ON THE DISCREPANCY OF CERTAIN SEQUENCES MOD 1

BY
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1. Introduction

For any set of N points in the unit square, and any point (x, y) in the square, let $S(x, y)$ denote the number of points of the set which are in the rectangle $0 \leq \zeta < x, 0 \leq \eta < y$. We shall call the difference $\Delta(x, y) = S(x, y) - Nxy$ the error at (x, y). We shall call the quantity $D = \sup |\Delta(x, y)|$, where the supremum is taken over all points (x, y) of the unit square, the discrepancy of the set of points.

K. F. Roth [3] proved that for any set of N points in the unit square, $D > c\sqrt{\log N}$, where c is a positive absolute constant. An analogous result holds in k-dimensional space: $D_k > c_k(\log N)^{(k-1)/2}$, where c_k is a positive constant depending only on k, and D_k is the discrepancy of a set of N points in the k-dimensional unit cube.

Roth also showed an example of 2^n points in the unit square for which the discrepancy $D \leq 2n + 1$. In 1960 J. H. Halton [2] studied a generalization of that example in the k-dimensional unit cube, and obtained an analogous result: $D_k \leq c_k(\log N)^{k-1}$.

Halton also considered the gap between $(\log N)^{(k-1)/2}$ in Roth's theorem, and $(\log N)^{k-1}$ in the examples, and he was led to state the "tentative conjecture" that the results for the examples could be improved to agree with Roth's theorem. In this paper, the example in the two-dimensional case is studied and it is shown that at least for the two-dimensional case such an improvement is not possible.

Let $J(R, n)$ denote the set of R^n points of the form

$$\left(\frac{t_1}{R} + \frac{t_2}{R^2} + \cdots + \frac{t_n}{R^n}, \frac{t_n}{R} + \frac{t_{n-1}}{R^2} + \cdots + \frac{t_1}{R^n}\right)$$

where $t_i = 0, 1, 2, \cdots, R - 1$. We assume without loss of generality that the set of points is ordered so the x coordinates form an increasing sequence.

Points of the unit square which are of the form $(k/R^n, l/R^n)$, where k and l are positive integers, will be called "lattice points" of the unit square (with respect to $J(R, n)$). $T(R, n)$ will denote the average error of $J(R, n)$ at the lattice points, and it is shown in Section 3 that $T(R, n) = nT(R, 1) = n(R - 1)(R + 1)/12R$.

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A brief abstract of this paper appears in [1].

$J(2, n)$ is an example studied by Roth. In Section 4 it is shown that the exact maximum error attained at any lattice point is $n/3 + \frac{1}{9}(1 - (-\frac{1}{2})^n)$. It is also shown that this maximum error, denoted by M_n, is attained at exactly two lattice points which are explicitly determined. The error is non-negative at all points of the unit square, and from this it follows that $D \leq M_n + 2$. Thus for $J(2, n)$ we have $n/3 < D < n/3 + 3$.

If $N \geq 2$, and n the integer such that $R^n < N \leq R^{n+1}$, the sequence of N points in the unit square studied by Halton is the sequence

$$J'(R, N) = \{(0, Y_1^{(n+1)}), (1/N, Y_2^{(n+1)}), \cdots, ((N-1)/N, Y_N^{(n+1)})\},$$

where $Y_i^{(n+1)}$ is the y coordinate of the i^{th} point of $J(R, n + 1)$.

For any point (x, y) in the unit square let $S'_N(x, y)$ and $\Delta'_N(x, y)$ refer to the sequence $J'(R, N)$ in the analogous way that $S(x, y)$ and $\Delta(x, y)$ were defined. Let $S_n(x, y)$ and $\Delta_n(x, y)$ refer to the sequence $J(R, n)$. Then if (x, y) is any point of the unit square, we may write $x = \alpha/R^n$, and $S_n(\alpha/R^n, y) = S'_N(\alpha/N, y)$, since for each $i \leq R^n$, $Y_i^{(n+1)} = Y_i^{(n)}$. Therefore, $\Delta'_N(\alpha/N, y) = \Delta_n(x, y)$, and so $\sup \Delta'_N(x, y) \geq \sup \Delta_n(x, y)$. For this reason we only study the error of $J(R, n)$ consisting of R^n points.

2. Preliminary results

The i^{th} point of $J(R, n)$ will be denoted by $(X_i^{(n)}, Y_i^{(n)})$, or simply (X_i, Y_i) if no confusion will result. We note that since the x coordinates of $J(R, n)$ form an increasing sequence, $X_i = (i - 1)/R^n$. $S_n(x, y)$, or simply $S(x, y)$, will denote the number of points of $J(R, n)$ in the rectangle $0 \leq \zeta < x$, $0 \leq \eta < y$. $\Delta_n(x, y)$, or simply $\Delta(x, y)$, will denote the error of $J(R, n)$ at (x, y).

LEMMA 1. *Let i, p, and q be integers, with $1 \leq i \leq R^n$, $0 \leq p \leq R^{n-1} - 1$, and $1 \leq q \leq R$. Then the following four statements are equivalent:*

(1) $i = pR + q$

(2) $X_i^{(n)} = X_{p+1}^{(n-1)} + (q - 1)/R^n$

(3) $Y_i^{(n)} = (q - 1)/R + (1/R)Y_{p+1}^{(n-1)}$

(4) $(q - 1)/R \leq Y_i^{(n)} < q/R$.

Proof. Let $i = pR + q$. Therefore

$$X_i^{(n)} = p/R^{n-1} + (q - 1)/R^n = X_{p+1}^{(n-1)} + (q - 1)/R^n.$$

Since $X_{p+1}^{(n-1)} = t_1/R + \cdots + t_{n-1}/R^{n-1}$, therefore

$$Y_i^{(n)} = (q - 1)/R + t_{n-1}/R^2 + \cdots + t_1/R^n = (q - 1)/R + (1/R)Y_{p+1}^{(n-1)},$$

and statement (4) follows.

The converse implications also follow immediately from the definitions.

LEMMA 2. *Let (x, y) be any points in the unit square, with $y \leq 1/R$. Then*

$$\Delta_n(x, y) = \Delta_{n-1}(x, Ry).$$

Proof. If $(X_i^{(n)}, Y_i^{(n)})$ is a point of $J(R, n)$ in the rectangle $0 \le \zeta < x$, $0 \le \eta < y$, then $0 \le Y_i^{(n)} < y \le 1/R$, and so by Lemma 1, $i = pR + 1$,

$$X_{p+1}^{(n-1)} = X_i^{(n)} < x \quad \text{and} \quad (1/R)Y_{p+1}^{(n-1)} = Y_i^{(n)} < y.$$

Thus, corresponding to each point of $J(R, n)$ in the rectangle $0 \le \zeta < x$, $0 \le \eta < y$, there is a point of $J(R, n-1)$ in the rectangle $0 \le \zeta < x$, $0 \le \eta < Ry$.

The converse is also true, and so $S_n(x, y) = S_{n-1}(x, Ry)$. Lemma 2 follows by subtracting $R^n xy$ from both sides of this equation.

To prove Theorem 1, we sum the errors at all lattice points on the horizontal line $y = l/R^n$. For this purpose, we prove the following:

LEMMA 3. *Let q be an integer such that $1 \le q \le R$, and let p be an integer such that $1 \le p \le R^{n-1} - 1$. Let $(k/R^n, l/R^n)$ be any lattice point with $(q-1)/R \le l/R^n \le q/R$.* Then the following statements are true:

(a) *If $0 \le k/R^n \le (q-1)/R^n$, then $\Delta_n(k/R^n, l/R^n) = k - kl/R^n$.*

(b) *If $((p-1)R + q)/R^n \le k/R^n \le (pR + q - 1)/R^n$, then*

$$\Delta_n(k/R^n, l/R^n)$$
$$= \Delta_n(k/R^n, (q-1)/R) + \Delta_{n-1}(p/R^{n-1}, (l-(q-1)R^{n-1})/R^{n-1})$$
$$- kl/R^n + k(q-1)/R + p(l-(q-1)R^{n-1})/R^{n-1}.$$

(c) *If $((R^{n-1} - 1)R + q)/R^n \le k/R^n \le 1$, then $\Delta_n(k/R^n, l/R^n) = l - kl/R^n$.*

Proof. (a) The first k points of $J(R, n)$ are exactly the points of $J(R, n)$ which are in the rectangle $0 \le \zeta < k/R^n, 0 \le \eta < l/R^n$. For if i is an integer such that $1 \le i \le k$, then $X_i = (i-1)/R^n < k/R^n$, and since $i - 1 < k \le q - 1 \le R - 1$,

$$Y_i = (i-1)/R < k/R \le (q-1)/R \le l/R^n.$$

Therefore, $S(k/R^n, l/R^n) = k$, and part (a) follows.

(b) We may write

$$S(k/R^n, l/R^n)$$
$$= S(k/R^n, (q-1)/R) + S(0 \le \zeta < k/R^n, (q-1)/R \le \eta < l/R^n),$$

and we first show that this last term is equal to

$$S(p/R^{n-1}, (l-(q-1)R^{n-1})/R^n).$$

If (X_i, Y_i) is a point of $J(R, n)$ such that

$$0 \le X_i < p/R^{n-1}, \qquad 0 \le Y_i < (l-(q-1)R^{n-1})/R^n,$$

then $Y_i < 1/R$, and so $i = p'R + 1$ for some integer p'. Therefore $X_i = p'/R^{n-1} < p/R^{n-1}$, and therefore $X_i \le (p-1)/R^{n-1}$. Also

$$Y_i = 0/R + t_{n-1}/R^2 + \cdots + t_1/R^n,$$

and so

$$X_i = t_1/R + \cdots + t_{n-1}/R^{n-1} + 0/R^n.$$

Therefore

$$X_{i+q-1} = X_i + (q-1)/R^n \leq ((p-1)R + q - 1)/R^n < k/R^n,$$

and

$$(q-1)/R \leq Y_{i+q-1} = (q-1)/R + Y_i < l/R^n.$$

Thus, corresponding to each point (X_i, Y_i) in the rectangle

$$0 \leq \zeta < p/R^{n-1}, \qquad 0 \leq \eta < (l - (q-1)R^{n-1})/R^n,$$

the point (X_{i+q-1}, Y_{i+q-1}) is in the rectangle $0 \leq \zeta < k/R^n$, $(q-1)/R \leq \eta < l/R^n$. The converse is proved in a similar manner, and this gives the result that

$$S(0 \leq \zeta < k/R^n, (q-1)/R \leq \eta < l/R^n)$$
$$= S(p/R^{n-1}, (l - (q-1)R^{n-1})/R^n).$$

Therefore we obtain

$$\Delta_n(k/R^n, l/R^n) = \Delta(k/R^n, (q-1)/R) + \Delta_n(p/R^{n-1}, (l-(q-1)R^{n-1})/R^n)$$
$$- kl/R^n + k(q-1)/R + p(l - (q-1)R^{n-1})/R^{n-1}.$$

Now $(l - (q-1)R^{n-1})/R^n \leq 1/R$, and therefore by Lemma 2,

$$\Delta_n(p/R^{n-1}, (l - (q - 1(R^{n-1})/R^n) = \Delta_{n-1}(p/R^{n-1}, (l - (q-1)R^{n-1})/R^{n-1}).$$

(c) The points of $J(R, n)$ which are in the rectangle $0 \leq \zeta < k/R^n$, $0 \leq \eta < l/R^n$, are exactly the l points of $J(R, n)$ whose y coordinates are $0, 1/R^n, 2/R^n, \cdots, l - 1/R^n$. For if not, then we must have a point (X_i, Y_i) with $0 \leq Y_i \leq (l-1)/R^n$ and $k/R^n \leq X_i$. Therefore we must have

$$((R^{n-1} - 1)R + q)/R^n \leq X_i = (i-1)/R^n,$$

and so

$$(R^{n-1} - 1)R + q < i \leq R^n.$$

We may therefore write $i = (R^{n-1} - 1)R + j$ where $q < j \leq R$, and by Lemma 1 this implies that $Y_i \geq (j-1)/R \geq q/R \geq l/R^n$ which is a contradiction. Therefore $S(k/R^n, l/R^n) = l$, and part (c) follows.

LEMMA 4. *Let j, q, k be integers such that $1 \leq j \leq R, 0 \leq q \leq R, 0 \leq k \leq R^n$, and $k \equiv j \pmod R$. Then $\Delta_n(k/R^n, q/R) = \Delta_1(j/R, q/R)$.*

Proof. We first note that the points of $J(R, 1)$ are $(0, 0)(1/R, 1/R), \cdots, ((R-1)/R, (R-1)/R)$, and therefore at each lattice point $(j/R, q/R)$, $S_1(j/R, q/R) = \min(j, q)$.

Now Lemma 4 is true for $q = 0$, and for $k = 0$. Suppose $q > 0$ and Lemma 4 is valid when q is replaced by any of the numbers $0, 1, \cdots, q-1$. We establish it for q itself by applying the previous lemma with $l/R^n = q/R$. We consider three cases:

Case 1. $0 < k/R^n \leq (q-1)/R^n$. Then $k = j$, and by Lemma 3(a),

$$\Delta_n(k/R^n, q/R) = j - jq/R = \min(j, q) - jq/R = \Delta_1(j/R, q/R).$$

Case 2. $q/R^n \leq k/R^n \leq ((R^{n-1}-1)R + q - 1)/R^n$. Let p be any integer such that $1 \leq p \leq R^{n-1} - 1$, and consider two subcases:

Case 2(a). $((p-1)R + q)/R^n \leq k/R^n \leq p/R^{n-1}$. Then $k = (p-1)R + j$ with $q \leq j \leq R$, and so by Lemma 3(b) and the induction hypothesis,

$$\begin{aligned}\Delta_n(k/R^n, l/R^n) &= \Delta_1(j/R, (q-1)/R) + \Delta_{n-1}(p/R^{n-1}, 1) + 1 - j/R \\ &= \min(j, q-1) - j(q-1)/R + 1 - j/R \\ &= q - jq/R = \Delta_1(j/R, q/R).\end{aligned}$$

$\Delta_{n-1}(p/R^{n-1}, 1) = 0$, since the first p points of $J(R, n-1)$ are exactly the points of $J(R, n-1)$ which are in the rectangle $0 \leq \zeta < p/R^{n-1}, 0 \leq \eta < 1$.

Case 2(b). $p/R^{n-1} < k/R^n \leq (pR + q - 1)/R^n$. Then $k = pR + j$ with $1 \leq j \leq q - 1$, and so by Lemma 3(b) and the induction hypothesis,

$$\begin{aligned}\Delta_n(k/R^n, l/R^n) = \Delta_1(j/R, (q-1)/R) - j/R &= j - j(q-1)/R - j/R \\ &= \Delta_1(j/R, q/R).\end{aligned}$$

Case 3. $((R^{n-1}-1)R + q)/R^n \leq k/R^n \leq 1$. Then $k = (R^{n-1}-1)R + j$ with $q \leq j \leq R$, and so by Lemma 3(c),

$$\Delta_n(k/R^n, q/R) = q - jq/R = \Delta_1(j/R, q/R).$$

LEMMA 5. *At each point of the unit square the error of $J(R, n)$ is nonnegative.*

Proof. It suffices to prove that at each lattice point the error is nonnegative. For if (x, y) is any point of the unit square, let $(k/R^n, l/R^n)$ be the lattice point such that $(k-1)/R^n < x \leq k/R^n$, $(l-1)/R^n < y \leq l/R^n$. Therefore

$$\Delta(x, y) \geq S(x, y) - kl/R^n = \Delta(k/R^n, l/R^n).$$

The error of $J(R, 1)$ is nonnegative at each lattice point since

$$\Delta_1(k/R, l/R) = \min(k, l) - kl/R \geq 0.$$

We assume the error of $J(R, n-1)$ is nonnegative at each lattice point, and prove it is true for $J(R, n)$.

Let $(k/R^n, l/R^n)$ be any lattice point, and let q be the integer such that $(q-1)/R < l/R^n \leq q/R$ with $1 \leq q \leq R$. Let $l = (q-1)R^{n-1} + t$, with $1 \leq t \leq R^{n-1}$. We again consider three cases:

Case 1. $0 < k/R^n \leq (q-1)/R^n$. Then $\Delta_n(k/R^n, l/R^n) = k - kl/R^n \geq 0$.

Case 2. $q/R^n \leq k/R^n \leq ((R^{n-1}-1)R + q - 1)/R^n$. Let p be any integer such that $1 \leq p \leq R^{n-1} - 1$, and consider two subcases:

Case 2(a). $((p - 1)R + q)/R^n \leq k/R^n \leq p/R^{n-1}$. Then

$$k = (p - 1)R + j$$

with $q \leq j \leq R$, and so by Lemmas 3(b) and 4, and the induction hypothesis,

$$\Delta_n(k/R^n, l/R^n)$$
$$= \Delta_1(j/R, (q - 1)/R) + \Delta_{n-1}(p/R^{n-1}, t/R^{n-1}) + t/R^n(R - j) \geq 0.$$

Case 2(b). $p/R^{n-1} < k/R^n \leq (pR + q - 1)/R^n$. Then $k = pR + j$ with $1 \leq j \leq q - 1$, and so by Lemmas 3(b) and 4, and the induction hypothesis,

$$\Delta_n(k/R^n, l/R^n) = j - j(q - 1)/R + \Delta_{n-1}(p/R^{n-1}, t/R^{n-1}) - jt/R^n \geq 0.$$

Case 3. $((R^{n-1} - 1)R + q)/R^n \leq k/R^n \leq 1$. Then $\Delta_n(k/R^n, l/R^n) = l - kl/R^n \geq 0$.

The following result is needed in Theorem 1.

LEMMA 6. *Let* $T(R, 1) = (1/R^2)\sum_{k=1}^{R} \sum_{l=1}^{R} \Delta_1(k/R, l/R)$. *Then*

$$T(R, 1) = (R - 1)(R + 1)/12R.$$

Proof. Since the points of $J(R, 1)$ are on the diagonal $y = x$, at each lattice point $(k/R, l/R)$ we have $S(k/R, l/R) = \min(k, l)$, and therefore the errors on the l^{th} row of lattice points are

$$\begin{aligned} \Delta_1(k/R, l/R) &= k - kl/R \qquad (k = 1, 2, \cdots, l) \\ &= l - kl/R \qquad (k = l + 1, \cdots, R). \end{aligned}$$

Therefore the sum of the errors on the l^{th} row of lattice points is

$$\begin{aligned} \sum_{k=1}^{R} \Delta_1(k/R, l/R) &= \sum_{i=1}^{l} (i - il/R) + \sum_{i=1}^{R-l} (l - (l + i)l/R) \\ &= \tfrac{1}{2}(Rl - l^2). \end{aligned}$$

Therefore

$$\begin{aligned} T(R, 1) &= (1/R^2)\sum_{l=1}^{R} \sum_{k=1}^{R} \Delta_1(k/R, l/R) \\ &= (1/2R^2)\sum_{l=1}^{R} (Rl - l^2) = (R - 1)(R + 1)/12R. \end{aligned}$$

The following two lemmas are needed for the proof of Theorem 2.

LEMMA 7. *At each lattice point* $(k/R^n, l/R^n)$,

$$\Delta_n(k/R^n, l/R^n) = \Delta_n(1 - k/R^n, 1 - l/R^n).$$

Proof. Let each point (X, Y) of $J(R, n)$ be shifted to the point

$$(X^*, Y^*) = (X + 1/2R^n, Y + 1/2R^n),$$

and denote this shift sequence by $J^*(R, n)$. The points of $J^*(R, n)$ are symmetric with respect to both diagonals of the unit square. For a point (X^*, Y^*) belongs to $J^*(R, n)$ if and only if the point (Y^*, X^*) belongs to $J^*(R, n)$, and if and only if the point $(1 - Y^*, 1 - X^*)$ belongs to $J^*(R, n)$.

Let $S^*(x, y)$ be defined for $J^*(R, n)$ in the same way that $S(x, y)$ was defined for $J(R, n)$. Then at each lattice point $(k/R^n, l/R^n)$, $S^*(k/R^n, l/R^n) = S(k/R^n, l/R^n)$, so it suffices to prove Lemma 5 for $J^*(R, n)$.

From the symmetry of $J^*(R, n)$ it follows that at each lattice point $(k/R^n, l/R^n)$,

$$\begin{aligned} S(k/R^n, l/R^n) &= S(1, l/R^n) - \{S(1, 1 - k/R^n) - S(1 - l/R^n, 1 - k/R^n)\} \\ &= l - (R^n - k) + S(1 - k/R^n, 1 - l/R^n) \\ &= \Delta(1 - k/R^n, 1 - l/R^n) + kl/R^n. \end{aligned}$$

LEMMA 8. *If $(k/2^n, y)$ is a point of the unit square, such that k is an odd integer and $y \le \frac{1}{2}$, then for the sequence $J(2, n)$,*

$$\Delta_n(k/2^n, y) = \Delta_n((k + 1)/2^n, y) + y.$$

Proof. If (X, Y) is the point of $J(2, n)$ with $X = k/2^n$, then, since k is odd,

$$X = t_1/2 + \cdots + t_{n-1}/2^{n-1} + 1/2^n,$$

and so $Y \ge \frac{1}{2}$. This implies that $S(k/2^n, y) = S((k + 1)/2^n, y)$, and the lemma follows.

3. The average error of $J(R, n)$ at the lattice points

THEOREM 1. *Let $T(R, n) = (1/R^{2n}) \sum_{k=1}^{R^n} \sum_{l=1}^{R^n} \Delta_n(k/R^n, l/R^n)$. Then*

$$T(R, n) = nT(R, 1) = n(R - 1)(R + 1)/12R.$$

Proof. Part 1. In the first part of the proof, let q and l be fixed integers such that $1 \le q \le R$ and $(q - 1)/R < l/R^n \le q/R$, and let $l = (q - 1)R^{n-1} + t$ with $1 \le t \le R^{n-1}$. We shall obtain an expression for the sum of the errors at the lattice points on the line $y = l/R^n$.

Part 1(a). $0 < k/R^n \le (q - 1)/R^n$.

$$\begin{aligned} &\Delta_n(k/R^n, l/R^n) \\ &\quad = k - kl/R^n = \min(k, q - 1) - k(q - 1)/R + k(q - 1)/R - kl/R^n \\ &\quad = \Delta_1(k/R, (q - 1)/R) - kt/R^n. \end{aligned}$$

Therefore

$$\sum_{k=1}^{q-1} \Delta_n(k/R^n, l/R^n) = \sum_{k=1}^{q-1} \Delta_1(k/R, (q - 1)/R) - tq(q - 1)/2R^n.$$

Part 1(b). $q/R^n \le k/R^n \le ((R^{n-1} - 1)R + q - 1)/R^n$. If p is any integer such that $1 \le p \le R^{n-1} - 1$, we may consider two subcases:

Case 1. $((p - 1)R + q)/R^n \le k/R^n \le p/R^{n-1}$. If we write

$k = (p - 1)R + j$ with $q \leq j \leq R$, then by Lemmas 3(b) and 4,

$$\Delta_n(k/R^n, l/R^n)$$
$$= \Delta_1(j/R, (q - 1)/R) + \Delta_{n-1}(p/R^{n-1}, t/R^{n-1}) + t/R^{n-1} - jt/R^n.$$

Therefore

$$\sum_{k=(p-1)R+q}^{pR} \Delta_n(k/R^n, l/R^n)$$
$$= \sum_{j=q}^{R} \Delta_n(((p - 1)R + j)/R^n, l/R^n)$$
$$= \sum_{j=q}^{R} \Delta_1(j/R, (q - 1)/R) + (R - q + 1)\Delta_{n-1}(p/R^{n-1}, t/R^{n-1})$$
$$+ (t/2R^n)(R^2 + R - 2qr - q + q^2).$$

Case 2. $p/R^{n-1} < k/R^n \leq (pR + q - 1)/R^n$. If we write $k = pR + j$ with $1 \leq j \leq q - 1$, then by Lemmas 3(b) and 4,

$$\Delta_n(k/R^n, l/R^n) = \Delta_1(j/R, (q - 1)/R) + \Delta_{n-1}(p/R^{n-1}, t/R^{n-1}) - jt/R^n.$$

Therefore

$$\sum_{k=pR+1}^{pR+q-1} \Delta_n(k/R^n, l/R^n)$$
$$= \sum_{j=1}^{q-1} \Delta_n((pR + j)/R^n, l/R^n)$$
$$= \sum_{j=1}^{q-1} \Delta_1(j/R, (q-1)/R) + (q-1)\Delta_{n-1}(p/R^{n-1}, t/R^{n-1}) - tq(q - 1)/2R^n.$$

Combining Cases 1 and 2 we obtain

$$\sum_{k=(p-1)R+q}^{pR+q-1} \Delta_n(k/R^n, l/R^n)$$
$$= \sum_{j=1}^{R} \Delta_1(j/R, (q - 1)/R) + R\Delta_{n-1}(p/R^{n-1}, t/R^{n-1})$$
$$+ (t/2R^n)(R^2 + R - 2qR).$$

Therefore

$$\sum_{k=q}^{(R^{n-1}-1)R+q-1} \Delta_n(k/R^n, l/R^n)$$
$$= \sum_{p=1}^{R^{n-1}-1} \sum_{k=(p-1)R+q}^{pR+q-1} \Delta_n(k/R^n, l/R^n)$$
$$= (R^{n-1} - 1) \sum_{j=1}^{R} \Delta_1(j/R, (q - 1)/R)$$
$$+ R \sum_{p=1}^{R^{n-1}-1} \Delta_{n-1}(p/R^{n-1}, t/R^{n-1})$$
$$+ (t/2)(1 + R - 2q) + (t/2R^n)(-R^2 - R + 2qR).$$

Part 1(c). $((R^{n-1} - 1)R + q)/R^n \leq k/R^n \leq 1$. If we write $k = (R^{n-1} - 1)R + j$ with $q \leq j \leq R$, then by Lemma 3(c),

$$\Delta_n(k/R^n, l/R^n) = l - kl/R^n = (q - 1) - j(q - 1)/R + (t/R^n)(R - j)$$
$$= \min(j, q - 1) - j(q - 1)/R + (t/R^n)(R - j)$$
$$= \Delta_1(j/R, (q - 1)/R) + (t/R^n)(R - j).$$

Therefore

$\sum_{k=(R^{n-1}-1)R+q}^{R^n} \Delta_n(k/R^n, l/R^n)$

$\quad = \sum_{j=q}^{R} \Delta_1(j/R, (q-1)/R) + (t/2R^n)(R^2 + R - 2qR - q + q^2).$

Combining parts 1(a), 1(b), and 1(c), we obtain

$\sum_{k=1}^{R^n} \Delta_n(k/R^n, l/R^n)$

$\quad = \sum_{k=1}^{q-1} \Delta_1(k/R, (q-1)/R) - tq(q-1)/2R^n$

$\qquad + (R^{n-1} - 1) \sum_{j=1}^{R} \Delta_1(j/R, (q-1)/R)$

$\qquad + R\sum_{p=1}^{R^{n-1}-1} \Delta_{n-1}(p/R^{n-1}, t/R^{n-1})$

$\qquad + (t/2)(1 + R - 2q) + (t/2R^n)(-R^2 - R + 2qR)$

$\qquad + \sum_{j=q}^{R} \Delta_1(j/R, (q-1)/R) + (t/2R^n)(R^2 + R - 2Rq - q + q^2)$

$\quad = R^{n-1}\sum_{j=1}^{R} \Delta_1(j/R, (q-1)/R) + R\sum_{p=1}^{R^{n-1}-1} \Delta_{n-1}(p/R^{n-1}, t/R^{n-1})$

$\qquad + (t/2)(1 + R - 2q).$

Part 2. We now find the sum of the errors at all lattice points of the unit square.

$\sum_{l=1}^{R^n} \sum_{k=1}^{R^n} \Delta_n(k/R^n, l/R^n)$

$\quad = \sum_{q=1}^{R} \sum_{l=(q-1)R^{n-1}+1}^{qR^{n-1}} \sum_{k=1}^{R^n}\Delta_n(k/R^n, l/R^n)$

$\quad = \sum_{q=1}^{R}\sum_{t=1}^{R^{n-1}} \sum_{k=1}^{R^n} \Delta_n(k/R^n, ((q-1)R^{n-1} + t)/R^n)$

$\quad = \sum_{q=1}^{R}\sum_{t=1}^{R^{n-1}} \{R^{n-1}\sum_{j=1}^{R} \Delta_1(j/R, (q-1)/R)$

$\qquad + R\sum_{p=1}^{R^{n-1}-1} \Delta_{n-1}(p/R^{n-1}, t/R^{n-1}) + (t/2)(1 + R - 2q)\}$

$\quad = \sum_{t=1}^{R^{n-1}} \{R^{n-1}\sum_{q=1}^{R}\sum_{j=1}^{R} \Delta_1(j/R, (q-1)/R)$

$\qquad + R^2\sum_{p=1}^{R^{n-1}-1} \Delta_{n-1}(p/R^{n-1}, t/R^{n-1}) + (t/2)(R + R^2 - 2\sum_{q=1}^{R} q)\}$

$\quad = R^{2n-2}\sum_{q=1}^{R}\sum_{j=1}^{R} \Delta_1(j/R, (q-1)/R)$

$\qquad + R^2\sum_{t=1}^{R^{n-1}} \sum_{p=1}^{R^{n-1}-1} \Delta_{n-1}(p/R^{n-1}, t/R^{n-1})$

$\quad = R^{2n-2}\sum_{q=1}^{R}\sum_{j=1}^{R} \Delta_1(j/R, q/R)$

$\qquad + R^2\sum_{t=1}^{R^{n-1}}\sum_{p=1}^{R^{n-1}} \Delta_{n-1}(p/R^{n-1}, t/R^{n-1}).$

The last equality follows since $\Delta_1(j/R, 0) = \Delta_1(j/R, 1) = \Delta_{n-1}(p/R^{n-1}, 1) = 0$.

Therefore $T(R, n) = T(R, 1) + T(R, n-1) = nT(R, 1)$, and the theorem is proved.

COROLLARY. *For the sequence* $J(R, n)$,

$$\int_0^1 \int_0^1 \Delta(x, y)\, dx\, dy = n\,\frac{(R-1)(R+1)}{12R} + \frac{1}{2} + \frac{1}{4R^n}.$$

4. The maximum error of $J(2, n)$

THEOREM 2. *Let M_n denote the maximum error of $J(2, n)$ at the lattice points of the unit square. Then*
(1) *M_n is attained at exactly two lattice points*

$$(A_n/2^n, B_n/2^n) \quad and \quad (1 - A_n/2^n, 1 - B_n/2^n),$$

where

$$A_n = (3 \cdot 2^{n-1} + (-2)^{n-1} - 1)/3 \quad and \quad B_n = (2^n - (-1)^n)/3.$$

(2) $M_n = n/3 + \frac{1}{9}(1 - (-\frac{1}{2})^n)$.

Proof. By constructing the sequence $J(2, 1)$, we see the theorem is true for $n = 1$. We assume it is true for $J(2, n - 1)$, and establish it for $J(2, n)$ where $n \geq 2$.

If we denote $M_n = \Delta_n(A_n/2^n, B_n/2^n)$, we have by Lemmas 7, 2, and 8,

$$\begin{aligned} M_{n-1} &= \Delta_{n-1}(A_{n-1}/2^{n-1}, B_{n-1}/2^{n-1}) \\ &= \Delta_{n-1}(2(2^{n-1} - A_{n-1})/2^n, 2(2^{n-1} - B_{n-1})/2^n) \\ &= \Delta_n(2(2^{n-1} - A_{n-1})/2^n, (2^{n-1} - B_{n-1})/2^n) \\ &= \Delta_n((2(2^{n-1} - A_{n-1}) - 1)/2^n, (2^{n-1} - B_{n-1})/2^n) - (2^{n-1} - B_{n-1})/2^n. \end{aligned}$$

We define A_n and B_n by the recursion formulas:

$$\begin{aligned} A_1 &= 1, \qquad A_n = 2(2^{n-1} - A_{n-1}) - 1 \qquad (n \geq 2). \\ B_1 &= 1, \qquad B_n = 2^{n-1} - B_{n-1} \qquad (n \geq 2). \end{aligned}$$

and obtain the recursion formula for M_n :

$$M_1 = \tfrac{1}{2}, \qquad M_n = M_{n-1} + B_n/2^n.$$

From these recursion formulas we obtain the formulas for A_n , B_n , and M_n stated in the theorm. We must show that at each lattice point $(k/2^n, l/2^n)$ different from $(A_n/2^n, B_n/2^n)$ and $(1 - A_n/2^n, 1 - B_n/2^n)$, we have $\Delta_n(k/2^n, l/2^n) < M_n$.

This is true for $J(2, 1)$ so we assume it is true for $J(2, n - 1)$ and establish it for $J(2, n)$. Let $(k/2^n, l/2^n)$ be any lattice point as described above. By Lemma 7 we may assume $l/2^n \leq \frac{1}{2}$.

Case 1. k even. Therefore by Lemma 2,

$$\Delta_n(k/2^n, l/2^n) = \Delta_{n-1}(\tfrac{1}{2}k/2^{n-1}, l/2^{n-1}) \leq M_{n-1} < M_n .$$

Case 2. k odd, and $l < B_n$. Therefore by Lemma 8 and Case 1,

$$\Delta_n(k/2^n, l/2^n) = \Delta_n((k + 1)/2^n, l/2^n) + l/2^n \leq M_{n-1} + l/2^n < M_n .$$

Case 3. k odd, $l = B_n$, and $k \neq A_n$. Therefore by Lemmas 8 and 2,

$$\Delta_n(k/2^n, l/2^n) = \Delta_n((k + 1)/2^n, B_n/2^n) + B_n/2^n$$

$$= \Delta_{n-1}(\tfrac{1}{2}(k+1)/2^{n-1}, (2^{n-1} - B_{n-1})/2^{n-1}) + B_n/2^n$$
$$< M_{n-1} + B_n/2^n = M_n .$$

The last inequality holds, for otherwise $\frac{1}{2}(k + 1) = 2^{n-1} - A_{n-1}$ which would imply $k = A_n$.

Case 4. k odd, and $l > B_n$. Therefore by Lemmas 8, 2, and 7,

$$\begin{aligned}\Delta_n(k/2^n, l/2^n) &= \Delta_n((k+1)/2^n, l/2^n) + l/2^n \\ &= \Delta_{n-1}(\tfrac{1}{2}(k+1)/2^{n-1}, l/2^{n-1}) + l/2^n \\ &= \Delta_{n-1}((2^{n-1} - \tfrac{1}{2}(k+1))/2^{n-1}, (2^{n-1} - l)/2^{n-1}) + l/2^n.\end{aligned}$$

We note that $(2^{n-1} - l)/2^{n-1} \leq \frac{1}{2}$ and consider two subcases:

Case 4(a). $2^{n-1} - \frac{1}{2}(k + 1)$ is even. Then by Case 1,

$$\begin{aligned}&\Delta_{n-1}((2^{n-1} - \tfrac{1}{2}(k+1))/2^{n-1}, (2^{n-1} - l)/2^{n-1}) + l/2^n \\ &\leq M_{n-2} + l/2^n \\ &= M_n - B_n/2^n - B_{n-1}/2^{n-1} + l/2^n \\ &= M_n - B_n/2^n - (2^{n-1} - B_n)/2^{n-1} + l/2^n < M_n .\end{aligned}$$

The last inequality follows since $B_n/2^n < l/2^n \leq \frac{1}{2}$.

Case 4(b). $2^{n-1} - \frac{1}{2}(k + 1)$ is odd. Then by Lemma 8 and a result shown in Case 4(a),

$$\begin{aligned}&\Delta_{n-1}((2^{n-1} - \tfrac{1}{2}(k+1))/2^{n-1}, (2^{n-1} - l)/2^{n-1}) + l/2^n \\ &= \Delta_{n-1}((2^{n-1} - \tfrac{1}{2}(k+1) + 1)/2^{n-1}, (2^{n-1} - l)/2^{n-1}) + l/2^n \\ &\quad + (2^{n-1} - l)/2^{n-1} \\ &\leq M_n - B_n/2^n - (2^{n-1} - B_n)/2^{n-1} + l/2^n + (2^{n-1} - l)/2^{n-1} < M_n .\end{aligned}$$

This completes the proof of Theorem 2.

COROLLARY. *For the sequence* $J(2, n)$, $n/3 < D < n/3 + 3$.

Proof. $D > M_n > n/3$. By Lemma 5, $D = \sup \Delta(x, y)$. Let (x, y) be any point of the unit square. We may assume $x \neq 0$ and $y \neq 0$, and we choose the lattice point $(k/2^n, l/2^n)$ such that

$$(k-1)/2^n < x \leq k/2^n, \qquad (l-1)/2^n < y \leq l/2^n.$$

Then $S(x, y) = S(k/2^n, l/2^n) \leq S((k - 1)/2^n, (l - 1)/2^n) + 2$, and so

$$\Delta(x, y) < \Delta((k-1)/2^n, (l-1)/2^n) + 2 \leq M_n + 2.$$

Therefore $D \leq M_n + 2 < n/3 + 3$.

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