

# SOME REMARKS ON EXTENSIBILITY, CONFLUENCE OF PATHS, BRANCHING PROPERTIES, AND INDEX SETS, FOR CERTAIN RECURSIVELY ENUMERABLE GRAPHS<sup>1</sup>

BY

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## 1. Introduction

This paper embraces three main sections, the contents of which are inter-related chiefly by their common placement or origin in the theory of regressive sets and regressing functions. In order to avoid yet another recital of notational and terminological conventions, we simply refer the reader to the recital given in §§2, 4 of [1] (which follows, in the main, the conventions established in [2]). A number of the results contained in this paper were either (a) found by someone other than the present author, or (b) found independently by the present author and someone else; we have attempted to be scrupulous in drawing attention to such cases.

## 2. Potential recursiveness of regressing functions

In [5] we gave a proof of the existence of retraceable sets having no general recursive retracing functions. The existence of such sets having been noted, a natural question is this: is there a retraceable set whose elements cannot be ordered so as to admit, relative to that ordering, a regressing function with general recursive extension? The answer, as we shall see below, is yes, even if a number of additional requirements are imposed. We shall, moreover, demonstrate (Theorem 2) the existence of a retraceable set  $\alpha$  such that  $\alpha$  has no general recursive retracing function but a suitable ordering of  $\alpha$  admits a regressing function with general recursive extension. We remark that Robinson [6] has announced a result—also obtained, independently and in stronger form, by A. H. Lachlan—from which a slightly weaker form of our *Theorem 1* follows as a rather easy corollary. However, we wish to state *Theorems 1* and *3* in the strongest formulations known to us; also, we wish to introduce *Lemma 1* into the literature as an item of independent interest (this lemma, previously unpublished, is due to D. A. Martin).

**LEMMA 1** (Martin).<sup>2</sup> *Let  $\alpha$  be an infinite set of natural numbers. The following condition on  $\alpha$  is necessary and sufficient for the existence of an infinite set  $\beta$  such that  $\beta \subset \alpha$  and  $\beta$  is retraced by a general recursive function: there is a general recursive function  $g(x, y)$  such that*

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<sup>2</sup> We are indebted to Martin for his permission to include a proof of *Lemma 1*. The proof given here is our own; we do not know Martin's proof, but conjecture it to be substantially the same as ours.

(1)  $(\forall i)(g(i, y) \text{ is the characteristic function of a set which intersects } \alpha)$ ,  
and

(2)  $(\forall i)(\forall j)(\{y \mid g(i, y) = 0\} \cdot \{y \mid g(j, y) = 0\} = \emptyset \text{ if } i \neq j)$ .

*Proof.* The proof of necessity is quite easy; suffice it to remark that it is very similar to the first few lines of the proof of *Corollary 1* below. Sufficiency is a little trickier; we shall now describe a construction for establishing it. Suppose then that  $g(x, y)$  is a recursive function related to  $\alpha$  as in (1) and (2). The idea is to use heavily the fact that for all  $j$ ,  $\{y \mid \exists i \leq j)(g(i, y) = 0)\}$  is recursive, in order to construct by stages a recursive function  $f$  which retraces an infinite subset of  $\alpha$ . Let  $\beta = \bigcup_n \{y \mid g(n, y) = 0\}$ , and let  $h$  be a 1-1 recursive function with range  $\beta$ .

*Stage 0.* Place in  $f$  all pairs  $(x, x)$  such that  $g(0, x) = 0$ . Compute  $h(0)$ . Let  $k_0$  be the least  $k$  such that  $k > 0$  and  $(\forall y)(g(k, y) = 0 \Rightarrow y > h(0))$ . Place in  $f$  all pairs  $(x, h(0))$  such that  $g(k_0, x) = 0$ . Then go to Stage 1.

*Stage  $s + 1$ .* Compute  $h(s + 1)$ . Let  $r_0$  be the least  $r$  such that (i) no member of  $\{y \mid g(r, y) = 0\}$  has been added to the domain of  $f$  at an earlier stage, and (ii)

$$(\forall y)(\forall t \leq s)(g(r, y) = 0 \Rightarrow y > h(0) + h(t + 1)).$$

Place in  $f$  all pairs  $(x, h(s + 1))$  such that  $g(r_0, x) = 0$ . Finally, if  $t < s$  and  $t$  has not yet been added to the domain of  $f$ , place  $(t, 0)$  in  $f$ ; then proceed to Stage  $s + 2$ .

It is evident that  $f$ , so defined, is a recursive function. Let  $a_0$  be a particular element of  $\alpha$  such that  $g(0, a_0) = 0$ . Then  $(a_0, a_0)$  enters  $f$  at stage 0. Now,  $a_0 = h(l)$  for some uniquely determined  $l$ . It follows from the construction that, at stage  $l$ , all pairs  $(x, a_0)$  for which  $g(l^*, x) = 0$  enter  $f$ , for some  $l^*$  determined by  $l$ . Let  $a_1$  satisfy  $g(l^*, a_1) = 0$  &  $a_1 \in \alpha$ . Then by the construction,

$$a_1 > a_0 \quad \& \quad a_1 > \max \{h(0), h(1), \dots, h(l)\}.$$

Hence there is a uniquely determined  $k, k > l$ , such that, at stage  $k$ ,  $a_1 = h(k)$  and all pairs  $(x, a_1)$  for which  $g(k^*, x) = 0$  enter  $f$ , for a certain  $k^*$  determined by  $k$ . Let  $a_2$  satisfy  $g(k^*, a_2) = 0$  &  $a_2 \in \alpha$ . Then

$$a_2 > a_1 \quad \& \quad a_2 > \max \{h(0), \dots, h(k)\}.$$

Continuing in this way, we obtain a sequence  $a_0, a_1, a_2, \dots$ , with  $a_i < a_{i+1}$  &  $a_i \in \alpha$  for all  $i \geq 0$ , such that  $f$  retraces  $\{a_0, a_1, \dots\}$ . ■

**COROLLARY 1.** *Let  $\alpha$  be an infinite set of numbers, and suppose that there exist a number  $n_0$  and two general recursive functions  $f$  and  $t$ , with  $t$  strictly increasing, such that, for every  $m$ ,  $\alpha$  contains a number  $q_m$  for which*

(i)  $f^{t(m)}(q_m) = n_0$  and

(ii)  $k < t(m) \Rightarrow f^k(q_m) \neq n_0$ .

(As usual,  $f^0(x) = x, f^{h+1}(x) = f(f^h(x))$ , for all  $x$  and  $h$ .) Then  $\alpha$  has an

infinite subset retraceable by a general recursive function (and hence, by [5, Lemma 6],  $\alpha$  can be split by a recursive set).

*Proof.* Let a function  $r(x, y)$  be defined as follows:

$$\begin{aligned} r(x, y) &= 0 \quad \text{if } f^{t(x)}(y) = n_0 \text{ and } (\forall k < t(x))(f^k(y) \neq n_0); \\ &= 1 \quad \text{otherwise.} \end{aligned}$$

Clearly,  $r(x, y)$  is general recursive; and, for each  $m$ ,  $r(m, y)$  is a characteristic function. Moreover, it is evident that

$$m \neq n \Rightarrow \{x \mid r(m, x) = 0\} \cdot \{x \mid r(n, x) = 0\} = \emptyset.$$

From the hypotheses concerning  $t$ , we see that

$$(\forall m)(\{x \mid r(m, x) = 0\} \cdot \alpha \neq \emptyset).$$

Hence, by Lemma 1,  $\alpha$  has an infinite subset  $\beta$  such that  $\beta$  is retraced by a general recursive function. ■

The next lemma is, like Lemma 1, an unpublished theorem of D. A. Martin. We shall indicate the construction needed for its proof, and we shall point out those properties of the construction which show it to have the desired effect; beyond this, however, we leave details of verification to the reader.

**LEMMA 2 (Martin).** *There exists an infinite set  $\alpha$ , a recursive sequence  $\{\omega_{\varphi(n)}\}$  of disjoint finite sets, and a recursive function  $r$ , such that*

- (1)  $\alpha$  cannot be split by a recursive set,
- (2)  $(\forall n)(\omega_{\varphi(n)} \cdot \alpha \neq \emptyset)$ , and
- (3)  $(\forall n)(r(n) > |\omega_{\varphi(n)}|)$ .

*Sketch of proof.*<sup>3</sup> Let  $R_0^s, R_1^s, R_2^s, \dots$  be a simultaneous enumeration of all recursive sets, carried out so that all but finitely many  $R_n^s = \emptyset$  for each  $s$  and, for each  $n$  and  $s$ , either

$$R_n^{s+1} - R_n^s = \emptyset \quad \text{or} \quad R_n^{s+1} - R_n^s = \{k\}$$

for some  $k$ , and, in the latter case,  $k > \max \{x \mid x \in R_n^s\}$ . For each number  $n$  and each subset  $S_{n_i}$  of  $\{0, 1, \dots, n\}$ , we set aside a finite set  $G_{n_i}$  of numbers which is sufficiently large to insure that at least  $n + 1$  members of  $G_{n_i}$  all belong to exactly the same sets  $R_j$  for  $j \in S_{n_i}$ ; this can easily be done, and is to be done, in such a way that the  $G_{n_i}$  form a strong array; i.e.,  $\{G_{n_i}\}$  is a recursive sequence of disjoint sets such that  $|G_{n_i}|$  is a recursive function of  $n$  and  $S_{n_i}$ . The sequence  $\{\omega_{\varphi(n)}\}$  is obtained from a certain constructed se-

<sup>3</sup> The construction here sketched is exactly the one used by Martin, in September 1963, to obtain Lemma 2. (Though Martin did not explicitly refer, in his communication to the author at that time, to property (3), there is negligible room for doubt as to his awareness of it. It is precisely property (3) which gives Lemma 2 an advantage, for our present purposes, over the Lachlan-Robinson result (which in other respects is stronger, but for which, to the author's knowledge, property (3) has not been verified).)

quence  $A_0, A_1, \dots$  as follows. We assume that an arrangement is being followed whereby each  $S_{n_i}$  and associated  $G_{n_i}$  is considered at each of infinitely many stages. At stage  $s$ , suppose we are considering  $G_{n_i}$ . If some  $n + 1$  of the members of  $G_{n_i}$  are, by inspection of  $R_0^s, R_1^s, \dots$ , proven to lie in exactly the same  $R_j$  for each  $j \in S_{n_i}$ , then we choose the least  $n + 1$  such elements of  $G_{n_i}$  and place one of them in each of  $A_0, \dots, A_n$ , with the following exception: If  $k \leq n$  and  $A_k$  already contains a number  $q$  such that  $q$  has by now been proven to be in exactly the same  $R_j$  as the  $n + 1$  numbers in question, for each  $j \leq k$  such that  $j \in S_{n_i}$ , then we abstain from giving  $A_k$  a new member. The sets  $A_k$  obtained from this procedure, as  $s \rightarrow \infty$ , are clearly nonempty, finite and disjoint; further, it is easily seen that there is a recursive function  $r$  such that  $(\forall n)(r(n) > |A_n|)$ . We let  $\varphi$  be a recursive function such that  $\omega_{\varphi(n)} = A_n$  for all  $n$ . Now suppose  $J_0, J_1, \dots, J_n$  is given, where each  $J_i, 0 \leq i \leq n$ , is either  $R_i$  or  $R'_i$ . Call  $J_0, J_1, \dots, J_n$  an  $n$ -sequence provided, for each  $k \leq n$ ,  $A_k$  contains a member of  $\bigcap_{j \leq k, R_j \text{ infinite}} J_j$ . It is easily seen from the construction that an  $n$ -sequence exists for every  $n$ . We order the  $n$ -sequences by the stipulation that  $J_0, J_1, \dots, J_n$  is lower than  $K_0, K_1, \dots, K_n$  just in case  $J_i = R'_i$  and  $K_i = R_i$  where  $i$  is the smallest integer such that  $J_i \neq K_i$ . For each  $n$ , let  $J_0^n, \dots, J_n^n$  be the lowest  $n$ -sequence. It is straightforward to verify that, for arbitrary  $n$ , there exists an  $m, m > n$ , such that  $q \geq m \Rightarrow J_n^q = J_n^m$ ; i.e.,  $\lim_m J_n^m$  exists for all  $n$ . Finally, it is easily seen from the foregoing discussion that each set  $A_k$  contains a least number  $a_k$  such that

$$a_k \in \bigcap_{j \leq k, R_j \text{ infinite}} \lim_m J_j^m;$$

letting  $\alpha = \{a_1, a_2, \dots\}$ , it can now be checked that  $\alpha$  is not split by any recursive set. ■

Before turning to *Theorem 1*, we prove an incidental corollary to *Lemma 2* which illustrates nicely the utility of property (3). In [4], Gonshor announced the following result in answer to a question of Dekker: there exist recursive equivalences between hyperimmune and non-hyperimmune sets. We point out that a stronger form of this assertion follows from *Lemma 2*. We first remark that, in Martin's terminology, an infinite set  $\alpha$  which does not satisfy the condition of *Lemma 1* is called *superimmune*; it is easy to show that any superimmune set is hyperimmune (i.e., not bounded by any recursive function), but not conversely. We further remark that, by *Lemma 1* and [5, *Lemma 6*], a set which cannot be split by a recursive set must be a superimmune; hence, the set  $\{a_1, a_2, \dots\}$  constructed for *Lemma 2* is superimmune.

**COROLLARY 2.** *There exists a superimmune set  $\alpha$  and a 1-1 partial recursive function  $f$  defined on  $\alpha$  such that  $f(\alpha)$  is not hyperimmune.*

*Proof.* Let  $\alpha$  be  $\{a_1, a_2, \dots\}$ , the set constructed for *Lemma 2*; and let  $r$  be a recursive function bounding the cardinalities of the sets  $A_k$  occurring in the proof of *Lemma 2*. Define a recursive function  $t$  as follows:

$$t(0) = r(0), \quad t(k+1) = t(k) + 1 + r(k+1).$$

Let  $p$  be any 1-1 partial recursive function with the following property:

$$\text{if } n \in A_0, \text{ then } p(n) \in \{0, \dots, r(0)\},$$

and

$$\text{if } n \in A_{k+1}, \text{ then } p(n) \in \{t(k) + 1, t(k) + 2, \dots, t(k+1)\}.$$

Then clearly, using the "array" definition of hyperimmunity (see [3]), we have that  $p(\alpha)$  is not hyperimmune. ■

We now proceed to state three definitions, and then *Theorem 1*. For the definitions of *basic* and *closed basic* retracing functions, the reader is referred to [9]; our definitions are the obvious flankers to Yates' notion of a closed basic retracing function.

**DEFINITION.** Let  $f$  be a basic retracing function; and let  $r$  be a recursive function enumerating the range of  $f$ .  $f$  is *bounded* if and only if there is a recursive function  $g$  such that, for every  $n$ ,  $g(n) > |f^{-1}(r(n))|$ .

**DEFINITION.** Let  $f$  be a basic retracing function; and let  $r$  be a recursive function enumerating the range of  $f$ .  $f$  is *weakly closed* if and only if there is a recursive function  $g$  such that, for every  $n$ ,  $f^{-1}(r(n)) \subset D_{g(n)}$ . (The sequence  $\{D_m\}$  is defined at the beginning of §3.)

**DEFINITION.** Let  $f$  be a closed basic retracing function.  $f$  is *strongly closed* if and only if its range is a recursive set.

**THEOREM 1.** *There exists an infinite set  $\beta$  of numbers such that*

(1)  *$\beta$  is retraced by a bounded basic retracing function; and*

(2) *neither  $\beta$  nor any of its infinite subsets can be sequentially ordered so as to admit a potentially recursive regressing function.*

*Proof.* Let  $\{\omega_{\varphi(n)}\}$  and  $\alpha$  be as in *Lemma 2*. We define a certain bounded, basic retracing function  $h$  by means of the construction used by Yates to prove [9, *Theorem 6*]; however, as is done in [7, Chapter 12], we apply this construction in a way appropriate to defining a *finite-to-one* function. Let  $r$  be a recursive function such that  $r(n) > |\omega_{\varphi(n)}|$  for all  $n$ , and let  $\Psi$  be a recursive function such that  $\rho\Psi = \bigcup_n \omega_{\varphi(n)}$  and  $\Psi^{-1}(k)$  is infinite for each  $k \in \bigcup_n \omega_{\varphi(n)}$ .

*Stage 0.* Place in  $h$  all pairs  $(x, x)$  such that  $x \in \omega_{\varphi(0)}$ .

*Stage  $s + 1$ .* Compute  $\Psi(s)$ . Let  $m_s$  be that number  $m$  such that, at the conclusion of stage  $s$ , precisely the sets  $\omega_{\varphi(0)}, \omega_{\varphi(1)}, \dots, \omega_{\varphi(m)}$  have made contributions to  $\delta h$ . If  $\Psi(s)$  is not yet in  $\delta h$ , or if  $\Psi(s)$  has previously entered  $\rho h$  other than at Stage 0, go on to Stage  $s + 2$ . Otherwise: place in  $h$  precisely those pairs  $(x, \Psi(s))$  such that

$$(i) \ x > \Psi(s) \quad \text{and} \quad (ii) \ x \in \bigcup_{m_s < k \leq m_s + \Psi(s) + 1} \omega_{\varphi(k)}.$$

This construction evidently determines  $h$  as a partial recursive function with all properties of a retracing function (see [9]) save possibly that of

retracing some infinite set. Moreover, in view of the fact that  $r(n) > |\omega_{\varphi(n)}|$  holds for all  $n$ , it is clear that, given a 1-1 recursive function  $g$  which enumerates  $\rho h$ , there is a recursive function  $\xi$  such that, for all  $n$ ,  $\xi(n) > |h^{-1}(g(n))|$ . The next step is to check that  $h$  retraces an infinite subset  $\beta$  of  $\alpha$ . But, some member  $b_0$  of  $\alpha$  is in  $\omega_{\varphi(0)}$ , and we therefore have  $(b_0, b_0) \in h$ . Then, at the first stage  $s$  for which  $\Psi(s) = b_0$ , we adjoin to  $h$  any pair  $(x, b_0)$  such that  $x > b_0$  &  $x \in \bigcup_{m_s < k \leq m_s + b_0 + 1} \omega_{\varphi(k)}$ . But each  $\omega_{\varphi(k)}$  contains some member of  $\alpha$ ; hence  $\bigcup_{m_s < k \leq m_s + b_0 + 1} \omega_{\varphi(k)}$  contains at least  $b_0 + 1$  members of  $\alpha$ , at least one of which, say  $b_1$ , is greater than  $b_0$ . Therefore  $(b_1, b_0)$  enters  $h$  at stage  $s$ . Then, since  $\Psi^{-1}(b_1)$  is infinite, there comes a first stage  $s_1 > s$  at which  $b_1 = \Psi(s_1)$ , etc., etc. Thus  $h$  retraces some infinite subset  $\beta$  of  $\alpha$ . But this is a suitable  $\beta$  for our theorem. For if  $\beta$  admitted, in any ordering, a potentially recursive regressing function, then, by *Corollary 1* (with  $t =$  the identity function,  $f =$  a recursive extension of a regressing function for  $\beta$ , and  $n_0 =$  the fixed point of  $\beta$  under  $f$ ), we would have that  $\beta$  can be split by a recursive set; this, however, contradicts *Lemma 2* in view of our choice of  $\alpha$ . Similarly for any infinite subset of  $\beta$ . ■

Our next theorem shows that the affirmative answer to the question posed at the beginning of this section is not simply a corollary to [5]. Let  $(\ , \ )$  be a 1-1 recursive "pairing" function mapping  $N \otimes N$  onto  $N$ ; here of course the symbol  $\otimes$  denotes cartesian product. We place the usual requirement on  $(\ , \ )$  that there be recursive functions  $h_1$  and  $h_2$  such that  $(h_1(x), h_2(x)) = x$  holds for every  $x$ .

**THEOREM 2.** *There is a continuum of retraceable sets  $\alpha$  such that*

- (1)  $\alpha$  admits a general recursive regressing function; but
- (2)  $\alpha$  admits no general recursive retracing function.

*Proof.* Let  $\{A_{f(n)}\}$  be an r.e. sequence of pairwise disjoint r.e. sets; and let  $\{S_n\}$  be a sequence of infinite sets such that  $(\forall n)(S_n \subset A_{f(n)})$  and  $\bigcup_n S_n$  cannot be recursively split. (For the existence of such sequences  $\{A_{f(n)}\}$  and  $\{S_n\}$ , we may cite either *Lemma 2* above (simply lump together the elements of disjoint recursive subsequences) or *Theorem 2* of [5].) Let

$$B = \{0\} \otimes \bigcup_n A_{f(n)}, \quad B_n = \{0\} \otimes A_{f(n)}, \quad B_n^* = \{0\} \otimes S_n.$$

We shall construct, in stages, a general recursive regressing function  $r$ . We introduce some terminology to be applied to our function  $r$ . For each natural number  $s$ ,  $r^{(s)} = \{(m, n) \mid (m, n) \text{ has been placed in } r \text{ by the conclusion of stage } s\}$ . Suppose that there exists a sequence  $y_1, \dots, y_k$  ( $k > 1$ ) such that  $y_k = (0, 0)$  and, for  $1 \leq t \leq k - 1$ ,  $(y_t, y_{t+1}) \in r^{(s)}$ . Then we denote this sequence by  $\hat{r}(y_1)$ , and we say that  $y_1$  is *complete at stage  $s + 1$*  (or simply *complete*, if there is no danger of confusion as to what stage we are referring to). Suppose that  $y_1$  is complete at stage  $s + 1$ ; and suppose, moreover, that  $\hat{r}(y_1)$  has the following form:

$$\hat{r}(y_1) = z_{11}, \dots, z_{1r_1}, \dots, z_{k1}, \dots, z_{kr_k}, 0,$$

where (i)  $k \geq 1$  &  $(1 \leq l \leq k \Rightarrow r_l \geq 1)$ , (ii) for  $1 \leq l \leq k$ , the block  $z_{11}, \dots, z_{1r_1}$  is of the form  $(0, j_l), (1, j_l), \dots, (r_l - 1, j_l)$ , and (iii) for  $1 \leq l < k$ ,  $z_{11} > z_{l+1, r_{l+1}}$ .

Under these circumstances, we shall say that  $\hat{r}(y_1)$  is *admissible at stage*  $s + 1$  (or simply *admissible* if there is no attendant danger of confusion). Finally, if  $y_1$  is in the domain of  $r^{(s)}$  and if, at the conclusion of stage  $s$ , there is a sequence  $z_1, \dots, z_t, t \geq 1$ , such that  $(y_1, z_1) \in r^{(s)}$  and

$$1 \leq m < t \Rightarrow (z_m, z_{m+1}) \in r^{(s)}$$

and  $z_t$  is not yet in the domain of  $r$ , then we say that  $z_t$  is the *s-end* of  $y_1$ , and we denote  $z_t$  by  $y_1^{*s}$ .

We are now ready to describe the construction of  $r$ . We assume, with no loss of generality, that  $0 \notin \bigcup_n A_{f(n)}$  (so that  $(0, 0) \notin B$ ), and we let  $g$  be a 1-1 recursive function with range  $B$ . We also assume that our pairing function is such that

$$(0, 0) = 0, \quad k \geq l \quad \& \quad m \geq r \Rightarrow (k, m) \geq (l, r).$$

*Stage 0.* Place  $((0, 0), (0, 0))$  in  $r$ , mark  $(1, 0)$ , and proceed to Stage 1.

*Stage s, s > 0.* There are two steps.

*Step A.* There are three cases.

*Case I.*  $g(s - 1) \in B_{g(0)}$ . Let  $g(s - 1) = (0, z)$ . If  $g(s - 1)$  is not as yet in the domain of  $r$ , place  $(g(s - 1), (0, 0))$  in  $r$ , mark  $(1, z)$ , and then go to Step B. Otherwise, let the  $(s - 1)$ -end,  $g(s - 1)^{*s-1}$ , be the number  $(k, l)$ . (When the description of Stage  $s$  is complete, it will be clear that  $l = z$  and  $k > 0$ .) Place  $((k, l), (0, 0))$  in  $r$ , mark  $(k + 1, l)$ , and go to Step B. (Again, when our description of Stage  $s$  is complete it will be plain that  $(k + 1, l)$  was never previously a marked pair.<sup>4</sup>)

*Case II.*  $g(s - 1) \in B_{g(t)}$  with  $t > 0$ ; moreover,  $g(t - 1)$  is complete and  $\hat{r}(g(t - 1))$  is admissible; and, finally,

$$g(s - 1) > \max \{(j, z) \mid g(t - 1) = (0, z) \& (j, z) \in \hat{r}(g(t - 1))\}.$$

Let  $n$  be that number, if one exists, such that, for some sequence  $r_1, \dots, r_j, j \geq 1$ , we have  $r_j = n$ ,

$$\begin{aligned} (g(s - 1), r_1) &\in r^{(s-1)}, \\ \sim(\exists y)((r_j, y) &\in r^{(s-1)}), \end{aligned}$$

and, for  $1 \leq q < j$  (if  $j > 1$ ),

$$(r_q, r_{q+1}) \in r^{(s-1)}.$$

Then (as will be clear when our description of Stage  $s$  is complete)  $n$  has the

<sup>4</sup> Indeed, it will be clear that no number is ever marked more than once.

form  $(j, z)$  for some  $z$ .<sup>5</sup> Mark  $(j + 1, z)$ , place  $(n, g(t - 1))$  in  $r$ , and proceed to Step B. If no such  $n$  exists, mark  $(1, z)$  (where  $g(s - 1) = (0, z)$ ), place  $(g(s - 1), g(t - 1))$  in  $r$ , and go on to Step B.

Case III.  $g(s - 1) \in B_{g(t)}$  with  $t > 0$ , but either  $g(t - 1)$  is not complete or  $\hat{r}(g(t - 1))$  is not admissible<sup>6</sup> or

$$g(s - 1) \leq \max \{(j, z) \mid g(t - 1) = (0, z) \text{ \& } (j, z) \in \hat{r}(g(t - 1))\}.$$

In this case, proceed directly to Step B.

Step B. There are four substeps to be carried out.

Substep B1. If there are no *marked* pairs not previously added to the domain of  $r$ , go directly to Substep B2. Otherwise, let  $(g, h)$  be the smallest marked pair not yet in the domain of  $r$ ; place  $((g, h), (g + 1, h))$  in  $r$ , then go to Substep B2.

Substep B2. If there are no *unmarked* pairs of the form  $(j, m)$ , with  $j \geq 1$ , such that  $(j, m)$  is already in the range of  $r$  but not yet in the domain of  $r$ , go directly to Substep B3. Otherwise, let  $(j, m)$  be the smallest such pair. Place  $((j, m), (j + 1, m))$  in  $r$ ; then go to Substep B3.

Substep B3. Let  $(0, m)$  be the least pair with first entry 0 which is not yet in the domain of  $r$ . Place  $((0, m), (1, m))$  in  $r$ . Then proceed to Substep B4.

Substep B4. Suppose there is a number  $s_0$ ,  $0 < s_0 < s$ , such that, at stage  $s_0$ ,  $g(s_0 - 1)$  was found to come under Case III of Step A, but such that  $g(t - 1)$  is complete at stage  $s$  and  $\hat{r}(g(t - 1))$  is admissible at stage  $s$  and

$$g(s_0 - 1) > \max \{(j, z) \mid g(t - 1) = (0, z) \text{ \& } (j, z) \in \hat{r}(g(t - 1))\}.$$

(Here  $t$  is the unique number satisfying  $t > 0$  &  $g(s_0 - 1) \in B_{g(t)}$ .) Take the *least*  $s_0$ , and proceed for  $g(s_0 - 1)$  as in Case II of Step A, except that instead of concluding with a return to the beginning of Step B, we conclude by terminating Stage  $s$ .

This completes the description of Stage  $s$  and so of the construction of  $r$ . It is clear from the construction that  $r(= \bigcup_s r^{(s)})$  is a general recursive function. We shall now argue that a suitable restriction  $\check{r}$  of  $r$  regresses (in the sense of [2]) a continuum of retraceable sets  $\alpha$  having property (2). We shall, in fact, explicitly define *one* such set; it will then be clear to the reader that there are continuously many others. Let  $b_0$  be the least element of  $B_{g(0)}^*$ . Then eventually, under Case I of Step A,  $b_0$  becomes complete and  $\hat{r}(b_0)$  admissible. Suppose  $b_0 = g(s_0)$ . Now let  $b_1$  be the least element  $b$  of  $B_{g(s_0+1)}^*$  such that

$$b > \max \{(j, z) \mid g(s_0) = (0, z) \text{ \& } (j, z) \in \hat{r}(g(s_0))\};$$

<sup>5</sup> It will also be seen that  $z$  must be such that  $g(s - 1) = (0, z)$ .

<sup>6</sup> When the construction is completely described it will be clear that (a) completeness of  $g(t - 1) \Rightarrow$  admissibility of  $\hat{r}(g(t - 1))$ , and (b)

$$g(s - 1) \leq \max \{\dots\} \Rightarrow g(s - 1) < \max \{\dots\}.$$

we let  $s_1$  be such that  $b_1 = g(s_1)$ . If  $b_0$  is complete and  $\hat{r}(b_0)$  admissible by Stage  $s_1 + 1$ , then, at that stage,  $b_1$  will become complete and  $\hat{r}(b_1)$  admissible, with  $b_0 \in \hat{r}(b_1)$ , in virtue of Case II of Step A. Otherwise, this will occur at a stage  $t > s_1 + 1$  in virtue of Substep B4. Suppose now that we have already defined  $b_0, b_1, \dots, b_m$ , with  $b_m = g(s_m)$ . We let  $b_{m+1}$  be the least element  $b$  of  $B_{g(s_{m+1})}^*$  such that

$$b > \max \{ (j, z) \mid g(s_m) = (0, z) \ \& \ (j, z) \in \hat{r}(g(s_m)) \};$$

we further let  $s_{m+1}$  be such that  $b_{m+1} = g(s_{m+1})$ . Just as in the case of  $b_0$  and  $b_1$ , we now see that we must eventually have  $b_{m+1}$  complete,  $\hat{r}(b_{m+1})$  admissible, and  $b_m \in \hat{r}(b_{m+1})$ . Thus, the sequence  $0, b_0, b_1, \dots, b_m, \dots$  thus defined is a strictly increasing sequence such that  $\bigcup_n \hat{r}(b_n)$  is regressed by  $r$  (or rather, if we use the definition of regression in [2], by a suitable restriction of  $r$ ); it remains to be shown that (a)  $\bigcup_n \hat{r}(b_n)$  is retraceable, and (b)  $\bigcup_n \hat{r}(b_n)$  admits no *general* recursive retracing function.

*Ad (a).* We define a partial recursive function  $p$ , such that  $p$  retraces  $\bigcup_n \hat{r}(b_n)$ , as follows.

Given a number  $x$ , allow the construction of  $r$  to run until the first stage  $s$  (if such exists) at which  $x$  is found to be complete. ( $p$  will be seen to be undefined for precisely those  $x$  which do not lie in any set of the form  $\hat{r}(g(t))$ , where  $t$  is such that  $g(t)$  is eventually complete and  $\hat{r}(g(t))$  eventually admissible; and this in turn is just the set of those  $x$  which never become complete as  $s \rightarrow \infty$ .) Then, as is clear from the construction of  $r$ , there will be a (unique) number  $g(t)$  such that  $g(t) \in \hat{r}(x)$  at Stage  $s$ ,

$$(0, z) \neq g(t) \Rightarrow \sim(\exists w, u)(r^w(x) = (0, z) \ \& \ r^u(0, z) = g(t)),$$

and  $\hat{r}(g(t))$  satisfies the conditions (i)–(iii) for *admissibility*. Let  $\hat{r}(g(t))$  be

$$z_{11}, \dots, z_{1r_1}, \dots, z_{k1}, \dots, z_{kr_k}, 0.$$

We place the following pairs in  $p$ :  $(0, 0)$ ,  $(z_{k1}, 0)$ ,  $(z_{11}, z_{l+1, r_{l+1}})$  for each  $l$  such that  $1 \leq l < k$ , and  $(z_{lj}, z_{l, j-1})$  for each  $l$  and  $j$  such that  $1 \leq l \leq k$ ,  $j > 1$ . By the definitions of completeness and admissibility together with the definition of the set  $\{0, b_0, b_1, \dots\}$ , it is clear that  $\bigcup_n \hat{r}(b_n)$  is retraced by  $p$ .

*Ad (b).* Suppose  $f$  were a general recursive function, some restriction  $f_0$  of which retraced  $\bigcup_n \hat{r}(b_n)$  in the sense of [2]; thus, in particular,

$$(\forall x)(\exists y)(f_0^{y+1}(x) = f_0^y(x)).$$

But then there must be a general recursive function  $f_1$  which *itself* meets the conditions of [2] for being a retracing function of  $\bigcup_n \hat{r}(b_n)$ . So,

$$(\forall x)(\exists y)(f_1^{y+1}(x) = f_1^y(x)).$$

But then, if  $R$  is any recursive set, we see easily that  $R \cdot \bigcup_n \hat{r}(b_n)$  must also be retraced by a *general* recursive function. Hence, in particular,  $(\{0\} \otimes N) \cdot \bigcup_n \hat{r}(b_n)$  is so retraced; therefore  $(\{0\} \otimes N) \cdot \bigcup_n \hat{r}(b_n)$  can be recursively split.

But

$$(\{0\} \otimes N) \cdot \bigcup_n \hat{r}(b_n) = \{0, b_0, b_1, \dots\} \subset \bigcup_n S_n;$$

hence  $\bigcup_n S_n$  can be recursively split: contradiction. ▀

*Remarks.* 1. The above proof of *Theorem 2* is, unfortunately, a good illustration of one of the less pleasant facets of recursion theory: intuitively simple arguments frequently fail to have correspondingly short statements at the written level.

2. It remains an open question whether, in *Theorem 2*, (1) can be strengthened thus: (1')  $\alpha$  admits a general recursive regressing function  $f$  such that

$$(\forall x)(\exists y)(f^{y+1}(x) = f^y(x)).$$

Certainly, no such construction as the one used in our above proof of *Theorem 2* can produce such an  $\alpha$ .

3. It is (in view of *Corollary 1*) an immediate corollary to our proof of *Theorem 2* that there is a continuum of pairs  $(\alpha, R)$  such that (i)  $\alpha$  is a retraceable set and  $R$  is a recursive set with  $\alpha \cdot R$  infinite, (ii)  $\alpha$  can be sequentially ordered in such a way as to admit regression by a function with general recursive extension, and (iii)  $\alpha \cdot R$  cannot be so ordered.

We conclude §2 by stating a theorem, the proof of which we shall omit, which shows that the condition in *Theorem 1* concerning all infinite subsets of  $\beta$  is not a consequence of the other conditions.

**THEOREM 3.<sup>7</sup>** *There is a pair  $\beta, \gamma$  of sets of natural numbers, such that the following statements hold:*

- (1)  $\beta$  and  $\gamma$  are recursively separated;
- (2)  $\beta$  is retraced by a bounded basic retracing function;
- (3)  $\gamma$  is retraced by a general recursive, basic retracing function;
- (4)  $\beta + \gamma$  is retraced by a bounded basic retracing function; and
- (5) no sequential ordering of  $\beta + \gamma$  admits a potentially recursive regressing function.

### 3. R.e. relations with regression-like properties, $k$ -immunity, and a characterization of recursively infinite number sets

As in §4 of [1], we use " $D_m$ " to denote the  $m$ -th term in the standard binary indexing of the class of all nonempty finite subsets of  $N$ . In the discussion which follows, we consider sequences  $\{D_{r(m)}\}$ ,  $r$  a recursive function, such that

$$m \neq n \Rightarrow D_{r(m)} \cdot D_{r(n)} = \emptyset$$

and there is a fixed number  $k \geq 1$  such that  $|D_{r(m)}| = k$  for all  $m$ . Such a sequence we call a *disjoint  $k$ -array*.

<sup>7</sup> We do not know whether "bounded" can be replaced by "closed" or "weakly closed" in *Theorems 1* and *3*. Certainly it cannot be replaced by "strongly closed".

DEFINITION. A set  $\alpha$  of natural numbers is  $k$ -immune ( $k$  a given number  $\geq 1$ )  $\Leftrightarrow \alpha$  is infinite and, if  $\{D_{r(m)}\}$  is any disjoint  $k$ -array, then  $(\exists l)(D_{r(l)} \subset \alpha')$ .

Our next theorem merely generalizes to all  $k \geq 1$  a pair of well-known elementary facts concerning ordinary 1-immunity.

THEOREM 4. (a)  $k$ -immunity is preserved under recursive equivalence.

(b) For each pair  $j, k$  of positive integers, there exists an infinite, co-infinite set  $\alpha$  such that  $\alpha$  is  $j$ -immune but not  $(j + 1)$ -immune while  $\alpha'$  is  $k$ -immune but not  $(k + 1)$ -immune.

Proof. (a) Suppose that  $\alpha$  is an infinite,  $k$ -immune set, and that  $\alpha \simeq \beta$  via the partial recursive function  $p$ . Suppose  $\beta$  is not  $k$ -immune. Let  $\{D_{r(m)}\}$  be a disjoint  $k$ -array all of whose terms meet  $\beta$ . Let  $j$  be the largest number such that  $1 \leq j \leq k$  and, for infinitely many  $m$ ,  $j$  elements of  $D_{r(m)}$  lie in  $\rho p$ ; since  $\beta \subset \rho p$  and each  $D_{r(m)}$  meets  $\beta$ , such a number must exist. Let  $b$  be a number such that  $m \geq b \Rightarrow$  not more than  $j$  elements of  $D_{r(m)}$  belong to  $\rho p$ . Now, it is clear that we may construct a recursive function  $q$  such that  $\{D_{q(n)}\}$  is a disjoint  $j$ -array consisting of precisely those sets  $D_{r(m)} \cdot \rho p$  for which  $m \geq b$  &  $|D_{r(m)} \cdot \rho p| = j$ . Since  $\beta \subset \rho p$ , it follows from the choice of  $b$  that each set  $D_{q(n)}$  meets  $\beta$ . Thus,  $\{p^{-1}(D_{q(n)})\}$  is a disjoint  $j$ -array each term of which meets  $\alpha$ . Since  $j \leq k$ , it is routine to modify  $\{p^{-1}(D_{q(n)})\}$  so as to obtain a disjoint  $k$ -array each term of which meets  $\alpha$ : contradiction. (a) follows.

(b) Let  $j \geq 1, k \geq 1$  be given. Let  $\{D_{r_1(m)}\}$  be a disjoint  $(j + 1)$ -array and  $\{D_{r_2(m)}\}$  a disjoint  $(k + 1)$ -array such that

$$(\forall m, n)(D_{r_1(m)} \cdot D_{r_2(n)} = \phi).$$

Let  $D_0^j, D_1^j, D_2^j, \dots$  be an enumeration of all disjoint  $j$ -arrays; and let  $D_0^k, D_1^k, D_2^k, \dots$  be an enumeration of all disjoint  $k$ -arrays. Let " $D_{m,r}^j$ " (resp. " $D_{m,r}^k$ ") denote the  $r$ -th term of  $D_m^j$  (resp.  $D_m^k$ ). We define  $\alpha, \alpha'$  by stages as follows:

Stage 0. Let

$$n_0 = (\mu n)(D_{0,0}^j \subset \bigcup_{l \leq n}(D_{r_1(l)} + D_{r_2(l)}));$$

let  $b_0, \dots, b_{n_0}$  be the least numbers in  $D_{r_1(0)}, \dots, D_{r_1(n_0)}$ , respectively, which are not in  $D_{0,0}^j$ ; and set

$$\alpha^{(0)} = \{b_0, \dots, b_{n_0}\},$$

$$\alpha'^{(0)} = [\bigcup_{l \leq n_0}(D_{r_1(l)} + D_{r_2(l)})] - \alpha^{(0)}.$$

Stage 1. Let

$$r_0 = (\mu r)(D_{0,r}^k \cdot \alpha'^{(0)} = \emptyset);$$

and let

$$n_1 = (\mu n)[n > n_0 \text{ \& } D_{0,r_0}^k \subset \bigcup_{l \leq n}(D_{r_1(l)} + D_{r_2(l)})].$$

Let  $c_0, \dots, c_{n_1-n_0-1}$  be the least numbers in  $D_{r_2(n_0+1)}, \dots, D_{r_2(n_1)}$ , respectively, which are not in  $D_{0,r_0}^k$ ; and set

$$\alpha'^{(1)} = \alpha'^{(0)} + \{c_0, \dots, c_{n_1-n_0-1}\},$$

$$\alpha^{(1)} = ([\bigcup_{n_0 < l \leq n_1} (D_{r_1(l)} + D_{r_2(l)})] - \alpha'^{(1)}) + \alpha^{(0)}.$$

Stage  $2(s+1)$ . Let

$$r_{2s+1} = (\mu r)(D_{s+1, r}^j \alpha^{(2s+1)} = \emptyset);$$

and let

$$n_{2(s+1)} = (\mu n)[n > n_{2s+1} \ \& \ D_{s+1, r_{2s+1}}^j \subset \bigcup_{l \leq n} (D_{r_1(l)} + D_{r_2(l)})].$$

Let  $d_0, \dots, d_{n_{2(s+1)}-n_{2s+1}-1}$  be the least numbers in  $D_{r_1(n_{2s+1}+1)}, \dots, D_{r_1(n_{2(s+1)})}$ , respectively, which are *not* in  $D_{s+1, r_{2s+1}}^j$ ; and set

$$\alpha^{(2(s+1))} = \alpha^{(2s+1)} + \{d_0, \dots, d_{n_{2(s+1)}-n_{2s+1}-1}\},$$

$$\alpha'^{(2(s+1))} = \alpha'^{(2s+1)} + ([\bigcup_{n_{2s+1} < l \leq n_{2(s+1)}} (D_{r_1(l)} + D_{r_2(l)})] - \alpha^{(2(s+1))}).$$

Stage  $2s+3$ . Let

$$r_{2(s+1)} = (\mu r)(D_{s+1, r}^k \alpha'^{(2(s+1))} = \emptyset);$$

and let

$$n_{2s+3} = (\mu n)[n > n_{2(s+1)} \ \& \ D_{s+1, r_{2(s+1)}}^k \subset \bigcup_{l \leq n} (D_{r_1(l)} + D_{r_2(l)})].$$

Let  $e_0, \dots, e_{n_{2s+3}-n_{2(s+1)}-1}$  be the least numbers in  $D_{r_2(n_{2(s+1)}+1)}, \dots, D_{r_2(n_{2s+3})}$ , respectively, which are *not* in  $D_{s+1, r_{2(s+1)}}^k$ ; and set

$$\alpha^{(2s+3)} = \alpha'^{(2(s+1))} + \{e_0, \dots, e_{n_{2s+3}-n_{2(s+1)}-1}\},$$

$$\alpha^{(2s+3)} = \alpha^{(2(s+1))} + ([\bigcup_{n_{2(s+1)} < l \leq n_{2s+3}} (D_{r_1(l)} + D_{r_2(l)})] - \alpha'^{(2s+3)}).$$

Let

$$\beta = \{x \mid (\forall n)(x \notin \alpha^{(n)} + \alpha'^{(n)})\}.$$

We set  $\alpha = \beta + \bigcup_n \alpha^{(n)}$ . Then  $\alpha' = \bigcup_n \alpha'^{(n)}$ ; and it is easily seen from the construction that  $\alpha, \alpha'$  have the properties required in part (b) of the theorem. ■

In *Theorem 5*, we shall exhibit necessary and sufficient conditions of “confluence” or “connectivity” under which an infinite, non- $k$ -immune set  $\alpha$  ( $k > 1$ ) is in fact non- $j$ -immune for some  $j < k$ . These conditions are intuitively rather simple, though their precise formal statement is a trifle messy. That they are necessary will be rather obvious; their sufficiency is somewhat less apparent. If  $R$  is a binary relation on  $N$ , we denote by “ $R \upharpoonright$ ” the set  $\{x \mid (\exists y)R(x, y)\}$ , by “ $R \upharpoonright$ ” the set  $\{x \mid (\exists y)R(y, x)\}$ , and by “ $R \upharpoonright \cdot R \upharpoonright$ ” the set  $R \upharpoonright \cdot R \upharpoonright$ .

**THEOREM 5.** *Let  $k > 1$  and let  $\alpha$  be a non- $k$ -immune set of numbers. Then, if  $1 \leq j < k$ ,  $\alpha$  is non- $j$ -immune if and only if there exist a binary r.e. relation  $R$  and a disjoint  $k$ -array  $\{D_{r(n)}\}$  such that the following conditions are satisfied:*

- (1)  $\{D_{r(n)}\}$  witnesses the non- $k$ -immunity of  $\alpha$ ;
- (2)  $(\forall x, m, \tau)[(x \in \alpha \cdot D_{r(m)} \ \& \ \tau \text{ is a } j\text{-element subset of } D_{r(m)} \cdot R \upharpoonright \ \& \ x \in \tau) \Rightarrow (\exists n)(\forall k \geq n)(\forall y)(u \in \alpha \cdot D_{r(k)} \Rightarrow D_{r(k)} \text{ possesses a } j\text{-element subset } \eta \text{ such that})$

$\eta \subset R \uparrow$  &  $y \in \eta$  &  $\tau \subset$  the  $R$ -posterity<sup>8</sup> of  $\eta$  &  $(\forall q \in \eta)(\tau \cdot [the\ R\text{-posterity of } q] \neq \emptyset))$ ];

(3)  $(\forall x, m, \tau)[(x \in \alpha \cdot D_{r(m)} \text{ \& } x \text{ a } j\text{-element subset } \xi \text{ of } D_{r(m)} \text{ \& } (\exists k) (\tau \text{ is a } j\text{-element subset of } D_{r(k)} \text{ for which } \tau \subset \text{the } R\text{-posterity of } \xi \text{ and } \tau \cdot [the\ R\text{-posterity of } x] \neq \emptyset)) \Rightarrow \tau \cdot \alpha \neq \emptyset]$ ;

(4)  $(\exists a)(\exists \mu) [\mu \text{ is a } j\text{-element subset of } D_{r(0)} \text{ \& } a \in \mu \cdot \alpha \text{ \& } \mu \subset R \uparrow]$ .

*Proof.* It is virtually trivial to show that if  $k > j$  and  $\alpha$  is both non- $k$ -immune and non- $j$ -immune, then (1)–(4) hold for suitable  $R$  and  $\{D_{r(n)}\}$ . For, suppose  $\{D_{t(n)}\}$  is a disjoint  $j$ -array witnessing that  $\alpha$  is non- $j$ -immune. There is clearly no loss of generality in assuming that  $(\bigcup_n D_{t(n)})'$  is infinite. Let  $\{D_{q(n)}\}$  be a disjoint  $(k - j)$ -array such that  $\bigcup_n D_{q(n)} \subset (\bigcup_n D_{t(n)})'$  and set  $D_{r(n)} = D_{t(n)} + D_{q(n)}$  for all  $n$ . Now define a r.e. (in fact, recursive) relation  $R$ , as follows:

$$R(x, y) \Leftrightarrow (\exists n)(x \in D_{r(n+1)} \text{ \& } y \in D_{t(n)}).$$

It is plain that for this choice of  $R$  and  $\{D_{r(n)}\}$ , (1)–(4) hold. Suppose, on the other hand, that  $\alpha$  is non- $k$ -immune, that  $\{D_{r(n)}\}$  is a disjoint  $k$ -array witnessing the fact, and that  $\{D_{r(n)}\}$  and a certain binary r.e. relation  $R$  satisfy (1)–(4), for a certain positive integer  $j < k$ . We must show that  $\alpha$  is not  $j$ -immune. We denote by “ $\hat{R}(\alpha)$ ” the  $R$ -posterity of a set  $\alpha$ ; “ $\hat{R}(n)$ ” shall be used in place of “ $\hat{R}(\{n\})$ ”. (Note that the relation  $m \in \hat{R}(n)$  is transitive.) Applying condition (4) let  $a_0$  be a particular element of  $\alpha \cdot D_{r(0)}$  and  $\mu_0$  a particular  $j$ -element subset of  $D_{r(0)} \cdot R \uparrow$  such that  $a_0 \in \mu_0$ . Given  $n \in \alpha$ ,  $1 \leq t \leq \binom{k}{j}$ , and a  $j$ -element subset  $J$  of  $D_{r(m)} \cdot R \uparrow$  such that  $n \in J$ , let  $K(n, J, t)$  be the set of all  $j$ -element sets  $\gamma$  satisfying the following condition:

$$\begin{aligned} & (\exists k)[\gamma \subset D_{r(k)} \cdot R \uparrow \text{ \& } J \subset \hat{R}(\gamma) \\ & \text{\& } (\exists l)(\exists \beta_1, \dots, \beta_l)(\forall i, h)(1 \leq i, h \leq t \\ & \Rightarrow (\beta_i \text{ is a } j\text{-element set} \\ & \text{\& } \beta_i \subset D_{r(i)} \cdot R \uparrow \text{ \& } (i \neq j \Rightarrow \beta_i \neq \beta_j) \\ & \text{\& } \gamma \subset \hat{R}(\beta_i) \text{ \& } (\forall q \in \beta_i)[\gamma \cdot \hat{R}(q) \neq \emptyset]) \\ & \text{\& } (\forall q \in \gamma)(J \cdot \hat{R}(q) \neq \emptyset)]. \end{aligned}$$

Clearly, (K) is an r.e. predicate. We see by (2) that there is a least number  $y$  such that  $(\forall k \geq y)(\forall z)(z \in \alpha \cdot D_{r(k)} \Rightarrow D_{r(k)} \text{ has a } j\text{-element subset } \eta \text{ such that } \eta \subset R \uparrow \text{ \& } z \in \eta \text{ \& } J \subset \hat{R}(\eta) \text{ \& } (\forall q \in \eta) \cdot (J \cdot \hat{R}(q) \neq \emptyset))$ . We denote

<sup>8</sup> By the “ $R$ -posterity of  $\eta$ ” we mean the set  $P_\eta$  defined inductively as follows:

$$\begin{aligned} P_\eta^{(0)} &= \{x \mid (\exists y)(y \in \eta \text{ \& } R(y, x))\}; \\ P_\eta^{(n+1)} &= \{x \mid (\exists y)(y \in P_\eta^{(n)} \text{ \& } R(y, x))\}; \\ P_\eta &= \bigcup_n P_\eta^{(n)}. \end{aligned}$$

by " $K^*(n, J, t)$ " this uniquely determined  $y$ . Now, it is readily seen from condition (2) that  $K(a_0, \mu_0, 1)$  is infinite. Let  $t_0$  be the largest  $t$ ,  $1 \leq t \leq \binom{k}{j}$ , such that  $K(a_0, \mu_0, t)$  is infinite. Let  $p_0$  be a number and  $J_{p_0}$  a set such that:  $J_{p_0}$  has  $j$  elements,  $J_{p_0} \subset D_{r(p_0)} \cdot R \uparrow$ ,  $J_{p_0} \cdot \alpha \neq \emptyset$ , every set belonging to  $\bigcup_{t_0 < s \leq \binom{k}{j}} K(a_0, \mu_0, s)$  is a subset of some  $D_{r(m)}$  where  $m < p_0$ , and finally,

$$\mu_0 \subset \hat{R}(J_{p_0}) \quad \& \quad (\forall q \in J_{p_0})(\mu_0 \cdot \hat{R}(q) \neq \emptyset);$$

the existence of such  $p_0$  and  $J_{p_0}$  is insured by (2). Suppose that there are infinitely many  $j$ -element sets  $L$  such that, given a number  $q \in J_{p_0} \cdot \alpha$ , we have (i)  $L \in K(q, J_{p_0}, t_0)$ , and (ii) among the numbers  $l$  (see (K)) witnessing that  $L \in K(q, J_{p_0}, t_0)$ , there are some which exceed  $K^*(q, J_{p_0}, t_0)$ . Then, clearly, the set of all such  $L$  (i.e., those satisfying (i) and (ii)) can be effectively enumerated in a list  $L_0, L_1, \dots$ . From this list, we can obviously then obtain a disjoint  $j$ -array  $\{D_{u(n)}\}$ , every term of which is one of the  $L_i$ ; our claim is that  $D_{u(n)} \cdot \alpha \neq \emptyset$  holds for every  $n$ . Suppose not: let  $n_0$  be such that  $D_{u(n_0)} \cdot \alpha = \emptyset$ . Let  $l$  be a number  $> K^*(q, J_{p_0}, t_0)$  such that, for some  $k$ ,

$$D_{u(n_0)} \subset D_{r(k)} \cdot R \uparrow \quad \& \quad J_{p_0} \subset \hat{R}(D_{u(n_0)})$$

& there are distinct  $j$ -element subsets  $\beta_i$ ,  $1 \leq i \leq t_0$ , of  $D_{r(l)} \cdot R \uparrow$  such that

$$D_{u(n_0)} \subset \hat{R}(\beta_i) \quad \text{and} \quad (\forall q \in \beta_i)(D_{u(n_0)} \cdot \hat{R}(q) \neq \emptyset)$$

hold for  $1 \leq i \leq t_0$ . Since we are assuming  $D_{u(n_0)} \cdot \alpha = \emptyset$ , it follows from (3) that  $\beta_i \cdot \alpha = \emptyset$  for  $1 \leq i \leq t_0$ . But  $D_{r(l)} \cdot \alpha \neq \emptyset$ ; hence, by (2) and the choice of  $l$ ,  $D_{r(l)} \cdot R \uparrow$  has a  $j$ -element subset  $\beta_{t_0+1}$  which meets  $\alpha$  and is such that

$$J_{p_0} \subset \hat{R}(\beta_{t_0+1}) \quad \& \quad (\forall q \in \beta_{t_0+1})(J_{p_0} \cdot \hat{R}(q) \neq \emptyset)$$

Since  $D_{u(n_0)} \in K(q, J_{p_0}, t_0)$  and the relation  $x \in \hat{R}(y)$  is transitive, we also have, for each  $i$  with  $1 \leq i \leq t_0$ , that

$$J_{p_0} \subset \hat{R}(\beta_i) \quad \& \quad (\forall q \in \beta_i)(J_{p_0} \cdot \hat{R}(q) \neq \emptyset).$$

But this implies that  $J_{p_0} \in K(a_0, \mu_0, t_0 + 1)$ , contradiction. If, on the other hand, there are only *finitely many*  $L$  satisfying (i) and (ii), then let  $t_1$  be the largest  $t$ ,  $1 \leq t < t_0$ , such that there are infinitely many elements  $L$  of  $K(q, J_{p_0}, t)$  for which some  $l$  witnessing  $L \in K(q, J_{p_0}, t)$  is  $> K^*(q, J_{p_0}, t_0)$ ; the existence of  $t_1$  is insured by (2). Let  $p_1$  be a number and  $J_{p_1}$  a set such that:  $J_{p_1}$  has  $j$  elements,  $J_{p_1} \subset D_{r(p_1)} \cdot R \uparrow$ ,  $J_{p_1} \cdot \alpha \neq \emptyset$ , every set belonging to  $\{L \mid (\exists t)(t_0 \geq t > t_1 \& L \in K(q, J_{p_0}, t) \text{ is witnessed by some } l > K^*(q, J_{p_0}, t_0))\}$  is a subset of some  $D_{r(m)}$  where  $m < p_1$ , and, finally,

$$J_{p_0} \subset \hat{R}(J_{p_1}) \quad \& \quad (\forall q \in J_{p_1})(J_{p_0} \cdot \hat{R}(q) \neq \emptyset);$$

the existence of such  $p_1$  and  $J_{p_1}$  is insured by (2). Now repeat the argument of the preceding paragraphs, replacing  $K(q, J_{p_0}, t_0)$  by  $K(w, J_{p_1}, t_1)$  where

$w \in J_{p_1} \cdot \alpha$ , and replacing  $K^*(q, J_{p_0}, t_0)$  by

$$K^*(w, J_{p_1}, t_1) + K^*(q, J_{p_0}, t_0).$$

It is clear (in view of (2)) that at most  $\binom{k}{j}$  repetitions of this procedure will lead to a disjoint  $j$ -array each term of which meets  $\alpha$ . ■

**COROLLARY** [1, Footnote (2)]. *An infinite set  $\alpha$  of natural numbers has an infinite recursive subset (i.e., is not 1-immune)  $\Leftrightarrow$  there is a binary r.e. relation  $R$ , and a disjoint  $j$ -array  $\{D_{r(n)}\}$  for some  $j \geq 1$ , such that*

- (i)  $(\forall n)(\alpha \cdot D_{r(n)} \neq \emptyset)$ ,
- (ii)  $[x \in \alpha \cdot \bigcup_n D_{r(n)} \ \& \ y \in \bigcup_n D_{r(n)} \ \& \ y \in \hat{R}(x)] \Rightarrow y \in \alpha$ , and
- (iii)  $y \in \alpha \cdot \bigcup_n D_{r(n)} \Rightarrow (\exists m)[(x > m \ \& \ x \in \alpha \cdot \bigcup_n D_{r(n)}) \Rightarrow y \in \hat{R}(x)]$ .

From this corollary follows, in turn, *Theorem 2* of [1].

#### 4. Topolectomy on a proof of Dekker and Myhill; classification of an index set

In their paper [3], Dekker and Myhill show, by the well-aimed firing of a battery of facts about metric spaces, that if  $R_p$  is the family of all sets retraced by a fixed partial number-theoretic function  $p$ , then, continuum hypothesis or no continuum hypothesis, we must have  $|R_p| = 2^{\aleph_0}$  or  $|R_p| = k$ , where  $k$  is any countable cardinal. In *Theorem 6* below, we prove this again but without any use of topology; nothing is required but a straight-forward examination of branching properties.<sup>9</sup> The method (if such it may be called) lends itself, further, to proving an adjunct to a recent result of Yates [8] on “basic” retracing functions. For the remainder of the paper, all (partial) functions are assumed to be “special”, i.e., (i)  $\text{range}(f) \subset \text{domain}(f)$ , and (ii)

$$(\forall x)(x \in \text{domain}(f) \Rightarrow (\exists y \geq 0)(f^{y+1}(x) = f^y(x))).$$

We furthermore assume (iii) there is at least one nonrepeating infinite sequence  $a_0, a_1, \dots$  of numbers such that  $\{a_0, a_1, \dots\} \subset \text{domain}(f)$  and  $a_0, a_1, \dots$  is “regressed” by  $f$ . (We use quotation marks since we are not requiring, here, that  $f$  be partial recursive.) If  $f$  satisfies (i)–(iii), we inquire whether the graph of  $f$  embraces a tree as in Fig. 1, where a relationship  $a \rightarrow b$  means that  $b \in f(a) - \{a\}$ ,<sup>10</sup> and where the ellipses indicate more of the same indefinitely (i.e., binary branching at each node). We use “CU” to denote the statement that a countable union of countable sets is countable; as usual, “ZF” means (choiceless) Zermelo-Fraenkel set theory.

<sup>9</sup> The chief topological theorem used by Dekker and Myhill in their proof is the result that an uncountable closed set in a separable complete metric space has the power of the continuum. What we do in our proof of *Theorem 6* is, essentially, to make use of the one (exceedingly simple) combinatorial idea contained in the standard proof of the cited topological theorem, while forgetting about the topology.

<sup>10</sup> “ $\rightarrow$ ” now is used as in [1].

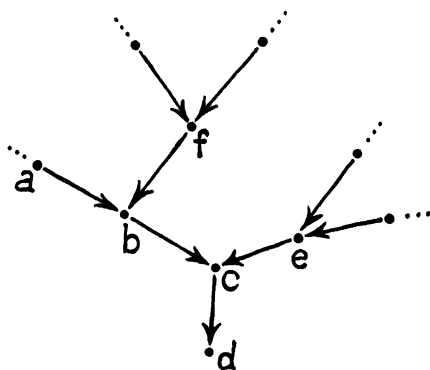


FIGURE 1

**THEOREM 6.** *It is provable in  $ZF + \{CU\}$  that the following three assertions are equivalent for a function  $f$  satisfying (i)–(iii):*

- (a)  $f$  regresses<sup>11</sup> uncountably many infinite sets;
- (b)  $f$  regresses<sup>11</sup>  $2^{\aleph_0}$  infinite sets;
- (c) the graph of  $f$  gives rise to at least one tree of the type shown in Fig. 1.

*Proof.* (c)  $\Rightarrow$  (b) and (b)  $\Rightarrow$  (a) are obvious; we concentrate on (a)  $\Rightarrow$  (c). Assume (a). Let  $\mathfrak{F}_0$  be the (countable) set of fixed points of  $f$ . By CU and (a), there must exist at least one such fixed point  $x$  for which

$$\{\alpha \mid \alpha \text{ is infinite \& } f \text{ regresses } \alpha \text{ with } x \text{ as fixed point}\}$$

is uncountable; let  $x_0$  be the least such  $x$ . Let

$$\mathfrak{F}_1 = \{y \mid y \in \text{domain}(f) \text{ \& } y \neq x_0 \text{ \& } f(y) = x_0\}.$$

$\mathfrak{F}_1$  must contain at least one number  $y$  such that

$$\{\beta \mid \beta \text{ is infinite \& } f \text{ regresses } \beta \text{ \& } y \in \beta\}$$

is uncountable. Suppose there is *only one* such  $y$ ; call it  $y_0$ . Then let

$$\mathfrak{F}_2 = \{z \mid z \in \text{domain}(f) \text{ \& } f(z) = y_0\}.$$

$\mathfrak{F}_2$  in turn must contain at least one  $w$  such that

$$\{\gamma \mid \gamma \text{ is infinite \& } f \text{ regresses } \gamma \text{ \& } w \in \gamma\}$$

is uncountable. If there is *only one* such  $w$ , say  $w_0$ , define  $\mathfrak{F}_3$  from  $w_0$  as  $\mathfrak{F}_2$  was defined from  $y_0$ . We claim that, eventually, we must arrive at a  $k > 0$  such that  $\mathfrak{F}_k$  contains *at least two* numbers,  $t_1$  and  $t_2$ , for which

$$\{\delta \mid \delta \text{ is infinite \& } f \text{ regresses } \delta \text{ \& } t_1 \in \delta\}$$

<sup>11</sup> Note, however, that for this theorem we are not requiring that  $f$  be partial recursive; thus “regresses” has here a much wider meaning than is usual.

and

$$\{\delta \mid \delta \text{ is infinite \& } f \text{ regresses } \delta \text{ \& } t_2 \in \delta\}$$

are both uncountable. For, if not, let  $r_0 = x_0, r_1 = y_0, r_2 = w_0, r_3, r_4, \dots$  be the sequence of *uniquely* determined numbers  $r_i$ , with  $r_i \in \mathfrak{F}_i$ , such that  $r_0 = x_0$  and, for  $k > 0$ ,  $r_k$  is the only  $r$  in  $\mathfrak{F}_k$  for which

$$\{\xi \mid \xi \text{ is infinite \& } f \text{ regresses } \xi \text{ \& } r \in \xi\}$$

is uncountable. Now, if  $\alpha$  is *any* infinite set regressed by  $f$  with  $x_0$  as fixed point, then either  $\alpha = \{r_0, r_1, r_2, \dots\}$  or else, for some  $w$  and some *least*  $t > 0$ ,  $f^*(w) = t = f^*(r_t)$ ,  $w \in \alpha$ , and  $w \neq r_t$ . (Thus,  $f(w) = r_{t-1}$ .) Hence, if  $\alpha \neq \{r_0, r_1, r_2, \dots\}$  then  $\alpha$  is one of the *countably many* sets regressed by  $f$  and containing  $w$ . But since, for each  $r_t$ ,  $t > 0$ , there are only countably many different numbers of  $f$ -height  $t$ , it follows that  $f$  regresses only countably many sets with fixed point  $x_0$ : contradiction. Let  $k_0$  be the least  $k$  such that  $k > 0$  &  $\mathfrak{F}_k$  has *two* numbers  $t_1$  and  $t_2$  of the required type; let  $t_1, t_2$  be fixed as the *least* two such numbers in  $\mathfrak{F}_k$ . Now our argument repeats itself *above each of*  $t_1, t_2$ ; thus, by limitless repetition of the above phenomena, we see that  $f$  yields a tree of the sort claimed in (c). Finally, it is not hard to see that all of this can be formalized in  $\text{ZF} + \{\text{CU}\}$ . ■

*Remarks.* (1) The above proof of *Theorem 6* is easily rewordable as a proof of *Lemma 3* on page 372 of [3]; the latter is the crucial lemma in the Dekker-Myhill proof of their T7.

(2) We have been informed (private communication to the author from Azriel Lévy) that  $\text{ZF} + \{\text{CU}\}$  does not  $\Rightarrow \text{ZF} + \{\text{CAC}\}$ , where CAC is the axiom of choice for countable families. Thus the above proof is formalizable in a rather weak set theory. Of course, the topological lemmas used by Dekker and Myhill can also be proved using only  $\text{ZF} + \{\text{CU}\}$ .

We now impose an additional restriction on our functions  $f$ , namely: (iv)  $f$  is finite-to-one on its domain. If  $f$  obeys all of (i)–(iv) and is partial recursive, it is said to be a *basic regressing function*. Such functions have been treated, with regressiveness specialized to *retraceability*, by Yates in [8] and [9]. In particular, in [8] Yates has, by means of a very interesting *ad hoc* construction, shown that there exist basic regressing functions (in fact, basic *retracing* functions) which do not regress any infinite set belonging to  $\Pi_2^0 + \Sigma_2^0$  (i.e., having a 2-quantifier form in the arithmetical hierarchy). It is, on the other hand, not very difficult to show, by a (König's Lemma)-type argument, that if  $f$  is a basic regressing function which regresses only one infinite set, then the infinite set regressed by  $f$  is in  $\Pi_2^0 \cdot \Sigma_2^0$ .

**LEMMA 3.**<sup>12</sup> *Suppose  $\beta$  is the unique infinite set regressed by a certain finite-to-one partial recursive function  $f$ . Then  $\beta \in \Pi_2^0 \cdot \Sigma_2^0$ .*

<sup>12</sup> The author originally noticed this for *retraceable* sets with immune complements; this he pointed out to Yates, who then in turn pointed out to the author that the im-

*Proof.* We shorten the work by taking advantage of a fact pointed out to us by Carl Jockusch: *if  $\alpha$  is any regressive set lying in  $\Sigma_k^0$ ,  $k > 1$ , then  $\alpha$  is also in  $\Pi_k^0$ .* There are at least two easy proofs of this, one of which goes as follows: the result for *retraceable* sets is a virtually immediate consequence of Post's Theorem ( $\Sigma_{n+1}^0 \cdot \Pi_{n+1}^0$  = the sets of degree  $\leq 0^{(n)}$ ) and the fact that  $\Sigma_{n+1}^0$  = the sets r.e. in sets of degree  $\leq 0^{(n)}$ . But ([2]) any regressive set is recursively equivalent to a retraceable set; and from this, since a recursive equivalence has domain and range of degrees  $\leq 0^{(1)}$ , the fact in question now follows. Thus, we need only show that  $\beta \in \Sigma_2^0$ . We first restrict  $f$  to a *basic* regressing function  $g$  (with a *unique* fixed point) such that  $g$  regresses  $\beta$ ; the procedure for obtaining  $g$  from  $f$  is evident. Let  $e$  be a Gödel number for  $g$ ; thus, in the standard Kleene notation,  $g(x) \simeq U(\mu y T_1(e, x, y))$ . Let  $b_0$  be the fixed point of  $\beta$  under  $g$ ; and, as in [1], for each  $x$  in the domain of  $g$  let  $g^*(x)$  denote  $\mu k (g^k(x) = b_0)$  and let  $\hat{g}(x)$  denote the set  $\{x, g(x), \dots, g^{g^*(x)}(x)\}$ . (Our use of the hat notation in the proof of *Theorem 2* was just a special case of this convention.) We define a predicate  $P(x, y, z)$  as follows (where, as usual,  $(x)_y$  is the power to which the  $y$ -th prime divides  $x$ ):

If  $(z)_0 < y$  or  $\sim T_1(e, (z)_0, (z)_1)$ , then  $P(x, y, z)$ ;

Otherwise,  $P(x, y, z) \Leftrightarrow x \in \hat{g}((z)_0)$ .

This predicate is plainly recursive,  $g$  being a basic regressing function. Our claim is that

$$x \in \beta \Leftrightarrow (\exists y)(\forall z)P(x, y, z).$$

Obviously,  $x \notin \beta \Rightarrow \sim(\exists y)(\forall z)P(x, y, z)$ . So let  $x_0 \in \beta$ , and suppose

$$\sim(\exists y)(\forall z)P(x_0, y, z).$$

For each natural number  $n$ , let  $H(g, n)$  be the set

$$\{x \mid g(x) \text{ is defined \& } g^*(x) = n\}.$$

Since  $g$  is finite-to-one, each  $H(g, n)$  is finite. Let  $n_0$  be such that  $x_0 \in H(g, n_0)$ . Now, since  $\sim(\exists y)(\forall z)P(x_0, y, z)$ , there must be infinitely many numbers  $w$  in the domain of  $g$  such that  $x_0 \notin \hat{g}(w)$ . Hence, since  $H(g, n_0)$  is finite, there must be some  $k \in H(g, n_0)$  such that  $k \neq x_0$  and, for infinitely many  $w$  in the domain of  $f$ ,  $k \in \hat{g}(w)$ . Let  $k_0$  be the least such  $k$  in  $H(g, n_0)$ . Then, since  $H(g, n_0 + 1)$  is finite, there is some  $k \in H(g, n_0 + 1)$  such that  $f(k) = k_0$  and, for infinitely many  $w \in$  the domain of  $f$ ,  $k \in \hat{g}(w)$ ; let  $k_1$  be the least such  $k$  in  $H(g, n_0 + 1)$ , etc., etc. Thus, since  $k_0 \in \hat{g}(k) \Rightarrow k \notin \beta$ , we see that  $f$

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munity of the complement is irrelevant. The present lemma is the generalization to *regressiveness* of Yates' observation. The proof, however, would be no different in the retraceable case; its main portion is just a verification of König's Lemma in a special situation. (It does not appear, however, that much space would be saved by formulating that part of the proof as an explicit *application* of König's Lemma.)

witnesses the regressiveness of an infinite set disjoint from  $\beta$ : contradiction. It follows that if  $x \in \beta$  then  $(\exists y)(\forall z)P(x, y, z)$ . ■

*Remarks.* 1. *Lemma 3* is easily extended to the case of a basic regressing function which regresses *finitely many* infinite sets.

2. As Jockusch has pointed out to the author, if  $\alpha$  is any set regressed by a basic regressing function, and  $\alpha \in \Pi_k^0$  with  $k > 1$ , then  $\alpha \in \Sigma_k^0$ . One way of seeing this is to “relativize” the above *Lemma*. First, note that if  $\alpha \in \Pi_2^0$  and  $\alpha$  is regressed by a regressing function  $f$ , then  $\alpha$  is the *unique* set regressed by some restriction of  $f$ ; this is not difficult to verify, and is proved explicitly in [8]. For the case  $k = 2$ , the claim therefore follows from *Lemma 3*; for  $k > 2$ , simply let all “regressing” functions be, not partial recursive, but *partial recursive in  $0^{(k-2)}$* , and use the thus-relativized versions of *Lemma 3* and the cited  $\Pi_2^0$  result. It follows from constructions of Yates in [8] and [9] that the word “basic” can *not* be omitted in stating this  $\Pi_k^0 \Rightarrow \Sigma_k^0$  relation; nor can it be omitted from *Lemma 3*.

Our next theorem shows that any  $f$  which is a basic regressing function and which does not regress at least one infinite element of  $\Pi_2^0 \cdot \Sigma_2^0$  must regress continuum many infinite sets.

**THEOREM 7.** *Let  $f$  be a basic regressing function which regresses exactly  $\aleph_0$  infinite sets. Then  $f$  regresses at least one infinite set  $\alpha$  such that  $\alpha \in \Pi_2^0 \cdot \Sigma_2^0$ .*

*Proof.* The idea is to isolate some one infinite set  $\alpha$  such that  $\alpha$  is the *unique* set regressed by some partial recursive subfunction of  $f$ ; then the above lemma about basic regressing functions which regress only one infinite set applies to yield  $\alpha \in \Pi_2^0 \cdot \Sigma_2^0$ . We note, for use below, the (easily verified) fact that if  $\beta, \gamma$  are *distinct* infinite sets regressed by  $f$ , then  $\beta \cdot \gamma$  is finite. There is no loss of generality in assuming that  $f$  has a *unique* fixed point, say,  $b$ . Consider the following assertion:

(P)  $(\forall n)((\exists \beta)(\beta \text{ is infinite \& } f \text{ regresses } \beta \text{ \& } n \in \beta) \Rightarrow (\exists \gamma, \delta) \cdot (\gamma, \delta \text{ are infinite \& } f \text{ regresses both } \gamma \text{ and } \delta \text{ \& } n \in \gamma \cdot \delta \text{ \& } \gamma \neq \delta))$ .

We claim that if (P) were true,  $f$  would regress too many infinite sets. For suppose (P) holds. Let

$$n_0 = (\mu n)(\exists \beta)(n \in \beta \text{ \& } \beta \text{ is infinite \& } f \text{ regresses } \beta);$$

clearly such  $n$  exist. Next let

$$n_1 = (\mu n)[(\exists \gamma)(n \in \gamma \text{ \& } \gamma \text{ is infinite \& } f \text{ regresses } \gamma \text{ \& } n_0 \notin \hat{f}(n) \text{ \& } n \notin \hat{f}(n_0))];$$

$n_1$  must exist by (P). Again define:

$$n_2 = (\mu n)[n_0 \in \hat{f}(n) - \{n\} \text{ \& } (\exists \beta)(n \in \beta \text{ \& } \beta \text{ is infinite \& } f \text{ regresses } \beta)],$$

$$n_3 = (\mu n)[n_1 \in \hat{f}(n) - \{n\} \text{ \& } (\exists \beta)(n \in \beta \text{ \& } \beta \text{ is infinite \& } f \text{ regresses } \beta)],$$

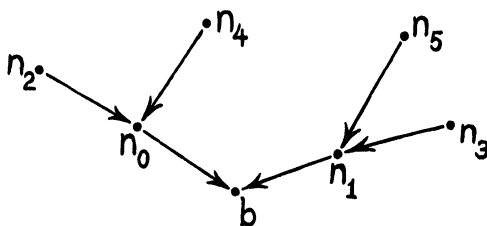


FIGURE 2

$$\begin{aligned}
 n_4 &= (\mu n)[n_0 \in \hat{f}(n) - \{n\} \ \& \ n_2 \notin \hat{f}(n) \\
 &\quad \& \ n \notin \hat{f}(n_2) \ \& \ (\exists \beta)(n \in \beta \ \& \ \beta \text{ is infinite} \ \& \ f \text{ regresses } \beta)], \\
 n_5 &= (\mu n)[n_1 \in \hat{f}(n) - \{n\} \ \& \ n_3 \notin \hat{f}(n) \\
 &\quad \& \ n \notin \hat{f}(n_3) \ \& \ (\exists \beta)(n \in \beta \ \& \ \beta \text{ is infinite} \ \& \ f \text{ regresses } \beta)].
 \end{aligned}$$

The existence of  $n_2, n_3, n_4, n_5$  is guaranteed by (P). At this point, we have obtained from the graph of  $f$  a graph as in Fig. 2. Clearly, by continuing to apply (P) in this manner, we obtain from  $f$  a graph of the type shown in Fig. 1. Thus  $f$  must regress  $2^{\aleph_0}$  infinite sets: contradiction. So, (P) is false; let  $m$  be the smallest number for which  $(\exists \beta)[(\beta \text{ is infinite} \ \& \ m \in \beta \ \& \ f \text{ regresses } \beta) \ \& \ (\forall n)(m \in \hat{f}(n) \Rightarrow (\forall \gamma)((\gamma \text{ infinite} \ \& \ n \in \gamma \ \& \ f \text{ regresses } \gamma) \Rightarrow \gamma = \beta))]$ . Let  $\beta_0$  be the unique such  $\beta$ , and define a partial recursive function  $h$  as follows:

$$\begin{aligned}
 h(x) &\simeq f(x) \quad \text{if } x \neq m, x \in \text{domain}(f), \text{ and } m \in \hat{f}(x); \\
 &\simeq m \quad \text{if } x = m; \\
 &\simeq \text{undefined otherwise.}
 \end{aligned}$$

Then  $h$  is, apart from its definition on  $m$ , a subfunction of  $f$ ; and, furthermore,  $h$  regresses *exactly one* infinite set, namely,  $(\beta_0 - \hat{f}(m)) + \{m\}$ . It follows that  $\beta_0 \in \Pi_2^0 \cdot \Sigma_2^0$ . ■

*Remark.* As Jockusch has pointed out to us, it can be shown that any partial recursive function which regresses only  $\aleph_0$  infinite sets in fact regresses only *hyperarithmetical* sets. (Indeed, Kripke has shown that if a  $\Sigma_1^1$  predicate of functions has a non-hyperarithmetical solution, it has  $2^{\aleph_0}$  solutions. The relativization of Kripke's result gives another, but rather more technically complicated, derivation of (a)  $\Rightarrow$  (c) of *Theorem 6*.) One might be tempted to conjecture that any hyperarithmetical set which is in fact regressive is actually regressed by some partial recursive function which regresses only countably many infinite sets (especially since the corresponding statement with "of degree  $\leq 0$ " for "hyperarithmetical" and "exactly one" for "only countably many" is true). However, this is not the case. By combining two of the theorems of [8] with a certain lemma on "common branches" which

we recently noticed, it is possible to produce a set  $\alpha$  such that  $\alpha \in \Pi_3^0 \cdot \Sigma_3^0$ ,  $\alpha$  is retraced by a basic retracing function (in fact, by a two-to-one general recursive function), and  $\alpha$  is not regressed by any partial recursive function (basic or otherwise) which regresses fewer than  $2^{\aleph_0}$  infinite sets.

Yates has shown, in [8], that any retracing function which retraces a set  $\alpha$  of degree  $\leq 0'$  has a basic restriction  $q$  such that  $\alpha$  is the unique set retraced by  $q$ ; this result holds equally well (and is no more difficult to prove) in the case of *regressive* sets of degree  $\leq 0'$ , relative to basic *regressing* functions. Let " $G(\text{Reg } (0'))$ " denote the set of all indices of basic regressing functions which have restrictions that regress a unique infinite set; by *Lemma 3* and the cited result of Yates (extended to the regressive case),  $G(\text{Reg } (0'))$  is the set of all indices of basic regressing functions that regress some set of degree  $\leq 0'$ . From the finite-to-one-ness property of basic functions, it is easily seen that  $G(\text{Reg } (0'))$  can be expressed at level  $\Sigma_4^0$ ; thus (where " $\{e\}$ " denotes the function with index  $e$ , and " $\{e\}^{-1}(x)$ " denotes the r.e. set  $\{y \mid x \in \{e\}^{\wedge}(y)\}$ ):

$$e \in G(\text{Reg } (0'))$$

$$\begin{aligned} &\Leftrightarrow (\exists x)[\{e\} \text{ is basic regressing} \ \& \ x \in \text{range } (\{e\}) \ \& \ \{e\}^{-1}(x) \text{ is infinite} \\ &\quad \& \ (\forall z)([\{e\}((z)_0) = x \ \& \ \{e\}((z)_1) \\ &\quad = x \ \& \ (z)_0 \notin \{e\}^{\wedge}((z)_1) \ \& \ (z)_1 \notin \{e\}^{\wedge}((z)_0)] \\ &\quad \Rightarrow [\{e\}^{-1}((z)_0) \text{ is finite} \vee \{e\}^{-1}((z)_1) \text{ is finite}]]. \end{aligned}$$

It is not hard to see that " $\{e\}$  is basic regressing" can be expressed in  $\Sigma_4^0$  form<sup>13</sup>; so, by obvious remarks concerning the forms of the remaining conjuncts together with the usual prenex form manipulations, we obtain the right-hand side of the above equivalence as a  $\Sigma_4^0$  predicate. It remains to prove the completeness portion of the following assertion:

**THEOREM 8.**  $G(\text{Reg } (0'))$  is a (many-one) complete set at level  $\Sigma_4^0$ .

*Proof.* Suppose  $P(w) \Leftrightarrow (\exists x)(\forall y)(\exists z)(\forall u)\Gamma(w, x, y, z, u)$ ,  $\Gamma$  recursive. We shall describe informally a procedure for obtaining a many-one reduction of  $\{w \mid P(w)\}$  to  $G(\text{Reg } (0'))$ . Our description will proceed with reference to Fig. 3; the argument will probably be clearer this way than if written out formally in terms of 3-tupling and un-3-tupling functions.

We begin by partitioning  $N$  into a recursive sequence of infinitely many infinite recursive cells, marked "0", "1",  $\dots$  in Fig. 3. Then each cell  $m$  is in turn partitioned into a recursive sequence of infinitely many infinite recursive subcells, labelled " $m0$ ", " $m1$ ",  $\dots$  in Fig. 3; this is to be done so that the collection of all subcells of all cells forms a recursive sequence. Given

<sup>13</sup> If we make the condition of having *finitely many* fixed points a part of the definition of basic regressing function, then, as Yates has independently noted, the set of all indices of basic regressing functions is complete in level  $\Pi_3^0$ .

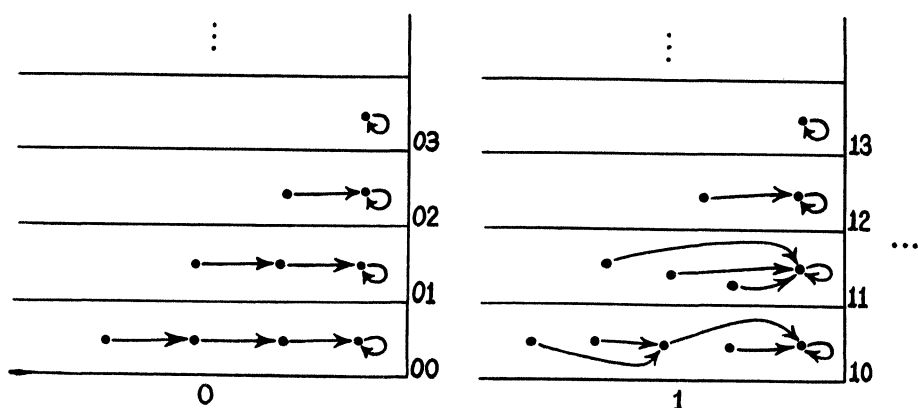


FIGURE 3

$m$  and  $n$ , we let  $a_{mn}^0, a_{mn}^1, a_{mn}^2, \dots$  be the elements of subcell  $mn$ , in increasing order. Our procedure, for a given number  $k$ , is now as follows.

*Stage 0.* Place  $(a_{00}^0, a_{00}^0)$  in  $g$ , give  $a_{00}^0$  a  $+$ , and proceed to Stage 1.

*Stage  $s + 1$ .* Let  $(s + 1)_0 = q, (s + 1)_1 = p$ ; we shall concentrate on cell  $qp$ . If  $(a_{qp}^0, a_{qp}^0)$  is not yet in  $g$ , put it there and give  $a_{qp}^0$  a  $+$ ; otherwise, begin at the next instruction. Now let  $r$  be the least number such that  $a_{qp}^r$  is not yet in the domain of  $g$ , and let  $a_{qp}^l$  be that element of cell  $qp$  which currently bears a  $+$  (when the description of the procedure is complete, it will be clear that, at any given stage, (a) at most one member of any given cell bears a  $+$  and (b) the particular cell being looked at does contain such a number). Consider the predicate  $\Gamma(k, q, p, z, u)$ . Let  $z^*$  be the least number  $z$  such that  $z$  has not yet been *rejected* for cell  $qp$ . If there is a number  $t \leq s$  such that  $\sim \Gamma(k, q, p, z^*, t)$ , then: (i) *reject*  $z^*$  for  $qp$ ; (ii) for each  $j \geq p$ , if any  $a_{qj}^w$  bears a  $+$ , remove the  $+$  from  $a_{qj}^w$  and place it on  $a_{qj}^0$ ; (iii) place  $(a_{qp}^r, a_{qp}^0)$  in  $g$ ; and (iv) for each  $j < p$ , if some  $a_{qj}^{w_0}$  bears a  $+$  and there is a sequence  $a_{qj}^{w_0}, a_{qj}^{w_1}, \dots, a_{qj}^{w_b}$  for which  $b > p, a_{qj}^{w_c} > a_{qj}^{w_{c+1}}$  for  $0 \leq c \leq b - 1, w_b = 0, (a_{qj}^{w_c}, a_{qj}^{w_{c+1}}) \in g$  for  $0 \leq c \leq b - 1$ , and  $(a_{qj}^0, a_{qj}^0) \in g$ , then remove the  $+$  from  $a_{qj}^{w_0}$  and place it on  $a_{qj}^{w_b-p}$ . Then go to Stage  $s + 2$ . If, on the other hand,  $\Gamma(k, q, p, z^*, t)$  holds for all  $t \leq s$ , place  $(a_{qp}^r, a_{qp}^l)$  in  $g$ , move  $+$  from  $a_{qp}^l$  to  $a_{qp}^r$ , and then proceed to Stage  $s + 2$ .

This completes the description of the construction; it is clear that this construction is uniformly effective with respect to  $k$ , and produces for each  $k$  a general recursive function  $g_k$ . Further, the reader should be able easily to convince himself, on the basis of the above description, of the following: if

$$(\exists x)(\forall y)(\exists z)(\forall u)\Gamma(k, x, y, z, u),$$

then  $g_k$  is basic and extends some regressing function regressing a unique infinite set; while if

$$\sim(\exists x)(\forall y)(\exists z)(\forall u)\Gamma(k, x, y, z, u),$$

then  $g_k$  does not regress any infinite set (and is everywhere lacking in adequate finite-to-one-ness). *Theorem 8* follows.

*Remark.*<sup>14</sup> By essentially the same argument as in the proof of *Theorem 8* (only the computation of the *upper* bound in the hierarchy changes from case to case), we can show that the following index sets are also  $\Sigma_4^0$ -complete:

- (a) the set of indices of partial recursive extensions of basic regressing functions;
- (b) the set of indices of general recursive extensions of basic regressing functions;
- (c) the set of indices of partial recursive restrictions of partial recursive extensions of basic regressing functions;
- (d) the set of indices of partial recursive restrictions of general recursive extensions of basic regressing functions;
- (e) each of (a)–(d) with “regressing” replaced by “retracing”;
- (f) the set  $G(\text{Ret}(0'))$  obtained by substituting “retracing” and “retrace” for “regressing” and “regress”, respectively, in the definition of  $G(\text{Reg}(0'))$ .

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<sup>14</sup> In reply to an inquiry, Yates has informed the author that he was independently aware of most of these classifications. However we do not know his proofs, and so do not know to what extent, if any, our proof of *Theorem 8* differs from his method of obtaining such classifications.