ON THE PROPAGATOR EQUATION

BY

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1. We are concerned here with weak and strong solutions of the evolution equation

$$(1.1) \qquad y' + A(t)y = f(t); \qquad y(\tau) = y_0$$

and related abstract equations. A number of observations and theorems will be given with no particular attempt at a "unified theory" (see [15] for a more complete discussion). Thus Part 2 is on strong solutions and propagators $G(t, s)$ solving (1.1) in the form

$$(1.2) \qquad y = G(t, \tau)y_0 + \int_\tau^t G(t, s)f(s)\, ds,$$

Part 3 contains some new results on weak solutions, and Part 4 is on some abstract problems. Some of the results have been announced in [16].

2. We suppose $A(t)$ is an unbounded linear operator in the separable Hilbert space H with domain $D_t = D(A(t))$ usually dense but this will be specified in each case. To begin with we suppose the problem (1.1) can be solved (uniquely) for

$$y_0 \in I_\tau \subset H \quad \text{and} \quad f(\cdot) \in F_\tau \subset L^2(H) = E$$

for some linear spaces I_τ and F_τ; furthermore we will deal with the finite interval case $\tau \leq t \leq T < \infty$ in general since all of the main features of the problem are exhibited there. We stipulate that all derivatives are in $D'(H)$ (see [33]) and the terms in (1.1) are in $L^2(H)$. First we note a somewhat stronger form of a lemma proved in [10] which is surely well known but seems not to have been written down in this form. Let $A(t)$ be accretive i.e., $\text{Re}\,(A(t)x, x) \geq 0$, and let y be a unique solution of (1.1) with $y_0 = 0$ which we write $y = K(f)$. Now

$$(2.1) \qquad \begin{aligned} \text{Re}\,(f, K(f))_E &= \text{Re} \int_\tau^T (y' + Ay, y)_H\, dt \\ &= \frac{1}{2} \int_\tau^T \frac{d}{dt} \| y \|^2\, dt + \text{Re} \int_\tau^T (Ay, y)\, dt \geq 0 \end{aligned}$$

(see [3] for integration theory in $L^2(H)$). If F_τ is dense and K extends by continuity to a continuous map $\bar{K} : E \to E$ then (2.1) extends to all E. But from (2.1) we can deduce also that

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$$\| y(t) \|^2 \leq 2 \| f \| \| Kf \|$$

from which follows the inequality

$$\| y \|_E = \| Kf \|_E \leq 2(T - \tau) \| f \|_E \leq 2c \| f \|_E .$$

Consequently K is a bounded operator $E \to E$ defined on a dense F_τ and hence is extendable.

PROPOSITION 1. *Let $A(t)$ be accretive in H and $y = K(f)$, $f \in F_\tau$, be a unique solution to* (1.1) *with $y_0 = 0$. If F_τ is dense in E then K extends by continuity to a continuous accretive operator $\bar{K} : E \to E$.*

Suppose now $(t, s) \to G(t, s) \in C^0(L_s(H))$ for $\tau \leq s \leq t \leq T$ (C^1 means K-times continuously differentiable functions and $L_s(H)$ is the space of bounded linear operators $H \to H$ with the topology of simple convergence or strong operator topology). Let

$$M(f) = \int_\tau^t G(t, s) f(s)\, ds.$$

This is well defined for any $f \in E$ (observe that $\varphi \in L^2$ implies $\varphi \in L^1$ on finite intervals). Since $\tau \leq s \leq t$, t fixed, is compact and $s \to G(t, s) \in C^0(L_s(H))$ we know $G(t, s)$ is simply bounded for $\tau \leq s \leq t$ and hence strongly bounded by Banach-Steinhaus (see [4]). In fact the same argument applies to $G(t, s)$ for $\tau \leq s \leq t \leq T$ by the continuity of $(t, s) \to G(t, s)$ and thus on this set $\| G(t, s) \| \leq C$. Hence

$$\begin{aligned} \| M(f)(t) \|_H &\leq C(t - \tau)^{1/2} \left(\int_\tau^T \| f(s) \|^2 ds \right)^{1/2} \\ &\leq C_1 \| f \|_E \end{aligned} \tag{2.2}$$

and consequently $\| M(f) \|_E \leq C_2 \| f \|_E$ and $M \cdot E \to E$ is continuous. This is of course well known (cf. [18]) and combined with Proposition 1 yields the result mentioned above.

PROPOSITION 2. *Let*

$$M(f) = \int_\tau^t G(t, s) f(s)\, ds$$

with $(t, s) \to G(t, s) \in C^0(L_s(H))$ for $\tau \leq s \leq t \leq T < \infty$. Then M is a bounded linear operator $E \to E$ and if $M(f)$ is the unique solution of (1.1) *with $y_0 = 0$ for $f \in F_\tau$ dense in E then the kernel $G(t, s)$ determines an accretive operator $M : E \to E$.*

We shall say (cf. [37]) that a family of operators $P(t, s) \in L(H)$, $\tau \leq s \leq t \leq T$, is a propagating family (or simply that $P(t, s)$ is a propagator) if $P(t, t) =$ identity and for $\tau \leq \xi \leq s \leq t \leq T$,

$$P(t, \xi) = P(t, s) P(s, \xi). \tag{2.3}$$

We write the solution of (1.1) with $f = 0$ now as

$$y(t) = P(t, \tau)y_0 . \tag{2.4}$$

We shall assume that solutions of (1.1) *are unique in this section unless otherwise stated.* Then observe that $y' \in L^2(H)$ implies y is continuous (cf. [18], [23]) and uniqueness implies that (2.4) holds for a linear operator $P(t, \tau)$ satisfying (2.3) on its domain of definition (cf. [5]). We shall assume that the operator K of Proposition 1 is an integral operator with kernel $G(t, s)$, $\tau \leq s \leq t \leq T$, but no assumptions about $G(t, s)$ are made unless explicitly stated below. Thus we consider only cases when the (unique) solution of (1.1) is given by a recovery formula

$$y(t) = P(t, \tau)y(\tau) + \int_\tau^t G(t, s)f(s)\, ds. \tag{2.5}$$

Standard results give $P(t, s) = G(t, s)$ but the proofs usually involve continuity and differentiability properties of $G(t, s)$ or $P(t, s)$. We want to see what relations persist under minimal hypotheses. A somewhat more abstract treatment of this was given in [11] with less detail.

We make first a few preliminary observations. First from the continuity of y follows the continuity of $t \rightarrow P(t, \tau)y_0$ for $y_0 \in I_\tau$. Next from the relation $y(t) = P(t, s)P(s, \tau)y(\tau)$ follows the solvability of (1.1) (with $f = 0$) for $t \geq s$ with initial value $y_s = P(s, \tau)y(\tau)$. Consequently

$$\tilde{I}_s = P(s, \tau)I_\tau \subset I_s .$$

Another useful fact is that if $P(t, s) \in L(H)$ is a propagator with $t \rightarrow P(t, s)$ and $s \rightarrow P(t, s)$ strongly continuous then in fact $(t, s) \rightarrow P(t, s)$ is strongly continuous. To see this simply write

$$\begin{aligned} &P(t', s') - P(t, s) \\ &\qquad = P(t, s') - P(t, s) + [P(t', \xi) - P(t, \xi)]P(\xi, s') \end{aligned} \tag{2.6}$$

for some suitable fixed ξ.

Under certain natural conditions the $P(t, \tau)$, defined on I_τ, have an immediate extension by continuity to operators $\bar{P}(t, \tau) \in L(H)$. For example if $y' + A(t)y = 0$ then following (2.1),

$$\frac{1}{2}(\| y(t) \|^2 - \| y(\tau) \|^2) + \mathrm{Re} \int_\tau^t (Ay, y)\, dt = 0 \tag{2.7}$$

and if $A(t)$ is accretive we have

$$\| y(t)\| = \| P(t, \tau)y(\tau)\| \leqq \| y(\tau)\|.$$

Hence $\| P(t, \tau)\| \leq 1$ on I_τ and if I_τ is dense $P(t, \tau)$ extends by continuity as indicated (regarding the density of I_τ see [27]). If I_s is dense for all $s \geq \tau$ then all $P(t, s)$ extend to operators $\bar{P}(t, s) \in L(H)$ and the propagator relation (2.3) extends by continuity as well. We can state

PROPOSITION 3. *Suppose* $\| P(t, s) \| \leq C$ *on* I_s *with* I_s *dense for all* $\tau \leq s \leq t \leq T$. *Then* $P(t, s)$ *extends by continuity to* $\bar{P}(t, s) \in L(H)$ *and* $t \to \bar{P}(t, s)$ *is strongly continuous.*

Proof. We know $t \to P(t, s)$ is strongly continuous on I_s dense. The set

$$Q = \{\bar{P}(t, s)\} \subset L(H),$$

$t \geq s$ (s fixed), is strongly bounded and hence equicontinuous by Banach-Steinhaus (see [4]). Moreover on Q the topology of simple convergence is equivalent to the topology of simple convergence over a dense set $I_s \subset H$. Hence $t \to \bar{P}(t, s)$ is strongly continuous since $\bar{P}(t, s) = P(t, s)$ on I_s, Q. E. D.

Now take again the situation of Proposition 3 and look at

$$s \to P(t, s)P(s, t)y_0$$

which is constant and hence continuous. We write then for $\Delta s > 0$

$$\begin{aligned} 0 = \bar{P}(t, s + \Delta s)[P(s + \Delta s, \tau) - P(s, \tau)]y_0 \\ + [\bar{P}(t, s + \Delta s) - P(t, s)]P(s, \tau)y_0. \end{aligned} \tag{2.8}$$

The first term vanishes as $\Delta s \to 0$ since $s \to P(s, \tau)y_0$ is continuous and $\| \bar{P}(t, s) \| \leq C$. Hence

$$[\bar{P}(t, s + \Delta s) - \bar{P}(t, s)]y_s \to 0$$

where $y_s = P(s, \tau)y_0$ is fixed. If $\Delta s < 0$ the same expression still makes sense and the same conclusion holds. Using the above remarks we have

COROLLARY 1. *In the situation of Proposition* 3 $s \to \bar{P}(t, s)$ *is strongly continuous on* $\tilde{I}_s = P(s, \tau)I_\tau$. *If each* $\tilde{I}_s$ *is dense then*

$$(t, s) \to \bar{P}(t, s) \in C^0(L_s(H)).$$

Note that $\tilde{I}_s$ dense implies I_s dense of course. Next suppose $t \to P(t, \tau)$ is strongly C^1 on I_τ for $t > \tau$ with then

$$\partial P(t, \tau)y_0/\partial t + A(t)P(t, \tau)y_0 = 0$$

(i.e., we suppose y' is continuous and that the derivative can be taken in the usual (vector-valued) sense). Since $P(t, s)P(s, \tau)$ is independent of s we have for $\Delta s > 0$ or $\Delta s < 0$

$$\begin{aligned} 0 = \bar{P}(t, s + \Delta s)\left[\frac{P(s + \Delta s, \tau) - P(s, \tau)}{\Delta s}\right]y_0 \\ + \left[\frac{\bar{P}(t, s + \Delta s) - P(t, s)}{\Delta s}\right]P(s, \tau)y_0. \end{aligned} \tag{2.9}$$

Suppose $s \to \bar{P}(t, s)$ is strongly continuous. Then for $s > \tau$ the first term tends to $\bar{P}(t, s)P_s(s, \tau)y_0$ as $\Delta s \to 0$ and hence $\bar{P}_s(t, s)$ exists strongly on elements $y_s = P(s, \tau)y_0$ and

$$\bar{P}_s(t, s)y_s = -\bar{P}(t, s)P_s(s, \tau)y_0 = \bar{P}(t, s)A(s)P(s, \tau)y_0 = \bar{P}(t, s)A(s)y_s .$$

Consequently

$$[\bar{P}_s(t, s) - \bar{P}(t, s)A(s)]y_s = 0. \tag{2.10}$$

PROPOSITION 4. *If $t \to P(t, \tau)$ is strongly C^1 on I_τ and $P(t, s)$ is extendable to $\bar{P}(t, s) \in L(H)$ then $s \to \bar{P}(t, s)$ strongly continuous implies $s \to \bar{P}(t, s)$ is strongly C^1 on $P(s, \tau)I_\tau = \tilde{I}_s$ and* (2.10) *holds.*

Now we look again at (2.5) and will show how to relate $P(t, \tau)$ and $G(t, \tau)$ under very weak hypotheses. We know only that $t \to P(t, \tau)$ is strongly continuous on I_τ and will assume

$$s \to G(t, s) \in C^0(L_s(H)).$$

Suppose further that on any $[s, T]$, $C^1(H) \cap C^0(D(A(t)))$ has a set $\hat{I}_s \subset I_s$ of initial values dense in H (i.e., assume the values $g(s)$ of functions $g \in L^2(H)$, such that $g' \in C^0(H)$ and $t \to A(t)g(t) \in C^0(H)$ for $s \leq t \leq T$, are dense in H). For information in this direction see e.g., [23]. Put now $Y_\xi = 1$ for $\tau - 1 \leq t < \xi$, $Y_\xi = 0$ for $t < \tau - 1$ and $t \geq \xi$ (ξ fixed) with $\tau < \xi < T$ and $y_n = (\varphi_n * Y_\xi)^\frown g$ with $g \in C^1(H) \cap C^0(D(A(t)))$, where $\frown$ denotes restriction to $[t, T]$, and $\varphi_n(x) = n\varphi(nx)$ with $\varphi \geq 0$, supp $\varphi \subset [-1, 1]$, and $\int \varphi(x)\, dx = 1$. Thus $\varphi_n(x) \to \delta(0)$ (Dirac measure) in say $E'(H)$ (see [34]). Taking derivatives in say $D'(H)$ on (τ, T) we obtain (picking n so that supp $\varphi_n(t - \xi) \subset (\tau, T)$)

$$\begin{aligned} f_n &= y_n' + A(t)y_n \\ &= -\varphi_n(t - \xi)g(t) + (\varphi_n * Y_\xi)^\frown A(t)g(t) + (\varphi_n * Y_\xi)^\frown g'(t) \end{aligned} \tag{2.11}$$

since $(\varphi_n * Y_\xi)^\frown$ acts like $(\varphi_n * Y_\xi)$ on test functions $\psi \in D(\tau, T)$ (the discontinuity of Y_ξ at $t = \tau - 1$ is not seen). Writing

$$d\nu_n(s) = -\varphi_n(s - \xi)\, ds, \qquad d\omega_n(s) = (\varphi_n * Y_\xi)^\frown(s)\, ds,$$

$$f(t, s) = G(t, s)g(s), \qquad h(t, s) = G(t, s)A(s)g(s) + G(t, s)g'(s)$$

we now *fix* t and write (2.5) with $y = y_n$ as

$$\begin{aligned} y_n(t) &= P(t, \tau)y_n(\tau) + \langle \nu_n(\cdot)f(t, \cdot)\rangle + \langle \omega_n(\cdot), h(t, \cdot)\rangle \\ &= P(t, \tau)y_n(\tau) + R_n(t, \xi), \end{aligned} \tag{2.12}$$

Let $C'[\tau, t]$ be the dual of $C^0[\tau, t]$ with the topology of uniform convergence on compact subsets of $C^0[\tau, t]$ and think of ν_n and ω_n as elements of $C'[\tau, t]$ (τ and t are fixed here). Let $\nu(s) = -\delta(\xi)$ if $\tau < \xi < t$ and $\nu(s) = 0$ if $t < \xi < T$. Then clearly

$$\langle \nu_n , \psi\rangle \to \langle \nu, \psi\rangle \quad \text{for } \psi \in C^0[\tau, t]$$

and using Banach-Steinhaus (cf. [4]) it follows that $\nu_n \to \nu$ in $C'[\tau, t]$ with the above topology. Similarly if $d\omega(s) = \tilde{Y}_\xi(s)\, ds$ then $\omega_n \to \omega$ in $C'[\tau, t]$. Using Proposition 14 of [6] we conclude that in H,

$$\langle \nu_n(\cdot), f(t, \cdot)\rangle \to \langle \nu(\cdot), f(t, \cdot)\rangle$$

and

$$\langle \omega_r(\cdot), h(t, \cdot)\rangle \to \langle \omega(\cdot), h(t, \cdot)\rangle$$

(observe that $f(t, \cdot)$ and $h(t, \cdot)$ belong to $C^0(H)$ on $[\tau, t]$). Consequently if $\xi < t$ we obtain $R_n(t, \xi) \to R(t, \xi)$ in H where

$$R(t, \xi) = -G(t, \xi)g(\xi) + \int_\tau^\xi G(t, s)(A(s)g(s) + g'(s))\, ds \tag{2.13}$$

while if $\xi > t$, $R(t, \xi)$ has the form

$$R(t, \xi) = \int_\tau^t G(t, s)(A(s)g(s) + g'(s))\, ds. \tag{2.14}$$

The functions $R_n(t, \xi)$ and $R(t, \xi)$ have not been defined for $\xi = t$ and there is no need to do so (recall also $\xi \neq \tau$ or T).

Thus let t be fixed and pick $\xi \in (\tau, T)$, $\xi \neq t$; then

$$R_n(t, \xi) \to R(t, \xi)$$

in H. Similarly

$$y_n(t) = (\varphi_n * Y_\xi)^\sim g(t) \to \tilde{Y}_\xi(t)g(t)$$

in H; this is seen from the expression

$$\varphi_n * Y_\xi(t) = \int_{\tau-1}^\xi \varphi_n(t - s)\, ds = \int_{t-1/n}^{t+1/n} \varphi_n(t - s)\, ds \tag{2.15}$$

which is either zero or one for n large enough (n depending on t of course). In particular

$$y_n(\tau) \to g(\tau)$$

and

$$P(t, \tau)y_n(\tau) = (\varphi_n * Y_\xi)^\sim(\tau)P(t, \tau)g(\tau) \to P(t, \tau)y(\tau)$$

n H (all that is used here about $P(t, \tau)$ is linearity). Hence taking pointwise limits in (2.12) we obtain for $\xi < t$

$$\begin{aligned} \tilde{Y}_\xi(t)g(t) = P(t, \tau)g(\tau) - G(t, \xi)g(\xi) \\ + \int_\tau^\xi G(t, s)(Ag(s) + g'(s))\, ds \end{aligned} \tag{2.16}$$

and letting $\xi \to \tau$ this yields for $t > \tau$

$$[P(t, \tau) - G(t, \tau)]g(\tau) = 0. \tag{2.17}$$

Consequently $P(t, \tau) = G(t, \tau)$ on $\hat{I}_\tau$ and $P(t, \tau)$ can be extended from $\hat{I}_\tau$

to $\bar{P}(t, \tau) = G(t, \tau)$; note however that if $\hat{I}_\tau \neq I_\tau$ then $G(t, \tau)$ may not be an extension of $P(t, \tau)$ under this argument. Hence we assume $P(t, \tau)$ on I_τ is extendable by continuity to H, in which case $G(t, \tau)$ must extend $P(t, \tau)$ when I_τ is dense. Then the propagator relation (2.3) will extend by continuity to $G(t, \tau)$.

THEOREM 1. *Let* (2.5) *give the unique solution of* (1.1) *with*

$$s \to G(t, s) \in C^0(L_s(H)).$$

Assume $C^1(H) \cap C^0(D(A(t)))$ *has a dense set of initial values* $\hat{I}_s \subset I_s$ *on any interval* $[s, T]$ *for* $s \geq \tau$ *and that* $P(t, \tau)$ *is extendable by continuity. Then* $P(t, s) = G(t, s)$ *on* $\hat{I}_s$ *and* $G(t, \tau)$ *is a propagator.*

The results of [11] yield a similar conclusion under a hypothesis

$$\bigcap D(A(t)) = D_0 \subset I_s$$

for all s (instead of the hypothesis on $\hat{I}_s$) and a substantially stronger continuity assumption on $G(\cdot, \cdot)$ (which could however be weakened to that of the present paper). For other abstract results involving a recovery formula (2.5) and general inversion formulas see [12], [13], [14], [29], [40].

3. In this section we will show that certain kinds of weak solutions of evolution equations are really strong solutions of another "intrinsic" differential problem and this leads to a new kind of uniqueness theorem for weak solutions. Some of these results were announced in [16] and we furnish details and some new material here. Let H be separable and $V(t) \subset H$ be a Lebesgue measurable family of Hilbert spaces (cf. [17]) with $V(t) \subset H$ algebraically and topologically and $V(t)$ dense in H. Let $W = \int^{\oplus} V(t)\, dt$, $\tau \leq t \leq T < \infty$, with scalar product

$$((u, v)) = \int_\tau^T ((u(t), v(t)))_t \, dt.$$

For each $t \in [\tau, T]$ let $a(t, \cdot, \cdot)$ be a continuous sesquilinear form on $V(t) \times V(t)$ with $\mathfrak{A}(t) \in L(V(t))$ the associated continuous linear operator defined by $a(t, x, y) = ((\mathfrak{A}(t)x, y))_t$ for $x, y \in V(t)$. Suppose the family $\mathfrak{A}(\cdot)$ is measurable (cf. [17]) and that $| a(t, u, v) | \leq M \| u \|_t \| v \|_t$ with M independent of t ($u, v \in V(t)$). Then if $u(\cdot), v(\cdot) \in W$ the function

$$t \to a(t, u(t), v(t))$$

is summable (cf. [23], [24]). We will consider functions $u \in W$ satisfying the weak problem

$$(3.1) \quad -\int_\tau^T (u, v')\, dt + \int_\tau^T a(t, u, v)\, dt + \lambda \int_\tau^T (u, v)\, dt = \int_\tau^T (f, v)\, dt + (u_0, v(\tau))$$

for all $v \in W$ with $v(T) = 0$ and $v' \in L^2(H)$. Here $(\ ,\ ) = (\ ,\ )_H$ and our hypotheses later imply $W \subset L^2(H)$. Derivatives are taken in $D'(H)$ on (τ, T); $f \in L^2(H)$ and $u_0 \in H$ are supposed given. One can assume λ is an arbitrarily large positive real number since a change of variables $v = we^{-\lambda t}$ can always be effected without changing the problem (cf. [23], [7], [8]). Supposing

$$\operatorname{Re} a(t, u, u) + \lambda \,|\, u \,|^2 \geq c \,\|\, u \,\|_t^2 \qquad \text{for } u \in V(t).$$

Lions proves existence of solutions of (3.1) in [23] and uniqueness theorems are proved under more hypotheses by Lions in [23], [24]; related theorems are proved in Kato-Tanabe [19] and Baiocchi [1], [2]. In case $V(t) = V$ is constant with $W = L^2(V)$ the hypotheses already indicated are enough for both existence and uniqueness (see Lions [23]); other proofs can be obtained by specializing Browder's more general non-linear results [7], [8] to the linear case and similar theorems under various hypotheses can be found in [18], [20], [25], [39], [31], [38], [26] and in many other articles too numerous to list here.

Now given a subhilbert space $V(t) \subset H$ as above one can describe it as the domain of a closed densely defined positive definite self adjoint operator $B^{1/2}(t)$ (called standard operator) where $B^{1/2}(t)$ maps $V(t)$ onto H with a continuous inverse. In fact $B(t) = \theta L(t)^{-1}$ where $\theta : H' \to H$ is the canonical antiisomorphisms and $L(t)$ is the Schwartz antikernel of $V(t)$ relative to H (see [13], [35], [36]). The Hilbert structure of $V(t)$ can then be described by

$$((u, v))_t = (B^{1/2}(t)u, B^{1/2}(t)v)$$

and if for example $V(t) = D(E(t))$ has the graph Hilbert structure of a closed densely defined operator $E(t)$ then $B(t) = 1 + E^*(t)E(t)$ (see [13]). Next we observe that if $x \in H$, $y \in V(t)$ then $y \to (x, y)_H$ determines an antilinear form on $V(t)$ which we write $((J(t)x, y))_t$ where $J(t) \in L(H, V(t))$ (see [23]). Using the standard operator $B^{1/2}(t)$ we obtain

$$(B^{1/2}(t)J(t)x, B^{1/2}(t)y) = (x, y)$$

and consequently $J(t) = B(t)^{-1}$ since $V(t)$ is dense where we think now of $J(t) \in L(H)$. Now (3.1) can be rewritten in the form

$$-\int_\tau^T (u, v')\, dt = \int_\tau^T ((\xi, v))_t\, dt + ((J(\tau)u_0, v(\tau))_\tau \tag{3.2}$$

where $\xi(t) = J(t)f(t) - \lambda J(t)u(t) - \mathfrak{A}(t)u(t)$ is a function. It is shown in [23] that in case $V(t) = V$ is constant with $W = L^2(V)$ then the problem of solving (3.1) or (3.2) is reduced to the "strong" problem of finding a function $u \in L^2(-\infty, T;\ V)$, $u = 0$ (almost everywhere) for $t < \tau$, with

$$\frac{d}{dt}(Ju) = \xi + \delta(\tau) \otimes Ju_0 \tag{3.3}$$

where d/dt is taken in $D'(V)$ on $(-\infty, T)$ and J refers to V (see also [7], [8] for related constructions). Thus even though $u'('$ in $D'(H)$ on $(\tau, T))$ is not a function in general it is true that $(Ju)' = (B^{-1}u)'$ $('$ in $D'(V)$ on $(-\infty, T))$ is a function for $t > \tau$ determined by (3.3). A related result with "measurable" variation of domain permitted but limited by $V \subset V(t) \subset K \subset H$ with V and K fixed Hilbert spaces, dense in H, V and $V(t)$ being closed subspaces of K, has been obtained by Baiocchi [1]. Again for V constant, Lions in [26] shows differentiability in a sense akin to the development which follows, by showing $u' \in L^2(\tau, T; V')$ with $'$ in $D'(V')$ (note that $(B^{-1/2}x, B^{1/2}y)_H = (x, y)_H = \langle x, y\rangle$ where $\langle\ ,\ \rangle$ denotes $V' - V$ conjugate linear duality).

We will give a new formulation for variable domains of this kind of relation between weak and strong problems by using the intrinsic characterization of $V(t)$ in terms of $B^{1/2}(t)$. Differentiability of the map $t \to B^{-1/2}(t)$ can be interpreted as smoothness in the variation of $V(t)$ and the strong equation will contain an additional term measuring the way in which the scalar product changes. We shall assume that $t \to B^{1/2}(t)v(t)$ is H measurable for the measurable vectors v of our family and this amounts to assuming

$$W = \{B^{-1/2}(\cdot)x(\cdot),\ x \in L^2(\tau, T; H)\};$$

the conditions on $\mathfrak{A}$ become simply $\mathfrak{A}(\cdot)u(\cdot) \in W$ for $u \in W$ with $\|\mathfrak{A}(t)\| \le M$ (see [15]). Thus let u be a solution of (3.1) or (3.2) and extend u for $t < \tau$ to a function $\tilde{u} = 0$ for $t < \tau$, $\tilde{u} = u$ for $\tau \le t \le T$. Similarly let $\tilde{f} = f$ for $\tau \le t \le T$ and $\tilde{f} = 0$ for $t < \tau$. Let $V(t) = V(\tau)$ for $t \le \tau$ and thus $B^{1/2}(t) = B^{1/2}(\tau)$ for $t \le \tau$. Then (3.2) implies ($\xi = J\tilde{f} - \lambda J\tilde{u} - \mathfrak{A}\tilde{u}$ where $J(t) = J(\tau)$ for $t \le \tau$ and $\mathfrak{A}(t) = 0$ for $t < \tau$).

$$(3.4) \qquad -\int_{-\infty}^{T} (\tilde{u}, v')\, dt = \int_{-\infty}^{T} ((\xi, v))_t\, dt + ((J(\tau)u_0, v(\tau)))_\tau$$

for all $v \in \tilde{W}$ with $v' \in L^2(-\infty, T; H)$ and $v(T) = 0$ where

$$\begin{aligned}\tilde{W} &= \{B^{-1/2}(\cdot)x(\cdot),\ x \in L^2(-\infty, T; H)\}\\ &= B^{-1/2}L^2(-\infty, T; H).\end{aligned}$$

Let now

$$D_{W}^{\sim} = \{\varphi(t) = B^{-1/2}(t)\psi(t) \text{ with } \psi \in D(H) \text{ on } (-\infty, T)\}$$

and assume $B^{-1/2}(\cdot)$ is piecewise strongly C^1 in the sense that for $x \in H$ and $t \ge \tau$, $(B^{-1/2}(t)x)'$ is continuous and equals $\dot{B}^{-1/2}(t)x$ for some strongly continuous operator $\dot{B}^{-1/2}(t) \in L(H)$. Then defining $\dot{B}^{-1/2}(t) = 0$ for $t < \tau$ (recall $B^{-1/2}(t)$ is constant for $t \le \tau$) we have left and right limits strongly at $t = \tau$ for $\dot{B}^{-1/2}(t)$. Moreover by Banach-Steinhaus $\|\dot{B}^{-1/2}(t)\| \le M$ for $-\infty < t \le T$ (cf. [4], [23]) and $\varphi \in D_W^{\sim}$ will be admissible as a v in (3.4) ($\varphi' = \dot{B}^{-1/2}\psi + B^{-1/2}\psi'$ for $t \ne \tau$ with appropriate limits as $t \to \tau^+$ or τ^-). Moreover $W \subset L^2(\tau, T; H)$ since $(B^{-1/2}(\cdot)x(\cdot), h) =$

$(x(\cdot), B^{-1/2}(\cdot)h)$ is measurable with $|B^{-1/2}(t)x(t)| \le M\,|x(t)|$ (cf. [3], [15]); similarly $\tilde{W} \subset L^2(-\infty, T; H)$. Thus putting φ in (3.4) one has

$$
\begin{aligned}
&-\int_{-\infty}^{T} (B^{-1/2}\tilde{u}, \psi')\,dt - \int_{-\infty}^{T} (\dot{B}^{-1/2}\tilde{u}, \psi)\,dt \\
&\qquad = \int_{-\infty}^{T} (B^{1/2}\tilde{\xi}, \psi)_H\,dt + (B^{-1/2}(\tau)u_0, \psi(\tau))_H
\end{aligned} \tag{3.5}
$$

(recall $J = B^{-1}$ and note that $\dot{B}^{-1/2}$ is self adjoint since $(\Delta B^{-1/2}x, y) = (x, \Delta B^{-1/2}y)$ where $\Delta B^{-1/2} = B^{-1/2}(t') - B^{-1/2}(t)$). But (3.5) means that

$$\langle (B^{-1/2}\tilde{u})', \psi\rangle - \langle \dot{B}^{-1/2}\tilde{u}, \psi\rangle = \langle B^{1/2}\tilde{\xi}, \psi\rangle + \langle \delta(\tau) \otimes B^{-1/2}(\tau)u_0, \psi\rangle$$

where $\langle\ ,\ \rangle$ denotes $D(H) - D'(H)$ conjugate linear duality on $(-\infty, T)$. Consequently we have, writing out the $B^{1/2}\tilde{\xi}$ term, an equation in $D'(H)$ on $(-\infty, T)$,

$$
\begin{aligned}
&(B^{-1/2}\tilde{u})' - \dot{B}^{-1/2}\tilde{u} + \lambda B^{-1/2}\tilde{u} + B^{1/2}\mathfrak{A}\tilde{u} \\
&\qquad = B^{-1/2}\tilde{f} + \delta(\tau) \otimes B^{-1/2}(\tau)u_0 .
\end{aligned} \tag{3.6}
$$

In particular for $t > \tau$ we see that $(B^{-1/2}\tilde{u})' = (B^{-1/2}u)'$ is a function in $L^2(\tau, T; H)$. This implies, by an elementary distribution argument, that $B^{-1/2}u$ is in fact continuous for $t \ge \tau$ (cf. [15], [28]) and from (3.6) we must have also $(B^{-1/2}u)(\tau) = B^{-1/2}(\tau)u_0$.

THEOREM 2. *Let $\mathfrak{A}(t) \in L(V(t))$ be a measurable family (as above) associated with the sesquilinear forms $a(t, \cdot, \cdot)$ on $V(t) \times V(t)$ where*

$$|a(t, u, v)| \le M\,\|u\|_t \|v\|_t$$

for $u, v \in V(t)$ with M constant ($\tau \le t \le T < \infty$). Let $B^{1/2}(t)$ be the standard operator for $V(t) \subset H$, H separable, and assume that $(B^{-1/2}(\cdot)x)$ is C^1 for $x \in H$ with derivative $\dot{B}^{-1/2}(t)x$ where $\dot{B}^{-1/2}(t) \in L(H)$ is strongly continuous. Then any solution $u \in W = B^{-1/2}L^2(\tau, T; H)$ of the weak problem (3.1) satisfies the strong problem (in $D'(H)$ on (τ, T) with terms in $L^2(\tau, T; H)$)

$$
\begin{aligned}
&(B^{-1/2}u)' - \dot{B}^{-1/2}u + \lambda B^{-1/2}u + B^{1/2}\mathfrak{A}u = B^{-1/2}f \\
&\qquad \text{with } (B^{-1/2}u)(\tau) = B^{-1/2}(\tau)u_0 .
\end{aligned} \tag{3.7}
$$

This includes the result of [16] as a special case. We remark that $u(\cdot)$ is not asserted to be continuous. It is interesting to note the relation of the smoothness possessed by $B^{-1/2}(\cdot)$ with a trace theorem of Lions [23]. Indeed one can think of W as $L^2(\tau, T;\ D(B^{1/2}(t)))$ and recall $(B^{1/2}(t)x, x) \ge |x|^2$ for $x \in V(t)$ with

$$\left|\left(\frac{d}{dt} B^{-1/2}(t)x, x\right)\right| = |(\dot{B}^{-1/2}(t)x, x)| \le c|x|^2$$

for $x \in H$ (by Banach-Steinhaus since $\dot{B}^{-1/2}(\cdot)$ is strongly continuous).

Even though c may not be less than 2, which is part of hypothesis (1.3) of [23, Theorem 4.1, p. 35], the first part of this theorem still remains valid and hence if $w \in W$ with $w' \in L^2(\tau, T; H)$ it follows that $w(\tau)$ makes sense and belongs to $D(B^{1/4}(\tau))$. Of course one knows that $w(\tau)$ makes sense simply from $w' \in L^2(\tau, T; H)$ (cf. [15], [28]). Thus in particular

$$(B^{-1/2}u)(\tau) \in D(B^{1/4}(\tau))$$

can be concluded automatically; this however is weaker than our result which uses the additional information contained in (3.6) and yields

$$(B^{-1/2}u)(\tau) = B^{-1/2}(\tau)u_0 \in D(B^{1/2}(\tau))$$

(see also [1], [2]).

Remark 1. Suppose $a(t, u, v) = (E(t)u, E^*(t)v)$ where $E(t) = A(t)^{1/2}$ with $A(t)$ for example a family of closed densely defined maximal accretive operators (see [21] for fractional powers). This arises in the weak problem associated with $u' + A(t)u = f$. If, for example,

$$V(t) = D(A^{1/2}(t)) \subset D(A^{1/2}(t)^*)$$

with the graph Hilbert structure of $A^{1/2}(t)$ on $V(t)$ and if say

$$| E^*(t)x |^2 \leq C(| E(t)x |^2 + | x |^2)$$

for $x \in V(t)$ then

$$| a(t, u, v) | \leq c \| u \|_t \| v \|_t .$$

Now $B(t) = 1 + E^*(t)E(t)$ and one can write

$$a(t, u, v) = (Eu, E^*v) = (Eu, E^*B^{-1/2}B^{1/2}v).$$

But

$$K(t) = E^*(t)B^{-1/2}(t) \in L(H)$$

by a lemma in [22] (observe $D(B^{1/2}(t)) = D(E(t)) \subset D(E^*(t))$ and $E^*(t)$ is closed). Hence one can write

$$a(t, u, v) = (B^{1/2}B^{-1/2}K^*Eu, B^{1/2}v)$$

and

$$\mathfrak{A}(t) = B^{-1/2}(t)K^*(t)E(t).$$

Hence (3.7) becomes

$$(3.8) \qquad (B^{-1/2}u)' - \dot{B}^{-1/2}u + \lambda B^{-1/2}u + K^*A^{1/2}u = B^{-1/2}f.$$

Of course when $u(t) \in D(A(t))$ the $K^*A^{1/2}u = B^{1/2}\mathfrak{A}u$ term becomes simply $B^{-1/2}Au$ (i.e., $\mathfrak{A} = B^{-1}A$ on $D(A)$ which follows immediately from writing out $(Au, v) = a(t, u, v) = ((\mathfrak{A}u, v)) = (B^{1/2}\mathfrak{A}u, B^{1/2}v)$ for $u \in D(A)$, $v \in D(B^{1/2})$).

Next we show that knowing how to write a weak solution of (3.1) as a strong solution of (3.7) leads to a new type of "intrinsic" uniqueness theorem for weak solutions. Related theorems can be found in [23], [24], [19], [1], [15], [16a], (see also Remark 3 for this). Setting $f = 0$ in (3.7) we take scalar products with $B^{-1/2}u$ to obtain

$$(3.9)\quad \frac{1}{2}\frac{d}{dt}|B^{-1/2}u|^2 + \lambda|B^{-1/2}u|^2 + \operatorname{Re}(B^{1/2}\mathfrak{A}u, B^{-1/2}u) - \operatorname{Re}(\dot{B}^{-1/2}u, B^{-1/2}u) = 0$$

We recall that λ may be chosen to be arbitrarily large and thus we can exploit an estimate of the form

$$(3.10)\quad |\operatorname{Re}(\dot{B}^{-1/2}u, B^{-1/2}u)| \leq c\,|B^{-1/2}u|^2.$$

A natural hypothesis is therefore to assume that $\dot{B}^{-1/2}(t)B^{1/2}(t)$, defined on $V(t)$, is extendable by continuity to a bounded operator on H (with a uniform bound; of c for $\tau \leq t \leq T$), that this would imply (3.10) follows immediately upon writing

$$(\dot{B}^{-1/2}u, B^{-1/2}u) = (\dot{B}^{-1/2}B^{1/2}B^{-1/2}u, B^{-1/2}u).$$

More simply one could merely require that $|\dot{B}^{-1/2}u| \leq c\,|B^{-1/2}u|$. Further let us assume that

$$(3.11)\quad \operatorname{Re}(\mathfrak{A}u, u) \geq -\gamma\,|B^{-1/2}u|^2.$$

Observe that this is *not* directly an assumption of monontonicity in the usual sense since $\operatorname{Re} a(t, u, u) = \operatorname{Re}(B^{1/2}\mathfrak{A}u, B^{1/2}u)$ (see also Remarks 2 and 3 below). Now with these assumptions we integrate (3.9) to obtain (setting $(B^{-1/2}u)(\tau) = 0$)

$$(3.12)\quad 0 \geq \tfrac{1}{2}\,|B^{-1/2}u|^2(T) + (\lambda - c - \gamma)\|B^{-1/2}u\|^2 \geq 0$$

where $\|\ \|$ denotes the $L^2(\tau, T; H)$ norm and λ is now taken so that $\lambda > c + \gamma$. Consequently $B^{-1/2}u = 0$ and we have

THEOREM 3. *Let the hypotheses of Theorem 2 hold and assume*

$$\operatorname{Re}(\mathfrak{A}u, u) \geq -\gamma\,|B^{-1/2}u|^2$$

for $u \in V(t)$. Let either $\dot{B}^{-1/2}(t)B^{1/2}(t) = Z(t)$ be extendable by continuity from $V(t)$ to an operator $\bar{Z}(t) \in L(H)$ with $\|\bar{Z}(t)\| \leq c$ on $[\tau, T]$ or let more simply

$$|\dot{B}^{-1/2}(t)u| \leq c\,|B^{-1/2}(t)u|.$$

Then solutions of (3.1) *are unique. Similarly* $|\dot{B}|^{-1/2}u| \leq c_1|u| + c_2|B^{-1/2}u|$ *with* $\operatorname{Re}(\mathfrak{A}u, u) \geq \alpha|u|^2 - \gamma|B^{-1/2}u|^2$ *gives the same result* (*and is "natural" by Remark* 2 *below*).

Remark 2. Take again the situation of Remark 1. Then condition (3.11) says that for $u \in D(B^{1/2})$

$$(3.13)\quad \operatorname{Re}(B^{-1/2}K^*A^{1/2}u, u) = \operatorname{Re}(A^{1/2}u, A^{*1/2}B^{-1}u) \geq -\gamma\,|B^{-1/2}u|^2$$

since $K = A^{*1/2}B^{-1/2}$. Suppose $B^{-1/2}$ commutes with $A^{1/2}$ and $A^{*1/2}$; then this becomes

$$\mathrm{Re}\,(A^{1/2}x, A^{*1/2}x) \geq -\gamma\,|\,x\,|^2$$

for $x = B^{-1/2}u \in D(B) = D(A^{*1/2}A^{1/2})$. Recalling that

$$D(A^{1/2}) \subset D(A^{*1/2}) \quad \text{with} \quad |\,A^{*1/2}x\,|^2 \leq c(|\,A^{1/2}x\,|^2 + |\,x\,|^2)$$

we will have

$$\mathrm{Re}\,(A^{1/2}x, A^{*1/2}x) \geq -\gamma\,|\,x\,|^2$$

for $x \in D(A^{1/2}) \supset D(B)$ if $D(A)$ is a core for $A^{1/2}$ and $\mathrm{Re}\,(Ay, y) \geq -\gamma\,|\,y\,|^2$ for $y \in D(A)$ ($D(A)$ a core—or determining set—for $A^{1/2}$ means $A^{1/2}$ is the closure of its restriction to $D(A)$). Thus there is a natural association in the commutative case indicated between (3.11) and standard monotonicity hypotheses for the strong problem (1.1) (cf. [23]) since it is known, for example, that $D(A(t))$ is a core for $D(A^{\alpha}(t))$, $0 < \alpha < 1$, when $A(t)$ is a closed densely defined maximal accretive operator (cf. [21]). Now in general $B^{-1/2}$ will not commute as above so we try to estimate the deviation as follows:

$$\begin{aligned} \mathrm{Re}\,(A^{1/2}u, A^{*1/2}B^{-1}u) &= \mathrm{Re}\,(A^{1/2}B^{-1/2}u, A^{*1/2}B^{-1/2}u) \\ &\quad + \mathrm{Re}\,(A^{1/2}u, Pu) + \mathrm{Re}\,(Qu, A^{*1/2}B^{-1/2}u) \end{aligned} \tag{3.14}$$

where

$$P = A^{*1/2}B^{-1} - B^{-1/2}A^{*1/2}B^{-1/2} \quad \text{and} \quad Q = B^{-1/2}A^{1/2} - A^{1/2}B^{-1/2}.$$

But by [22], $A^{*1/2}B^{-1/2} \in L(H)$, $A^{*}B^{-1} \in L(H)$, and if $B^{-1/2}A^{1/2}$ is extendable by continuity from $D(B^{1/2})$ to H as a bounded operator then Q extends to $\bar{Q} \in L(H)$ and $A^{*1/2}P \in L(H)$. Hence

$$|\,\mathrm{Re}\,(A^{1/2}u, Pu)\,| \leq |\,(u, A^{*1/2}Pu)\,| \leq c_1\,|\,u\,|^2, \tag{3.15}$$

$$|\,\mathrm{Re}\,(Qu, A^{*1/2}B^{-1/2}u)\,| \leq |\,\bar{Q}u\,|\,|\,A^{*1/2}B^{-1/2}u\,| \leq c_2\,|\,u\,|^2. \tag{3.16}$$

In this case we require a somewhat stronger estimate; for example let

$$\mathrm{Re}\,(A^{1/2}x, A^{*1/2}x) \geq \alpha\,|\,B^{1/2}x\,|^2 - c\,|\,x\,|^2 \tag{3.17}$$

for $x \in D(B^{1/2})$. Then from (3.14)

$$\mathrm{Re}\,(A^{1/2}u, A^{*1/2}B^{-1}u) \geq (\alpha - \tilde{c})|\,u\,|^2 - c\,|\,B^{-1/2}u\,|^2 \tag{3.18}$$

where $\tilde{c} = c_1 + c_2$. Thus if $\alpha \geq \tilde{c}$ we again obtain (3.13). Condition (3.17) is in turn a standard type monotonicity condition for the weak problem (3.1) (cf. [23]).

Remark 3. The uniqueness result of Lions [24] can be stated as follows. One assumes $K \subset H$ is a dense separable Hilbert space with continuous injection and that $V(t) \subset K$ is a closed subspace of K for $t \in (-\infty, T]$ dense in H. The sesquilinear forms $a(t, \cdot, \cdot)$ on $K \times K$ are given for

$\tau \leq t \leq T$ with $t \to a(t, x, y)$ measurable and bounded and

$$\operatorname{Re} a(t, u, u) + \beta \mid u \mid^2 \geq \alpha \parallel u \parallel^2 \qquad \text{for } u \in V(t).$$

If $P(t)$ is the orthogonal projection $P(t) : K \to V(t)$ it is assumed that

$$t \to P(t) \in C^0(L_s(K)),$$

for $u \in K$ $(1/\Delta t)(\Delta P)u \to P'(t)u$ weakly in K with $t \to P'(t)$ weakly continuous in K, and $\parallel P'(t) \parallel \leq c$. Then for $W = L^2(\tau, T; V(t))$ there is uniqueness in problem (3.1).

Now let us compare a hypothesis of the form (3.11) with a condition $(u \in V(t))$

$$\text{(3.19)} \qquad \operatorname{Re}(B^{1/2}\mathfrak{A}u, B^{1/2}u) \geq \alpha \mid B^{1/2}u \mid^2 - \beta \mid u \mid^2$$

indicated in (3.17) and in the theorem just cited. Thus consider

$$\text{(3.20)} \quad \operatorname{Re}(\mathfrak{A}u, u) = \operatorname{Re}(B^{1/2}\mathfrak{A}B^{-1/2}u, u) + \operatorname{Re}((\mathfrak{A} - B^{1/2}\mathfrak{A}B^{-1/2})u, u)$$

The second term is of course

$$a(t, B^{-1/2}u, B^{-1/2}u) = (B^{1/2}\mathfrak{A}B^{-1/2}u, B^{1/2}B^{-1/2}u)$$

and if (3.19) holds with $\mathfrak{A} - B^{1/2}\mathfrak{A}B^{-1/2} = R$ extendable by continuity to $\bar{R} \in L(H)$ (i.e., if $\mathfrak{A}$ is extendable) satisfying $\parallel R \parallel \leq \alpha$ then (3.11) holds with $\gamma = \beta$ (verification immediate). We recall that in Remark 1, $\mathfrak{A} = B^{-1/2}K^*A^{1/2}$ and this gives a good idea of what extendability of $\mathfrak{A}$ to $\bar{\mathfrak{A}} \in L(H)$ will involve (see also [26]). On the other hand it is evident that our hypotheses on $B^{-1/2}(t)$ of strong differentiability can be weakened somewhat to resemble those on $P(t)$ above but we omit details (see [15], [16a]). Thus the smoothness requirements on the way $V(t)$ changes in Lions theorem and in Theorem 3 with weak differentiability would be apparently of the same "order of magnitude". We require in addition a relation between $\dot{B}^{-1/2}(t)$ and $B^{-1/2}(t)$ whereas Lions requires $V(t) \subset K$ as a closed subspace; thus our theorem seems to allow more freedom in the "nature" of $V(t)$ but this remains to be clarified.

A different kind of uniqueness theorem is proved by Baiocchi in [1] where the smoothness conditions on $V(t)$ are apparently weakened in some respects to conditions of the following form. It is assumed that $V \subset V(t) \subset K$ with V and $V(t)$ closed subspaces of K, V dense in H (i.e. $V \subset H = H' \subset V'$ and $V' \subset K'$ as a closed subspace), $V(t)$ depends measurably on t in K (i.e. $t \to P(t)x$ is measurable with values in K for $x \in K$), H is the interpolation space $[K, V']_{1/2}$, and the space of test functions of the type

$$v \in L^2(-\infty, \infty; V(t)), \qquad v' \in L^2(-\infty, \infty; H)$$

is to be dense in $L^2(-\infty, \infty; V(t)) \cap H^{1/2}(H)$ and

$$H^{1/2}(H) = [H^1(H), H^0(H)]_{1/2}$$

$(H^K(H) = \{u \in L^2(-\infty, \infty; H), u^{(s)} \in L^2(-\infty, \infty; H) \text{ for } s = 0, \cdots, k\})$.

The other hypotheses are substantially the same as those of [24] mentioned above.

4. We give in this section some further results toward the solution of the problem $Su = Lu + Au = f$ with L and A closed densely defined linear operators in a Hilbert space E. The model problem is (1.1) (with $\tau = 0$ for convenience) where $L = L_0 = d/dt$ or $L = L_0 + \varepsilon$ defined on

$$D(L) = D(L_0) = \{u \in E = L^2(0, T; H), u' \in L^2(H), u(0) = 0\}$$

or alternatively a globalized version in $L^2_\mu(0, \infty; H)$, $d\mu = \exp(-2\gamma t)\, dt$, which involves $D(L) \subset D(L^*)$ and is consequently more manageable in certain respects (cf. [23], [14], [7]); all derivatives are in $D'(H)$. We consider here the finite interval case and remark that it is easy to see that L_0 is closed, densely defined, and maximal accretive (accretive = monotone). First observe that

$$\text{(4.1)} \qquad \operatorname{Re}(L_0 u, u)_E = \int_0^T \operatorname{Re}(u', u)\, dt = \frac{1}{2} |u|^2(T) \geq 0.$$

Next recall that a densely defined L_0 is maximal accretive if it is accretive and $L_0 + \lambda$ maps onto E for $\lambda > 0$ (see [30]). Hence L_0 is evidently maximal accretive. Clearly $L_0^* = -d/dt$ with

$$D(L_0^*) = \{u \in E, u' \in E, u(T) = 0\}$$

and by [30] L_0^* is also maximal accretive since it is accretive ($\operatorname{Re}(L_0^* u, u) = \frac{1}{2} |u|^2(0)$). As a model for A we suppose A defined by a suitable family $A(t)$ of operators in H with

$$D(A) = \{u \in E, u(t) \in D(A(t)) \text{ almost everywhere}, A(\cdot)u(\cdot) \in E\}.$$

If the $A(t)$ are accretive in H then A is accretive in $E = L^2(0, T; H)$. Since we can always add a term λy to (1.1) (as indicated earlier) for problems based on this model we can always take L strongly monotone (e.g. $L = L_0 + \varepsilon$ with $\operatorname{Re}(Lu, u) \geq \varepsilon \| u \|^2$). Now there are several immediate (unsolved) problems.

Problem 1. Let L and A be closed, densely defined, strongly monotone operators in a Hilbert space E. When is $L + A = S$ closed and when does $R(S) = E$?

Some answers to this are given in [14] and we will generalize those results here and give some variations (see also [17a]). A first comment is that S is not always onto or closed (cf. [14]) and thus some additional hypotheses are always necessary. The situation of [14] involved operators $L' \otimes 1$ and $1 \otimes A'$ in a certain Hilbert completion $E = H \hat{\otimes}_\sigma F$ of two Hilbert spaces H and F and led to the case of commuting operators L^{-1} and A^{-1} in E. Note also for example that $L = L_0 + \varepsilon$ is 1-1 onto with continuous inverse in our model above. We give first a generalization of Theorem 1 of [14] (part II) for this commutative case.

THEOREM 4. *Let L and A be strongly monotone linear operators in a Hilbert space E mapping onto E, with L^{-1} and A^{-1} permutable. Then $R(L+A)$ is dense.*

Proof. Let $x \perp R(L+A) = R(S)$ and set

$$y = L^{-1}A^{-1}x = A^{-1}L^{-1}x \in D(S) = D(L) \cap D(A).$$

Then for $w = A^{-1}x$, $z = L^{-1}x$ we have, since $y \in D(S)$

$$\begin{aligned} 0 &= \operatorname{Re}\,((L+A)L^{-1}A^{-1}x, x) \\ &= \operatorname{Re}\,(A^{-1}x, x) + \operatorname{Re}\,(L^{-1}x, x) \qquad (4.2) \\ &= \operatorname{Re}\,(w, Aw) + \operatorname{Re}\,(z, Lz) \geq c\,\|w\|^2 + c_1\,\|z\|^2. \end{aligned}$$

Consequently $w = z = 0$ and $x = 0$, Q. E. D.

COROLLARY 1. *Let L be strongly monotone and A monotone with L^{-1} and A^{-1} defined on E and permutable. Then $R(L+A)$ is dense.*

Proof. As in Theorem 4 we get $z = 0$ and hence $x = 0$, Q. E. D.

Theorem 2 of [14] (part II) can also be generalized in various ways and we will give two versions. The basic argument was due to Lions [23] but we modified this by exploiting monotonicity in [14] (and in the present paper we again use monotonicity in one version of the results). The main point of this section however is to reveal the operator theoretical nature of the hypotheses and results both in the Lions version and in the version based on monotonicity, and moreover to indicate the relations between the two versions. First we sketch the situation when commutativity prevails as in [14]. Let $A = A_1 + A_2$ with A_1 self adjoint onto E, $D(A_2) \supset D(A_1) = D(A)$, and $\|A_2 u\| \leq \beta \|A_1 u\|$ for $u \in D(A)$ with $\beta < 1$. Note that

$$\|Au\| \leq \|A_1 u\| + \|A_2 u\| \leq (1+\beta)\|A_1 u\|$$

and

$$\|Au\| \geq \|A_1 u\| - \|A_2 u\| \geq (1-\beta)\|A_1 u\|.$$

Following Lions [23] let $F = D(A) = D(A_1)$ with the graph Hilbert structure of say A_1 and set

$$\Phi = \{\varphi \in F, \varphi^* = A_1\varphi \in D(L^*)\}$$

with the induced Hilbert structure. Note that Φ will *not* be required to be dense in F nor complete in the following. Set then

$$E(u, \varphi) = \tilde{E}(u, \varphi^*) = (u, L^*\varphi^*) + (Au, \varphi^*) \qquad (4.3)$$

$$P(\varphi) = \tilde{P}(\varphi^*) = (f, \varphi^*) \qquad (4.4)$$

where $f \in E$, $u \in F$, $\varphi \in \Phi$. Clearly $E(\cdot, \cdot)$ is a sesquilinear form on $F \times \Phi$ with $u \to E(u, \varphi) : F \to C$ continuous. Note that A_1 must be 1-1 with continuous inverse A_1^{-1} and $\| u \| \leq c \| A_1 u \|$ (cf. [9]); consequently convergence in the graph topology of A_1 on F for u_n is equivalent to the convergence of $\| A_1 u_n \|$. We have not assumed A_1 strongly monotone although that is a good way to realize our hypotheses. Similarly $\varphi \to P(\varphi) : \Phi \to C$ is a continuous semi-linear form since Φ has the topology of F. As in [23] we set $X = \text{Re}\,(A\varphi, \varphi^*)$ and $Y = \text{Re}\,(\varphi, L^*\varphi^*)$ and then

$$X = \| A_1 \varphi \|^2 + \text{Re}\,(A_2 \varphi, A_1 \varphi) \geq (1 - \beta) \| A_1 \varphi \|^2. \tag{4.5}$$

We suppose now L^* is monotone and $A_1^{-1}L \subset LA_1^{-1}$, from which follows

$$A_1^{-1}L^* \subset L^* A_1^{-1} \quad \text{and} \quad A_1^{-1/2}L^* \subset L^* A_1^{-1/2}$$

(see [32] and assume A_1 positive). Then

$$Y = \text{Re}\,(A_1^{-1}\varphi^*, L^*\varphi^*) = \text{Re}\,(A_1^{-1/2}\varphi^*, L^*A_1^{-1/2}\varphi^*) \geq 0. \tag{4.6}$$

Consequently $\text{Re}\, E(\varphi, \varphi) \geq (1 - \beta) \| A_1 \varphi \|^2 \geq c \| \varphi \|_\Phi^2$ and by the Lions projection theorem (cf. [23, Chap. 3]) there exists $u \in F$ such that for all $\varphi \in \Phi$, $E(u, \varphi) = P(\varphi)$. Hence for any $\psi \in D(L^*)$ set $\varphi = A_1^{-1}\psi$ and we have $\tilde{E}(u, \psi) = \tilde{P}(\psi)$ which means

$$(u, L^*\psi) + (Au, \psi) = (f, \psi). \tag{4.7}$$

Consequently $u \in D(L)$ with $Lu = f - Au$ since L is closed.

Theorem 5. *Let A and L be closed densely defined linear operators in a Hilbert space E with $A = A_1 + A_2$, A_1 positive self adjoint onto (e.g. A_1 strongly monotone),*

$$D(A) = D(A_1) \subset D(A_2) \quad \text{and} \quad \| A_2 u \| \leq \beta \| A_1 u \|$$

for $u \in D(A_1)$ where $\beta < 1$ If L^* is monotone and $A_1^{-1}L \subset LA_1^{-1}$ then $S = L + A$ is onto.

Note that neither A nor L need be monotone in this theorem. Next we give a variation on this for the noncommutative case based on the model $E = L^2(H)$, $A = \{A(t)\}$ with

$$D(A) = \{u \in E, u(t) \in D(A(t)) \text{ almost everywhere}, A(\cdot)u(\cdot) \in E\}$$

(we occasionally do not distinguish between functions and equivalence classes when the meaning is clear). If $A(t) = A\ (t) + A_2(t)$ with $A_1(t)$ and $A_2(t)$ as in Theorem 5 for each t then $A = A_1 + A_2$ as in Theorem 5 but A_1^{-1} and L will not commute in general. In this context $F = D(A_1)$ with norm given by

$$\| u \|^2 = \int_0^T (| u |^2 + | A_1(t)u(t) |^2)\, dt$$

and $\Phi \subset F$ involves functions $\varphi \in F$ such that $\varphi^*(\cdot) = A_1(\cdot)\varphi(\cdot) \in D(L^*)$.

The previous constructions can be repeated except that Y must be handled differently. We record here the following formula (cf. [9]), where $L = L_0$ and $L^* = -d/dt$ with zero boundary condition at T.

$$\begin{aligned} 2Y &= 2 \operatorname{Re} \int_0^T (A_1^{-1}(t)\varphi^*(t), L^*\varphi^*(t))_H \, dt \\ &= -\int_0^T (A_1^{-1}\varphi^*, \varphi^*)' \, dt + \int_0^T ((A_1^{-1})'\varphi^*, \varphi^*) \, dt \qquad (4.8) \\ &= (A_1^{-1}\varphi^*, \varphi^*)(0) + \int_0^T ((A_1^{-1})'\varphi^*, \varphi^*) \, dt. \end{aligned}$$

The first term is positive but the second requires additional hypotheses on $(A_1^{-1})'$ so as to lead to an eventual estimate $\operatorname{Re} E(\varphi, \varphi) \geq c \| \varphi \|_\Phi^2$ (cf. [9]). Such hypotheses modeled on [23] will be given in an abstract form below which shows their real content from the point of view of functional analysis. Also the role of monotonicity in this kind of existence theory will be disclosed, using somewhat different hypotheses and the two versions of an existence theorem will be related.

As usual we can add a term ku with arbitrarily large k to the equation $Su = f$ when L arises from differentiation and it is this kind of problem which we shall mainly consider in order to concentrate on the abstract nature of the hypothesis of differentiability of $A_1(t)^{-1}$. Thus let again L and A be closed densely defined linear operators in a Hilbert space E with $A = A_1 + A_2$, $D(A) = D(A_1) \subset D(A_2)$, A_1 self adjoint onto (e.g. self adjoint strongly monotone—observe that lower semi-bounded A_1 will obviously lead to an equivalent theory by adding a ku term) and consider $(S + k)u = f$ or $(L + A + k)u = f$. Let $F = D(A_1)$ again with graph Hilbert structure and let Φ be as before. Define X and Y as before and assume now for $\beta < 1$ and $u \in D(A_1)$,

$$\| A_2 u \| \leq \beta \| A_1 u \| + C \| A_1^{1/2} u \|. \qquad (4.9)$$

Then setting $X_1 = X + k(\varphi, \varphi^*)$ we estimate the term $(A_2 \varphi, A_1 \varphi)$ by the Schwartz inequality, using (4.9), and exploit the relation

$$ab \leq \tfrac{1}{2}(a^2\eta + (1/\eta)b^2)$$

with $a = \| A_1 \varphi \|$, $b = \| A_1^{1/2}\varphi \|$ to obtain

$$X_1 \geq \left(1 - \beta - \frac{c\eta}{2}\right) \| A_1 \varphi \|^2 + \left(k - \frac{c}{2\eta}\right) \| A_1^{1/2} \|^2. \qquad (4.10)$$

Now let us give an abstract version of (4.8). The last term in (4.8) can be written formally as $(\tilde{R}\varphi^*, \varphi^*)$ where

$$\tilde{R} = A_1^{-1}L^* - L^*A_1^{-1}. \qquad (4.11)$$

Thus $\tilde{R}$ corresponds to $(A_1^{-1})'$ and we are saying in particular that $\psi \in D(L^*)$

implies $A_1^{-1}\psi \in D(L^*)$ in our model problem. Then (4.8) becomes, setting $\varphi^* = \psi$,

$$2Y = (A_1^{-1}\psi, L^*\psi) + (L^*\psi, A_1^{-1}\psi) = Z + (\tilde{R}\psi, \psi) \tag{4.12}$$

and $Z = (A_1^{-1}\psi, L^*\psi) + (L^*A_1^{-1}\psi, \psi)$. The condition $Z \geq 0$ is therefore somewhat "peculiar" as an operational hypothesis. Note however that if $\psi \in D(L) \cap D(L^*)$ and $L^* = -L$ then $Z = 0$. Now the abstract hypotheses modeled on [23] would be $Z \geq 0$ and

$$| (\tilde{R}\psi, \psi) | \leq 2\alpha \| \psi \|^2 + 2\tilde{c} \| A_1^{-1/2}\psi \|^2; \tag{4.13}$$

then evidently

$$Y \geq -\alpha \| A_1 \varphi \|^2 - \tilde{c} \| A_1^{1/2}\varphi \|^2 \tag{4.14}$$

and combining this with (4.10) and setting

$$E_1(\varphi, \varphi) = E(\varphi, \varphi) + k(\varphi, A_1 \varphi),$$

we obtain $\operatorname{Re} E_1(\varphi, \varphi) \geq \gamma \| \varphi \|_\Phi^2$ provided $\alpha + \beta < 1$ and $k \geq c/2\eta + \tilde{c}$ (observe that η is arbitrary). If $\alpha + \beta < 1$ this can always be achieved since k is arbitrary. Then by the Lions projection theorem again (see [23]) we have

THEOREM 6. *Assume L and A are closed densely defined linear operators in a Hilbert space E with $A = A_1 + A_2$, $D(A_1) \subset D(A_2)$, A_1 positive, self adjoint onto, and suppose $\psi \in D(L^*)$ implies $A_1^{-1}\psi \in D(L^*)$. Let (4.9) and (4.13) hold with $Z \geq 0$ and $\alpha + \beta < 1$. Then for suitably large k, $S + k$ is onto.*

It is natural to examine what role monotonicity can play in the noncommuting case (in view of Theorem 5). Let us estimate X_1 as in (4.10) but try to replace (4.13) and the condition $Z \geq 0$ by something involving the more natural hypothesis of monotonicity. Assume again that A_1 is positive explicitly (since we want to use $A_1^{1/2}$ again) and write formally for $\psi = \varphi^*$,

$$\begin{aligned} Y &= \operatorname{Re} (A_1^{-1/2}\psi, A_1^{-1/2}L^*\psi) \\ &= \operatorname{Re} (A_1^{-1/2}\psi, L^*A_1^{-1/2}\psi) + \operatorname{Re} (A_1^{-1/2}\psi, R\psi) \end{aligned} \tag{4.15}$$

where

$$R = A_1^{-1/2}L^* - L^*A_1^{-1/2} \tag{4.16}$$

corresponds to $(A_1^{-1/2})'$ (cf. (4.11)). It is only necessary here to assume that $\psi \in D(L^*)$ implies $A_1^{-1/2}\psi \in D(L^*)$ which is reasonable (and true in the model problem). Then if L^* is monotone we can assume

$$\operatorname{Re} (A_1^{-1/2}\psi, R\psi) \geq -\alpha \| \psi \|^2 - \tilde{c} \| A_1^{-1/2}\psi \|^2 \tag{4.17}$$

which will imply (4.14).

THEOREM 7. *Assume L and A are closed densely defined linear operators in a Hilbert space E with* $A = A_1 + A_2$, $D(A) = D(A_1) \subset D(A_2)$, A_1 *self-adjoint positive onto, and suppose* $\psi \in D(L^*)$ *implies* $A_1^{-1/2}\psi \in D(L^*)$. *Let* (4.9) *and* (4.17) *hold with* L^* *monotone and* $\alpha + \beta < 1$. *Then for suitably large k,* $S + k$ *is onto.*

One kind of connection between these last two theorems is given by the formula

$$\tilde{R} = A_1^{-1/2}R + RA_1^{-1/2}. \tag{4.18}$$

Putting this in (4.13) one would have on the left a term

$$(R\psi, A_1^{-1/2}\psi) + (RA_1^{-1/2}\psi, \psi)$$

which is not equal to 2 Re $(A_1^{-1/2}\psi, R\psi)$ in general unless $R = R^*$. Thus a direct comparison of the hypotheses is not apparent. In the model problem of course (4.15) simply reads

$$\begin{aligned} &-\mathrm{Re} \int_0^T (A_1^{-1/2}\psi, A_1^{-1/2}\psi')\, dt \\ &= -\mathrm{Re} \int_0^T (A_1^{-1/2}\psi, (A_1^{-1/2}\psi)')\, dt + \mathrm{Re} \int_0^T (A_1^{-1/2}\psi, (A_1^{-1/2})'\psi)\, dt. \end{aligned} \tag{4.19}$$

The hypotheses (4.17) with α arbitrarily small can be realized for example if

$$\| R\psi \| \le c_1 \| A_1^{-1/2}\psi \| + c_2 \| \psi \|. \tag{4.20}$$

Indeed in this event

$$\begin{aligned} | (A_1^{-1/2}\psi, R\psi) | &\le \| A_1^{-1/2}\psi \| \, \| R\psi \| \\ &\le c_1 \| A_1^{-1/2}\psi \|^2 + \tfrac{1}{2}c_2(\eta \| \psi \|^2 + (1/\eta)\| A_1^{-1/2}\psi \|^2) \end{aligned} \tag{4.21}$$

and $\alpha = c_2 \eta/2$ can be made as small as desired.

COROLLARY 1. *Let the first sentence of Theorem* 7 *be true and suppose* (4.9) *holds with* $\beta < 1$, (4.20) *holds, and* L^* *is monotone. Then for suitably large* k, $S + k$ *is onto*

We observe that a hypothesis of the form

$$\| \tilde{R}\psi \| \le c_1 \| A_1^{-1/2}\psi \| + c_2 \| \psi \|$$

does not imply (4.13) with small α unless c_2 is itself small. Thus the situation of Corollary 1 to Theorem 6 seems particularly nice.

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