2). In general the state space E need not have an algebraic structure. In this section we assume E is a linear space over the real or complex numbers with an invariant metric d, so that

$$d(x_1, x_2) = d(x_1 - x, x_2 - x) \qquad \text{for all} \quad x_1, x_2, x \quad \text{in} \quad E.$$

We assume that the process X satisfies hypothesis (F), regularity condition (1) and the following regularity condition:

(2′) The potential kernel $U^\lambda(x, y)$ depends only on the difference $y - x$ for all $\lambda > 0$; we write $U^\lambda(x, y) = U^\lambda(y - x)$ for all x, y in E.

As remarked before, $U^\lambda(x, y)$ belongs to $C_0(E)$ as a function of one variable with the other variable fixed. Then by regularity condition (2′) the function of one variable $x \to U^\lambda(x)$ also belongs to $C_0(E)$. The function $U^\lambda(\cdot)$ is thus uniformly continuous on E.

It is now easy to verify condition (2) that

$$\lim_{y \to x_0} U^\lambda(x, y) = U^\lambda(x, x_0)$$

uniformly in x, for all x, y, x_0 in E. For by the uniform continuity of $U^\lambda(\cdot)$ and condition (2′), given $\varepsilon > 0$ there is a $\delta > 0$ such that if

$$d(y, x_0) = d(y - x, x_0 - x) < \delta$$

then

$$| U^\lambda(x, y) - U^\lambda(x, x_0)| = | U^\lambda(y - x) - U^\lambda(x_0 - x)| < \varepsilon$$

for all x. Therefore the hypotheses of Theorem 4.1 are fulfilled.

6. Examples

In this section we give examples of large classes of processes which satisfy hypothesis (F) and the two sets of regularity conditions.

Let X be a real-valued process with stationary independent increments and right continuous paths. Then as is well known (see [1, p. 64]),

$$E_x(e^{iyX(t)}) = e^{iyx}e^{-t\psi(y)}$$

where

$$\psi(y) = \mathrm{im}\, y + \frac{\sigma^2}{2} y^2 + \int_{-\infty}^{\infty} \left[1 - e^{iyu} - \frac{iyu}{i + u^2}\right] \nu(du)$$

with m a real number, $\sigma^2 \geqq 0$ and ν a measure such that

$$\int_{-\infty}^{\infty} x^2(1 + x^2)^{-1}\nu(dx) < \infty.$$

For simplicity assume $\sigma^2 = m = 0$. Denoting the real part of ψ by ψ_R, we assume for all $\lambda > 0$ that

$$\int_{-\infty}^{\infty} (\lambda + \psi_R(x))^{-1}\, dx < \infty. \tag{6.1}$$

If in Hunt's hypothesis (F) we let ξ be Lebesgue measure then X is a Hunt process that satisfies hypothesis (F). Moreover, the λ-potential kernel for

X is given by

$$U^{\lambda}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iz(y-x)}}{\lambda + \psi(z)} dz. \tag{6.2}$$

By assumption (6.1) this integral exists absolutely and $U^{\lambda}(x, y)$ is bounded and continuous in x and y. Also each point is regular for itself (see [1, p. 64]). Hence X satisfies regularity condition (1). It is clear that condition (2′) is satisfied. It is easy to see that condition (2) is satisfied directly by considering the real and imaginary parts of the integrand in the integral defining $U^{\lambda}(x, y)$.

The stable processes of index α, $1 < \alpha \leqq 2$, on the real line are included in the above example. The potential kernel is given by (6.2) where

$$\Psi(z) = | z |^{\alpha} \left[1 + i\beta \left(\frac{z}{| z |} \tan \frac{\pi\alpha}{2} \right) \right], \qquad \text{with} \quad | \beta | \leqq 1. \tag{6.3}$$

If $\alpha = 2$ then X is one-dimensional Brownian motion. It is clear that (6.1) is satisfied since $\psi_R(z) = | z |^{\alpha}$ and $\alpha > 1$.

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