THE BAR CONSTRUCTION AND ABELIAN H-SPACES

BY

R. JAMES MILGRAM

If X is an associative H-space with unit Dold and Lashof [2] have given a method for constructing a classifying space B_X which generalizes the classifying spaces for topological groups. In this note we present a construction which has three advantages over that of [2]:

(1) if X is abelian B_X is also an abelian associative H-space with unit (Section 1),

(2) if X is a CW complex and the multiplication is a cellular map then B_X is also a CW complex and the cellular chain complex of B_X is isomorphic to the bar construction on the cellular chain complex of X (Section 2),

(3) there is an explicit diagonal approximation $D : B_X \to B_X \times B_X$ which is cellular and in the cell chain complex of B_X induces exactly Cartan's diagonal approximation for the bar construction (Section 3).

These properties are all easily established and once obtained are applied in Section 4 to give elementary constructions for the Eilenberg-Maclane spaces and to deduce the algebraic and geometric preliminaries to Cartan's calculations of $H^*(K(\pi, n))$.

Remark. The recent results of J. C. Moore and S. Eilenberg [1] and N. Steenrod and E. Rothenberg [8] which are applied to study $H^*(B_X)$ from information on the homology algebra $H_*(X)$ may also be easily developed using our techniques.

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1. The construction

In this section we define the i-classifying spaces $B^i(X)$ for a given associative H-space X with identity $*$ and prove some of their more important properties. Most of these results are well known in one form or another [2], [5], [6], the only novel results being 1.6, 1.7, which exhibit the abelian multiplication in B_X and play a vital role in the applications.

Let σ^n be the Euclidian n-simplex represented as the set of points $(t_1, \cdots, t_n)$ in R^n with

$$0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq 1.$$

It has faces σ_i^n (for which $t_i = t_{i+1}$) $1 \leq i < n$, σ_0^n $(t_1 = 0)$, and σ_n^n $(t_n = 1)$.

Let $A^i(X)$, $0 \leq i \leq \infty$, be the disjoint union $\sum_{j=0}^{i} X \times \sigma^j \times X^j$, ($X^j$ is the j-fold Cartesian product). In A^i generate an equivalence relation by

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means of generators of two kinds:

$$(1)\quad (x, t_1, \cdots, t_n, x_1, \cdots, x_n) \sim (x, t_1, \cdots, \hat{t}_i, \cdots, t_n, x_1, \cdots, \hat{x}_i, (x_i \cdot x_{i+1}), \cdots, x_n)$$

if $t_i = t_{i+1}$ or $x_i = *$ (for $i = n$ delete the last coordinates if $t_n = 1$ or $x_n = *$).

$$(2)\quad (y, 0, t_2, \cdots, t_n, x_1, \cdots, x_n) \sim (yx_1, t_2, \cdots, t_n, x_2, \cdots, x_n).$$

$E^i(X)$ is defined to be $A^i(X)/R$. It is topologized by the quotient topology and π_i is the projection

$$\pi_i : A^i \rightarrow E^i.$$

There are two things to notice about $E^i(X)$.

PROPOSITION 1.1. *X acts continuously and associatively as a set of "left transformations" $X \times E^i(X) \rightarrow E^i(X)$.*

Proof. Let $y \in X$, then $y : A^i(X) \rightarrow A^i(X)$ is given by

$$y(x, t_1, \cdots, t_n, x_1, \cdots, x_n) = (yx, t_1, \cdots, t_n, x_1, \cdots, x_n).$$

This action is continuous in X and $A^i(X)$, takes equivalence classes into equivalence classes and hence induces the desired action in $E^i(X)$, Q.E.D.

Let $E^\infty(X)$ be the union of the $E^i(X)$ with the weak topology.

PROPOSITION 1.2. *$E^i(X)$ is contained and contractible in $E^{i+1}(X)$. In particular $E^\infty(X)$ is contractible.*

Proof. The inclusion $A^i \subset A^{i+1}$ respects equivalence. The contraction is induced by the map

$$F_t : A^i \rightarrow A^{i+1}$$

given on points by

$$F_t(x, t_1, \cdots, t_j, x_1, \cdots, x_j) = (*, \bar{t}, \overline{t + t_j}, \cdots, \overline{t + t_1}, x, x_1, \cdots, x_j)$$

(where $\bar{\alpha} = \max(1, \alpha)$) which also respects equivalence.

DEFINITION 1.3. $B^i(X)$ is the set of equivalence classes of points of $E^i(X)$ under the action of X. It has the quotient topology, and ρ_i is the projection

$$\rho_i : E^i(X) \rightarrow B^i(X).$$

From 1.2 it would seem that $B^\infty(X)$ is in some sense a classifying space for X. This is justified by the following two results.

THEOREM 1.4. *Let G be a topological group with identity $*$, N an open neighborhood of $*$, h_t a homotopy $h_t(G, *) \rightarrow (G\, *)$ with $h_0 =$ id, $h_t(N) \subset N$ and $h_1(N) = *$; then $\rho_i\, E^i(G) \rightarrow B^i(G)$ is a Steenrod fiber bundle with fiber and group G.*

Proof. For $i = 0$, $E^0 = G$, $B^0 = *$ and the result is true. The proof now goes by induction. Assume the theorem for $j \leq i$. Since E^i is closed in E^{i+1} and $E^{i+1} - E^i$ is equivalent to a product $G \times (B^{i+1} - B^i)$ it suffices to show the theorem in some open set M with

$$B^i \subset M \subset B^{i+1}.$$

Let $S \subset G^{(i+1)}$ be the set of points $(g_1 \cdots g_{i+1})$ with some $g_j = *$. Our assumption on G implies there is a homotopy id $\times k_t$ taking

$$(G \times \sigma^{i+1} \times G^{(i+1)}, \qquad G \times \sigma^{i+1} \times S \cup G \times \partial\sigma^{i+1} \times G^{i+1})$$

into itself with $k_0 = \text{id}$. Deforming this homotopy we have that id $\times k_1'$ is a fiber preserving retraction of some neighborhood U of

$$G \times (\sigma^{i+1} \times S \cup \partial\sigma^{i+1} \times G^{(i+1)})$$

into

$$G \times (\sigma^{i+1} \times S \cup \partial\sigma^{i+1} \times G^{(i+1)}).$$

Set $M = \rho_{i+1}\,\pi_{i+1}(U)$, $V = \pi_{i+1}(U)$. Then the constructed retraction induces retractions

$$l : V \to E^i, \qquad m : M \to B^i \qquad\qquad (\rho_{i+1}\, l = m).$$

Moreover, since G is a group l maps fibers homeomorphically onto fibers. Thus we may extend the local product structure in E^i into $\rho^{-1}(M)$, finally comparing the structure in $\rho^{-1}(M)$ with that in $E^{i+1} - E^i$ we see that they differ only by left translation by elements of G, Q.E.D.

THEOREM 1.5. *If X is a connected, associative H-space with identity $*$ and a homotopy h_t as in* 1.4, *then $\rho_i : E^i(X) \to B^i(X)$ is a quasifibration with fiber X.*

(For the definition and major properties of quasifibrations see [7].)

Proof. This is identical to that of 1.4 up to the last paragraph. k_t need not now map fibers homeomorphically. However, since X is connected it is a homotopy equivalence on fibers. The proof is now completed by means of 2.2, 2.10 and for $B^\infty(X)$ 2.15 of [7].

In case X is abelian there is additional structure in $B^\infty(X)$.

THEOREM 1.6. *If X is abelian then $B^\infty(X)$ is an associative, abelian H-space with unit $*$.*

Proof. Define a mapping $u : B^\infty(X) \times B^\infty(X) \to B^\infty(X)$ by

$$u\{(t_1, \cdots, t_n, x_1, \cdots, x_n), (t_{n+1}, \cdots, t_{n+m}, x_{n+1}, \cdots, x_{n+m})\}$$
$$= (t_{\alpha(1)}, \cdots, t_{\alpha(n+m)}, x_{\alpha(1)}, \cdots, x_{\alpha(n+m)})$$

where α is any element of the symmetric group S_{n+m} for which

$$t_{\alpha(1)} \leq t_{\alpha(2)} \leq \cdots \leq t_{\alpha(n+m)}\,.$$

As a consequence of the first identification relation u is well defined and it is easy to verify its continuity.

COROLLARY 1.7. *If G is an abelian topological group the same is true of* $B^\infty(G)$.

Proof. It suffices to exhibit inverses. In fact

$$(t_1, \cdots, t_n, x_1, \cdots, x_n)^{-1} = (t_1, \cdots, t_n, x_1^{-1}, \cdots, x_n^{-1}).$$

Example 1.8. Let $SP^\infty(Y)$ be the infinite symmetric product of Y. It is an associative abelian H-space and

$$E^\infty(SP^\infty(Y)) = SP^\infty(cY), \qquad B^\infty(SP^\infty(Y)) = SP^\infty(\Sigma Y),$$

these equivalences being *multiplicative homeomorphisms.* (Here cY is the reduced cone on Y, and ΣY is the reduced suspension.) This may be seen by identifying the point

$$\langle (t_1, a_1), \cdots, (t_1 a_i), (t_2 a_{i+1}), (t_2 a_{i+j}), \cdots, (t_n a_s), \cdots, (t_n a_{s+k}) \rangle$$

of $SP^{s+k}(cX)$ with the point

$$\langle u(t_1, \langle a_1 \cdots a_i \rangle), (t_2 \langle a_{i+1} \cdots a_{i+j} \rangle), \cdots, (t_n, \langle a_s \cdots a_{s+k} \rangle) \rangle$$

of $E^n(SP^\infty(X))$. In fact, in this case the fibration is given in [7] and plays an important part in the proof of the main theorem.

Remark 1.9. This construction was originally given by Stasheff in [6] in a form equivalent to that in which it appears here. He shows that it is homotopy equivalent to those given by Milnor [5] and Dold-Lashof [7], and from this it follows that for X connected $E^i(X)$ is i-connected.

2. The cellular construction

X is now assumed to be a CW complex with cellular multiplication and $*$ a 0-cell.

Remark 2.1. If X is an associative H-space, then $S(X)$, the singular polytope of X, is an associative H-space with cellular multiplication. Thus, using the equivalence between X and $S(X)$, our restriction on X is always satisfied, at least up to weak H-equivalence.

The k topology on X is defined by letting U be k-open only if $U \cap C$ is relatively open in C for every compact subset C of X. In general the k topology is finer than the original, hence the identity

$$i : (X, k) \to X$$

is continuous.

Since (X, k) has the same compact sets as X it follows that i induces isomorphisms of the respective singular complexes. Hence i is always a weak homotopy equivalence.

Finally, if X, Y are CW-complexes then

(1) $(X, k) = X$

(2) $(X \times Y, k)$ is the "product" CW complex of X and Y, that is its cells are products of cells of X and Y.

Let $\bar{A}^i$ be (A^i, k) and $\bar{E}^i = \bar{A}^i/R$ with the quotient topology. Then 1.1, 1.2 hold without change and $\bar{B}^i = \bar{E}^i/X$ is defined. In fact, with the exception of 1.4, all the results of Section 1 continue to hold. Thus by use of excision and the Eilenberg-Zilber theorem it follows that the identity map

$$i : \bar{B}^j \to B^j$$

induces isomorphisms in homology.

Now we digress shortly to recall the definition of the bar construction. Given a D.G.A. algebra A over a commutative ground ring Γ with unit, and with augmentation $\varepsilon : A \to \Gamma$, let $\bar{A} = \ker \varepsilon$, then the bar construction $B(A)$ is

$$\Gamma + \bar{A} + \bar{A} \otimes_\Gamma \bar{A} + \cdots + \bar{A} \otimes_\Gamma \cdots \otimes_\Gamma \bar{A} + \cdots .$$

A typical generator $a_1 \otimes \cdots \otimes a_n$ is written $[a_1 | \cdots | a_n]$, and has dimension $(\sum_{j=1}^n \dim a_j) + n$. $B(A)$ is a graded chain complex with boundary operator defined by

$$\begin{aligned}\partial[a_1 | \cdots | a_n] = & \textstyle\sum_{j=1}^{n-1} (-1)^j [a_1 | \cdots | a_j \cdot a_{j+1} | \cdots | a_n] \\ & + \textstyle\sum_{j=1}^n (-1)^{n+\delta(j)} [a_1 | \cdots | \partial a_j | \cdots | a_n] \\ & + \varepsilon(a_1)[a_2 | \cdots | a_n] + (-1)^n \varepsilon(a_n)[a_1 | \cdots | a_{n-1}]\end{aligned}$$

where $\delta(j) = \sum_{k<j} \dim a_k$.

A generator of the form $[a_1 | \cdots | a_n]$ is said to have degree n, and $F^n(A)$ is the Γ-submodule of $B(A)$ generated by the elements of degree $\leq n$. Clearly, the $F^n(A)$ give a filtration of $B(A)$. The resulting spectral sequence was studied by Eilenberg-Moore in [1].

Returning to classifying spaces recall that the CW chain complex of X is given by

$$C_i(X) = H_i(X_i, X_{i-1} ; Z)$$

and the boundary is that in the exact sequence of the triple (X_i, X_{i-1}, X_{i-2}).

PROPOSITION 2.2. There is a natural chain isomorphism

$$J : C_\#(X) \otimes C_\#(Y) \to C_\#(X \times Y, k)$$

(The proof is direct.)

As a result the map $ui : (X \times X, k) \to X$ (u is the multiplication) induces a chain map

$$u_\# i_\# J : C_\#(X) \otimes C_\#(X) \to C_\#(X)$$

which makes $C_{\#}(X)$ into a D.G.A. algebra. We can now state the main result of this section.

THEOREM 2.3. *$\bar{B}^{\infty}(X)$ is a CW-complex and there is an isomorphism*

$$T : B((C_{\#}(X)) \to C_{\#}(\bar{B}^{\infty}(X)).$$

Proof. A cell $\sigma^n \times e^1 \times \cdots \times e^n$ of $\bar{B}^{\infty}(X)$ is defined to be the set of all points $(t_1, \cdots, t_n, x_1, \cdots, x_n)$ with $x_i \in e^i$. It is easy to verify that this decomposition gives $\bar{B}^{\infty}(X)$ the structure of a CW complex.

The boundary operator on such a cell is given by

$$\partial[\sigma^n \times e^1 \times \cdots \times e^i] = [\partial\sigma^n \times e^1 \times \cdots \times e^i] + (-1)^n[\sigma^n \times \partial(e^1 \times \cdots \times e^i)].$$

From the identifications of Section 1 it follows that

$$[\sigma_j^n \times e^1 \times \cdots \times e^n] = [\sigma^{n-1} \times e^1 \times \cdots \times (e^j \cdot e^{j+1}) \times \cdots \times e^n] \quad \text{for} \quad 1 \leq j < n.$$

Similarly,

$$[\sigma_0^n \times e^1 \times \cdots \times e^n] = \varepsilon(e^1)[\sigma^{n-1} \times e^2 \times \cdots \times e^n],$$

$$[\sigma_n^n \times e^1 \times \cdots \times e^n] = \varepsilon(e^n)[\sigma^{n-1} \times e^1 \times \cdots \times e^{n-1}].$$

Thus if we identify $\sigma^n \times e^1 \times \cdots \times e^n$ with $|e^1|\cdots|e^n|$ in $B(C_{\#}(X))$ the desired isomorphism is obtained, Q.E.D.

COROLLARY 2.4. *The chain complex $C_{\#}(\bar{B}^i(X))$ is chain isomorphic under T with $F^iB(C_{\#}(X))$.*

If X is abelian the multiplication

$$\bar{B}^{\infty}(X) \times \bar{B}^{\infty}(X) \to \bar{B}^{\infty}(X)$$

given in Section 1 is still continuous and is, moreover, cellular. In fact

$$\sigma^n \times e^1 \times \cdots \times e^n \cdot \sigma^m \times e^{n+1} \times \cdots \times e^{n+m} = \sum\nolimits_{\alpha \in S(n,m)} (-1)^{\gamma(\alpha)} \sigma^{m+n} \times e^{\alpha^{-1}(1)} \times \cdots \times e^{\alpha^{-1}(n+m)}.$$

Here, α is an n, m, shuffle, $\gamma(\alpha) = \alpha + m \sum_{k \leqq n} \dim(e^k) + s$ where s is the sign associated with the map

$$\text{Shuff } \alpha : (C_{\#}(X))^{n+m} \to (C_{\#}(X))^{n+m}.$$

This is shown by subdividing $\sigma^n \times \sigma^m$ into simplexes via the standard decomposition (see p. 68 of [3]), and observing that the restriction to the interior of each of these simplexes is a homeomorphism onto the interior of its image cell, in the composition

$$\sigma^{n+m} \times e^1 \times \cdots \times e^{n+m} \to (\sigma^n \times e^1 \times \cdots \times e^n) \times (\sigma^m \times e^{n+1} \times \cdots \times e^{n+m}) \to \bar{B}^i(X).$$

3. The diagonal map

A homotopy of the diagonal map $\Delta : \sigma^n \to \sigma^n \times \sigma^n$ is defined by

$$f_t(t_1, \cdots, t_n) = \{\overline{(1+t)t_1}, \cdots, \overline{(1+t)t_n}; \underline{(1+t)(t_1-1)+1}, \cdots, \underline{(1+t)(t_n-1)+1}\}$$

where $\bar{a} = \min(a, 1)$, $\underline{b} = \max(b, 0)$. On a typical point $(t_1, \cdots, t_n)$ with $t_i \leq \frac{1}{2} \leq t_{i+1}$

$$f_1(t_1, \cdots, t_n) = \{(2t_1, \cdots, 2t_i, 1, \cdots, 1), (0, \cdots, 0, 2t_{i+1}-1, \cdots, 2t_n - 1)\},$$

and on the chain level

$$f_1(\sigma^n) = \sum (F_j\sigma^n) \otimes (L_{n-j}\sigma^n)$$

the usual diagonal approximation.

Shuff $(\Delta \times f_t \times \Delta) : A^i \to A^i \times A^i$ commutes with the identifications and hence induces a homotopy

$$F_t : \bar{B}^i(X) \to \bar{B}^i(X) \times \bar{B}^i(X).$$

For a typical point

$$F_1(t_1, \cdots, t_n, x_1, \cdots, x_n) = \{(2t_1, \cdots, 2t_i, x_1, \cdots, x_i), (2t_{i+1}-1, \cdots, 2t_n - 1, x_{i+1}, \cdots, x_n)\},$$

and an easy calculation now gives

THEOREM 3.1.

$$F_{1\#}[e^1 | \cdots | e^n] = \sum_{j=0}^{n} (-1)^{\tau(j)} [e^1 | \cdots | e^j] \otimes [e^{j+1} | \cdots | e^n]$$

is a diagonal approximation where $\tau(j) = (n-j)(\sum_{k \leq j} \dim E^k)$.

4. Application to Eilenberg-MacLane spaces

Let π be a group. Give it the discrete topology. Then it is a CW complex consisting of 0-cells and the multiplication is cellular. Hence we can apply the theory of the last three sections.

In particular $E^\infty(\pi)$ is the universal covering space of $B^\infty(\pi)$ with π as group of cover transformations. Hence $B^\infty(\pi)$ is a $K(\pi, 1)$, as is also $\bar{B}^\infty(\pi)$.

If π were abelian then $\bar{B}^\infty(\pi)$ would be an abelian topological group with cellular multiplication, and we could iterate the construction obtaining $\bar{B}^\infty(\bar{B}^\infty(\pi)) = K(\pi, 2)$, etc. thus we have

THEOREM 4.1. *If π is an abelian group then there is a $K(\pi, n)$ for each $n = 0, 1, 2, \cdots$ which is a topological abelian group and a CW complex with cellular multiplication. Moreover,*

$$C_{\#}(K(\pi, n)) \cong B(C_{\#}(K(\pi, n-1)),$$

the isomorphism being of D.G.A.-algebras.

Taking into account the diagonal maps we have

THEOREM 4.2. (i) $H_*(K(\pi, n)) \cong H_*(B(C_{\#}(K(\pi, n-1))))$
(ii) $H^*(K(\pi, n)) \cong H^*(B(C_{\#}(K(\pi, n-1))))$,
the isomorphisms being ring homomorphisms respectively of Pontrjagin and cohomology rings.

Other results on suspension and transgression may be obtained from the fact that $B^1(X) = \Sigma(X)$, but we do not make them explicit here.

5. The H-type of an abelian H-space.

Suppose again that X is an abelian associative H-space with unit. There is then a map

$$F : X \to \Omega(B^{\infty}(X))$$

defined by $[F(x)]t = (1 - t, x) \in \Sigma X \subset B^{\infty}(X)$. Suppose we define an abelian multiplication in the space E of paths by $(\gamma \cdot \tau)t = \gamma(t) \cdot \tau(t)$, then F may be extended to a map $G : E^{\infty}(X) \to E$ as follows: We first define a path x_s by

$$x_s(t) = (1 - (1 - s)t, x),$$

then $G : (x, t_1, \cdots, t_n, x^1, \cdots, x^n) = F(x) \cdot x^1_{t_1} \cdots x^n_{t_n}$ and it is easy to check that

(1) G is a continuous homomorphism,

(2)
$$\begin{array}{ccc} E^{\infty} & \xrightarrow{\ G\ } & E \\ & {\scriptstyle\rho}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & B^{\infty}(X) & \end{array}$$

(3) $G \mid X = F$.

Thus, from the homotopy exact sequences of the 2 quasifibrations it follows that F is a weak homotopy equivalence (if X is connected). Moreover, as $B^{\infty}(X)$ is again a connected, associative, abelian H-space with unit we may iterate the constructions and we have

THEOREM 5.1. *X (as above) is weakly homotopic to an n^{th} loop space $\Omega^n(Y)$ for $n = 1, 2, 3, \cdots$. Moreover, there is an H-structure on $\Omega^n(Y)$ so that X is actually H-equivalent to $\Omega^n(Y)$.*

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The University of Minnesota
Minneapolis, Minnesota
Princeton University
Princeton, New Jersey