ON THE STABLE HOMOTOPY GROUPS AND THE STABLE MOD-2 HOMOTOPY GROUPS OF Z_2-MOORE SPACES

BY

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Introduction

For each simply connected, reduced s.s. complex X, D. M. Kan [4] has defined as s.s. free group GX which serves as a loop complex for X. The lower central series $\{\Gamma_r GX\}$ of GX was studied by E. B. Curtis [1] who proved that the associated spectral sequence converges to the homotopy groups of X.

Here we confine ourselves to the Z_2-Moore spaces. In the following we shall derive a spectral sequence which converges to the stable mod-2 homotopy groups of Z_2-Moore spaces from the Curtis spectral sequence. An algebra structure is introduced in the E^1 term with the multiplication defined by composition of maps. We carefully study the derivations of the algebra E^1 and calculate the E^2 term in low dimensions. It is found that all the part of the E^2 term with dimensions ≤ 7 survives in the E^∞ term. Henceforth the stable mod-2 homotopy groups of Z_2-Moore spaces in dimensions ≤ 7 are obtained. Through the universal coefficient theorem and the stable version of the Blakers-Massey theorem applied to the cofibration

$$S^q \to S^q U_2 e^{q+1} \to S^{q+1},$$

the structure of the stable homotopy groups of Z_2-Moore spaces with dimensions ≤ 7 follows very easily.

1. Preliminaries

Since the statements in the following sections will be in terms of s.s. Lie rings, we recall some definitions and fundamental theorems which will be used later.

1.1. An s.s. complex X is a sequence of sets X_n for $n \geq 0$, with face operators $d_i : X_n \to X_{n-1}$ and degeneracy operators $s_i : X_n \to X_{n+1}$, $0 \leq i \leq n$ which satisfy the usual identities [4, p. 283]. If all the X_n and the d_i, s_i are objects and morphisms in a category C, X will be called an s.s. object over C.

THEOREM 1.2. *Let A, B be s.s. abelian groups and*

$$f_0, f_1 : A \to B$$

be s.s. homomorphisms. Then f_0, f_1 are homotopic if and only if Nf_0 and Nf_1 are chain homotopic, where N is the normalization functor defined by

$$(NG)_n = \textstyle\bigcap_{i=1}^n \ker d_i \approx G_n/DG_{n-1},$$

DG_{n-1} is generated by $s_i G_{n-1}$ for $0 \leq i \leq n-1$.

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THEOREM 1.3. *Let G be an s.s. abelian group, then the inclusion map $NG \to \operatorname{Tot} G$, $\operatorname{Tot} G = \{G_n, d_n = \sum_{i=0}^{n}(-1)^i d_i\}$, is a chain homotopy equivalence.*

1.4. Given an abelian group M, let R_M be the free non-associative ring on M, i.e., $R_M = \sum_r R_M^r$, where $R_M^1 = M$, $R_M^r = \sum_{i=1}^{r-1} R_M^i \otimes R_M^{r-i}$ for $r > 1$.[1] Let I_M be the two sided ideal generated by the elements

$$x \otimes x, \quad (x \otimes y) \otimes z + (y \otimes z) \otimes x + (z \otimes x) \otimes y$$

for all x, y, z in R_M. For every positive integer r, let $L^r M = R_M^r/(R_M^r \cap I_M)$. Then the quotient $LM = R_M/I_M = \sum_{r=1}^{\infty} L^r M$ is a Lie ring, called the free Lie ring on M.

1.5. Given a simply connected s.s. complex X and a finite-dimensional s.s. complex Y, consider the filtration of the function complex $(GX)^Y$ by

$$(GX)^Y \supset (\Gamma_2(GX)^Y \supset \cdots \supset (\Gamma_r GX)^Y \supset \cdots .$$

For each $r \geq 1$, there is an s.s. fibration

$$(\Gamma_{r+1} GX)^Y \to (\Gamma_r GX)^Y \to (\Gamma_r GX/\Gamma_{r+1} GX)^Y.$$

The homotopy exact couple yields a spectral sequence. E. Curtis [1] proved the following

THEOREM 1.6 [1]. *Let X and Y be as above, then there is a spectral sequence $\{E_{r,q}^s(Y, X)\}$ having*

$$E_{r,q}^1 = \sum_i H^i(S^q Y; \pi_i(\Gamma_r GX/\Gamma_{r+1} GX)),$$

and for each $q \geq 1$, $\sum_r E_{r,q}^{\infty}$ is the graded group associated with certain filtration on the group $\pi_{q+1}(Y, X)$.[2]

By Witt's theorem $\Gamma_r GX/\Gamma_{r+1} GX \approx L^r(GX/\Gamma_2 GX)$. One sees that the L-functor plays an important role in computing $\pi_n(Y, X)$.

2. A spectral sequence

In this section, we are going to derive a spectral sequence for the groups of the homotopy classes of stable maps between Z_2-Moore spaces from the Curtis spectral sequence.

Let (Z_2, n) be the s.s. Z_2-vector space with one nondegenerate basis element $\bar{x}_n$ such that $d_i \bar{x}_n = 0$ for $0 \leq i \leq n$, and let $(Z_2, n)^F$ be the free s.s. abelian group with two nondegenerate basis elements y_{n+1}, x_n such that $d_0 y_{n+1} = 2x_n$, $d_i y_{n+1} = 0$ for $1 \leq i \leq n+1$, and $d_j x_n = 0$ for $0 \leq j \leq n$. Let $K'(Z_2, n+1)$ be an s.s. Z_2-Moore space. Since both

$$GK'(Z_2, n+1)/\Gamma_2 GK'(Z_2, n+1) \quad \text{and} \quad (Z_2, n)^F$$

[1] Here "+" means direct sum.

[2] $\pi_q(Y, X)$ is defined as the set of homotopy classes of maps from $S^q Y$ to X [3], and GX is defined in [4].

are free s.s. abelian groups and have the same homotopy groups, by a theorem of Dold [2], they are of the same homotopy type. Thus by (1.6) the knowledge of $(Z_2, n)^F$ is essential in setting up such a spectral sequence. By an argument on the basis of free Lie rings, the following is immediate.

PROPOSITION 2.1. *The sequence*

$$L(Z_2, n)^F \xrightarrow{2} L(Z_2, n)^F \xrightarrow{\iota} L((Z_2, n) + (Z_2, n+1)) \tag{2.2}$$

is exact where ι is the natural projection with $\iota x_n = \bar{x}_n$ and $\iota y_{n+1} = \bar{x}_{n+1}$.

Passing (2.2) to the long exact sequence of homotopy groups, we obtain

PROPOSITION 2.3. *The following sequence*

$$\pi_k L(Z_2, n)^F/2\pi_k L(Z_2, n)^F \xrightarrow{\iota *} \pi_k L((Z_2, n) + (Z_2, n+1)) \xrightarrow{\partial} {}_2\pi_{k-1} L(Z_2, n)^F \tag{2.4}$$

is exact where ${}_2A$ denotes the ker $(A \xrightarrow{2} A)$.

PROPOSITION 2.5. *There is an isomorphism*

$$\textstyle\sum_i H^i(K'(Z_2, m), E^1_{r,i}(K'(Z_2, n+1))) \to \pi_{m+1} L^r((Z_2, n) + (Z_2, n+1)),$$

where $K'(Z_2, \ \ )$ is the Z_2-Moore space and $E^1_{r,i}(X) = \pi_i L^r(GX/\Gamma_2 GX)$.

Proof. Since $GK'(Z_2, n+1)/\Gamma_2 GK'(Z_2, n+1)$ and $(Z_2, n)^F$ are of the same homotopy type, we have

$$\pi_i L^r(GK'(Z_2, n+1)/\Gamma_2 GK'(Z_2, n+1)) = \pi_i L^r(Z_2, n)^F.$$

Therefore

$$\begin{aligned}\textstyle\sum_i H^i(K'(Z_2, m), E^1_{r,i}(K'(Z_2, n+1)))& \\ &= \operatorname{Hom}(Z_2, \pi_m L^r(Z_2, n)^F) + \operatorname{Ext}(Z_2, \pi_{m+1} L^r(Z_2, n)^F) \\ &= {}_2\pi_m L^r(Z_2, n)^F + \pi_{m+1} L^r(Z_2, n)^F/2\pi_{m+1} L^r(Z_2, n)^F \\ &\approx \pi_{m+1} L^r((Z_2, n) + (Z_2, n+1)).\end{aligned}$$

Let $[(Z_2, k)^F, L(Z_2, n)^F]$ denote the group of the homotopy classes of s.s. homomorphisms from $(Z_2, k)^F$ into $L(Z_2, n)^F$. Clearly any s.s. homomorphism $(Z_2, k)^F \to L(Z_2, n)^F$ can be expressed by $\binom{u}{v}$ such that $d_0 u = 2v$, $u \in (NL(Z_2, n)^F)_{k+1}$ and $\binom{u}{v} y_{k+1} = u$, $\binom{u}{v} x_k = v$.

PROPOSITION 2.6. *$\binom{u}{v}$ and $\binom{u'}{v'}$ are homotopic iff there exists an element s in $(NL(Z_2, n)^F)_{k+2}$ such that $u - u' - d_0 s$ lies in $2L(Z_2, n)^F$.*

Proof. This is immediate from Theorem 1.2.

THEOREM 2.7. *Let*

$$\theta : [(Z_2, k)^F, L(Z_2, n)^F] \to \pi_{k+1} L((Z_2, n) + (Z_2, n+1))$$

be the homomorphism defined by

$$\theta\left(\mathrm{cl}\begin{pmatrix}u\\v\end{pmatrix}\right) = \mathrm{cl}\,(\iota u),$$

where cl *denotes the homotopy class. Then θ is an isomorphism.*

Proof. By Proposition 2.6, θ is well defined. Let x be an element in $(NL((Z_2, n) + (Z_2, n+1)))_{k+1}$ with $d_0 x = 0$. Since ι is epic, $N\iota$ is also epic. Hence there is an element u in $(NL(Z_2, n)^F)_{k+1}$ with $\iota u = x$. But $d_0\,\iota u = \iota(d_0 u) = 0$, $d_0 u = 2v$. Therefore

$$\theta\left(\mathrm{cl}\begin{pmatrix}u\\v\end{pmatrix}\right) = \mathrm{cl}\,(\iota u) = \mathrm{cl}\,(x),$$

i.e., θ is epic. Now let $\mathrm{cl}\,\binom{u}{v}$ be in $\ker\theta$, that is $\mathrm{cl}\,(\iota u) = 0$. Hence there is an element c in $(NL((Z_2, n) + (Z_2, n+1)))_{k+2}$ with $d_0 c = \iota u$. Let s be in $(NL(Z_2, n)^F)_{k+2}$ with $\iota s = c$. Thus $\iota(d_0 s - u) = 0$, i.e., $d_0 s - u$ is in $2L(Z_2, n)^F$. By Proposition 2.6, $\binom{u}{v}$ is homotopic to $\binom{0}{0}$, hence θ is monic.

2.8. *Suspension homomorphism.* Let $WK(Z_2, n)^F$ be the free s.s. abelian group with nondegenerate basis elements x_n, y_{n+1}, x_{n+1}, y_{n+2} such that $d_0 y_{n+1} = 2x_n$, $d_0 x_{n+1} = x_n$, $d_0 y_{n+2} = 2x_{n+1} - y_{n+1}$ and all the other face operators on these cells are trivial. Clearly we have an exact sequence

$$(2.9)\qquad (Z_2, n)^F \xrightarrow{i} WK(Z_2, n)^F \xrightarrow{\zeta} (Z_2, n+1)^F$$

where i is the inclusion and ζ is the natural projection. Since $WK(Z_2, n)$ is contractible, $L(WK(Z_2, n))$ is also contractible. Henceforth given any

$$(Z_2, n)^F \xrightarrow{f} L(Z_2, m)^F,$$

there are f_1 and f_2 which make the following diagrams

$$(2.10)\qquad \begin{array}{ccccc}
(Z_2, n)^F & \xrightarrow{i} & WK(Z_2, n)^F & \xrightarrow{\zeta} & (Z_2, n+1)^F \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\
L((Z_2, m)^F) & \xrightarrow{Li} & L(WK(Z_2, m)^F) & \xrightarrow{L\zeta} & L((Z_2, m+1)^F)
\end{array}$$

commute. Then we define the suspension homomorphism σ by $\sigma(\mathrm{cl}\,(f)) = \mathrm{cl}\,(f_2)$. It is quite easy to verify that σ is well defined. We define $WK((Z_2, n) + (Z_2, n+1))$ to be $WK(Z_2, n)^F/2WK(Z_2, n)^F$ and the sus-

PROPOSITION 2.11. *The following diagrams*
pension homomorphism σ is an analogous manner.

$$
\begin{array}{ccccc}
(Z_2, n)^F & \longrightarrow & WK(Z_2, n)^F & \xrightarrow{\zeta} & (Z_2, n+1)^F \\
\downarrow \iota & & \downarrow \iota & & \downarrow \iota \\
\begin{matrix}(Z_2, n) + \\ (Z_2, n+1)\end{matrix} & \longrightarrow & \begin{matrix}WK((Z_2, n) + \\ (Z_2, n+1))\end{matrix} & \xrightarrow{\zeta} & \begin{matrix}(Z_2, n+1) + \\ (Z_2, n+2)\end{matrix}
\end{array}
\tag{2.12}
$$

are commutative.

Proof is easy.

PROPOSITION 2.13. $\sigma\theta = \theta\sigma$.

Proof. Given $\mathrm{cl}\,\binom{u}{v}$ in $[(Z_2, k)^F, L(Z_2, n)^F]$, by definition $\theta(\mathrm{cl}\,\binom{u}{v}) = \mathrm{cl}\,(\iota u)$. Let y, x in $NL(WK(Z_2, n)^F)$ such that $d_0 x = v$. $d_0 y = 2x - u$. Clearly

$$\sigma \mathrm{cl} \begin{pmatrix} u \\ v \end{pmatrix} = \mathrm{cl} \begin{pmatrix} L(\zeta)y \\ L(\zeta)x \end{pmatrix}, \qquad \theta\sigma \mathrm{cl} \begin{pmatrix} u \\ v \end{pmatrix} = \mathrm{cl}(L(\iota\zeta)y).$$

However $d_0 L(\iota)y = L(\iota)u = \iota u$, $\sigma\theta\,\mathrm{cl}\,\binom{u}{v} = \mathrm{cl}\,(L(\zeta\iota)y)$. By Proposition 2.11, $\zeta\iota = \iota\zeta$, hence $\sigma\theta = \theta\sigma$.

Dold [2] has defined suspension homomorphism σ for any functor T between two abelian categories which satisfies $T(0) = 0$. σ is a natural transformation $\sigma_T : \pi_* T(X) \to \pi_{*+1} T(SX)$ for s.s. objects. It is easy to see that our definition here is compatible with that defined in [2].

PROPOSITION 2.14 [2]. *Let T be a functor on two variables such that $T(A, 0) = T(0, B) = 0$; then the total suspension homomorphism $\sigma_T = 0$.*

COROLLARY 2.15 [2]. *Let T_2 be the cross effect of a functor T between the category of abelian groups such that $T(0) = 0$; then $\sigma_{T_2} = 0$.*

2.16. *Composition of maps.* Given a in $[(Z_2, k)^F, L(Z_2, n)^F]$ and b in $[(Z_2, j)^F, L(Z_2, m)^F]$ such that $j \geq n$, then the composition of a and b, i.e., ba is defined by $\mathrm{cl}\,(f)$, where

$$f = (Z_2, k + j - n)^F \xrightarrow{h} L(Z_2, j)^F \xrightarrow{L(g)} LL(Z_2, m)^F \to L(Z_2, m)^F,$$

$\mathrm{cl}\,(h) = \sigma^{j-n}a$, $\mathrm{cl}\,(g) = b$, and the last one is induced by the multiplication of Lie ring.

PROPOSITION 2.17. $\sigma(ba) = \sigma(b)\sigma(a)$.

Proof. Given h and g as above. Then there exist h_1, h_2, g_1, g_2 which make the following diagrams commute:

$$\begin{array}{ccccccc}
(Z_2, k+j-n)^F & \xrightarrow{h} & L(Z_2, j)^F & \xrightarrow{L(g)} & LL(Z_2, m)^F & \rightarrow & L(Z_2, m)^F \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
WK(Z_2, k+j-n)^F & \xrightarrow{h_1} & L(WK(Z_2, j)^F) & \xrightarrow{L(g_1)} & LL(Z_2, m)^F) & \rightarrow & L(WK(Z_2, m)^F \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(Z_2, k+j-n+1)^F & \xrightarrow{h_2} & L(Z_2, j+1)^F & \xrightarrow{L(g_2)} & LL(Z_2, m+1)^F & \rightarrow & L(Z_2, m+1)^F.
\end{array}$$

But $\operatorname{cl} g_2 = \sigma b$, $\operatorname{cl} h_2 = \sigma^{j-n}(\sigma a)$. It is clear that the bottom line represents both $\sigma(ba)$ and $\sigma(b)\sigma(a)$.

2.18. *The limit algebra.* Through Propositions 2.5 and 2.6, we may identify the group $\sum_i H^i(K'(Z_2, k), E^1_{r,i}(K'(Z_2, m+1)))$ with

$$[(Z_2, k)^F, L(Z_2, m)^F]$$

and through θ with

$$\pi_{k+1} L((Z_2, m) + (Z_2, m+1)).$$

Let $M(m) = \sum_k [(Z_2, k)^F, L(Z_2, m)^F]$. Then the suspension homomorphism σ induces

$$\sigma : M(m) \to M(m+1) \qquad \text{for} \quad m \geq 1.$$

Let $\{M, \sigma_m : M(m) \to M\}$ be the direct limit of the system

$$M(1) \xrightarrow{\sigma} M(2) \xrightarrow{\sigma} \cdots.$$

Similarly we use $\pi_i L(Z_2)$ to denote

$$\xrightarrow[n]{\lim} \pi_{i+n} L(Z_2, n).$$

In order to know $\pi_i L(Z_2)$ completely, Schlesinger [7] has introduced an operation [[,]] in arbitrary s.s. Lie ring.

DEFINITION 2.19. Let

$$(\gamma, \delta) = (\gamma(0), \cdots, \gamma(p-1), \delta(0), \cdots, \delta(q-1))$$

be a shuffle permutation of type (p, q). The degeneracy operation S^γ (or S^δ) is obtained from the word $S_{p+q-1} \cdots S_1 S_0$ by deleting those symbols S_j whose subscripts j are in γ (or δ):

$$S^\gamma = S_{p+q-1} \cdots \hat{S}_{\gamma(p-1)} \cdots \hat{S}_{\gamma(0)} \cdots S_0.$$

DEFINITION 2.20. Let x and y be two simplexes of dimensions p and q in L. We define the double bracket:

$$[[x, y]] = \sum (-1)^{(\gamma, \delta)}[S^\gamma x, S^\delta y]$$

where the sum is over all shuffles of type (p, q) and $(-1)^{(\gamma,\delta)}$ denotes the sign of the permutation (γ, δ).

When $\dim x = \dim y$, $\frac{1}{2}[[x, y]]$ means that the summation runs over all (γ, δ) with the identification $(\gamma, \delta) \equiv (\delta, \gamma)$.

Given any sequence $I = (i_1, \cdots, i_r)$ of non-negative integers, we call r the length and $\sum_{j=1}^{r} i_j$ the dimension of I. I is said to be admissible if if $2i_{j-1} \geq i_j$ for $2 \leq j \leq r$.

PROPOSITION 2.21 [5]. *$\pi_i L^2(Z_2)$ is generated by w_i, where $w_i = \sigma_i(\bar{w}_i)$, $\bar{w}_i = \mathrm{cl}\,(\frac{1}{2}[[\bar{x}_i, \bar{x}_i]])$.*[3]

PROPOSITION 2.22 [5]. *$\pi_n L^{2^r}(Z_2)$ has a basis $w_I = w_{i_1} w_{i_2} \cdots w_{i_r}$ such that I is admissible of length r, $\dim I = n$, i_j positive for $1 \leq j \leq r$ and $\pi_n L^s(Z_2) = 0$ if s is not a power of 2.*

PROPOSITION 2.23. *Let*

$$M_{r,i} = \xrightarrow[m]{\lim} [(Z_2, m + i)^F, L^{2^r}(Z_2, m)^F].$$

θ induces an isomorphism $M_{r,i} \to \pi_{i+1} L^{2^r}(Z_2) + \pi_i L^{2^r}(Z_2)$.

Proof. We have

$$M_{r,i} = \xrightarrow[m]{\lim} [(Z_2, m + 1)^F, L^{2^r}(Z_2, m)^F].$$

But $\sigma\theta = \theta\sigma$. Hence

$$M_{r,i} = \xrightarrow[m]{\lim} \pi_{1+m+i} L^{2^r}((Z_2, m) + (Z_2 m + 1)).$$

By Corollary 2.15,

$$\xrightarrow[m]{\lim} \pi_{1+m+i} L^{2^r}((Z_2, m) + (Z_2, m + 1))$$
$$= \xrightarrow[m]{\lim} \pi_{1+m+i} L^{2^r}(Z_2, m) + \xrightarrow[m]{\lim} \pi_{1+m+i} L^{2^r}(Z_2, m + 1).$$

Thus $M_{r,i} \to \pi_{i+1} L^{2^r}(Z_2) + \pi_i L^{2^r}(Z_2)$.

Given a, b in M such that $a = \sigma_n(\bar{a})$, $b = \sigma_n(\bar{b})$. Then define $ba = \sigma_{s+m}((\sigma^s \bar{b})\bar{a})$ with s large enough to ensure $(\sigma^s \bar{b})\bar{a}$ being defined.

THEOREM 2.24. *M is a Z_2-algebra.*

Proof. By the universal property of free Lie ring, given any homomorphism

$$A \xrightarrow{k} LB$$

[3] The notation L^r here is just L_r^u in [5].

the following diagram

$$\begin{array}{ccc} LLA & \xrightarrow{L(Lk)} & L(LLB) \\ \lambda\big\downarrow & & \big\downarrow\lambda L\lambda \\ LA & \xrightarrow{\lambda Lk} & LB \end{array}$$

is commutative, where λ is induced by the multiplication of Lie ring. Hence given

$$(Z_2\,,\,j)^F \xrightarrow{f} L(Z_2\,,\,m)^F, \quad (Z_2\,,\,m)^F \xrightarrow{g} L(Z_2\,,\,n)^F,$$

$$(Z_2\,,\,n)^F \xrightarrow{h} L(Z_2\,,\,p)^F,$$

then $\lambda \circ L\lambda \circ L(Lh) \circ L(g) \circ f = \lambda \circ Lh \circ \lambda \circ L(g) \circ f$. Hence

$$(\mathrm{cl}\, h \circ \mathrm{cl}\, g) \circ \mathrm{cl}\, f = \mathrm{cl}\, h \circ (\mathrm{cl}\, g \circ \mathrm{cl}\, f).$$

This gives the associative law. The right distribution law is an immediate consequence of the definition of multiplication. Given

$$(Z_2\,,\,r)^F \xrightarrow{e} L(Z_2\,,\,j)^F, \quad (Z_2\,,\,j)^F \xrightarrow{f} L(Z_2\,,\,m)^F,$$

$$(Z_2\,,\,j)^F \xrightarrow{f'} L(Z_2\,,\,m)^F,$$

$$\mathrm{cl}\,(f + f') \circ \mathrm{cl}\,(e) = \mathrm{cl}\,(\lambda \circ L(f + f') \circ e).$$

However $L(f + f') = L(f) + L(f') + L(f, f')$, where

$$L(f, f') = L(Z_2\,,\,j)^F \rightarrow L((Z_2\,,\,j)^F, (Z_2\,,\,j)^F) \rightarrow LL(Z_2\,,\,m)^F.$$

By an argument similar to Corollary 2.15, $\sigma(\mathrm{cl}(\lambda \circ L(f, f') \circ e)) = 0$. Hence left distribution law holds.

PROPOSITION 2.25. *$[(Z_2\,,\,m)^F, L(Z_2\,,\,n)^F] = 0$ for $m < n - 1$ and $[(Z_2\,,\,n - 1)^F, L(Z_2\,,\,n)^F]$ is generated by* cl $\binom{x_n}{0}$ *and* σ cl $\binom{x_n}{0} =$ cl $\binom{x_{n+1}}{0}$.

Proof is easy.

We use the symbol β to denote both cl $\binom{x_n}{0}$ and the corresponding element in M. One shows easily that $\beta^2 = 0$. Consider the isomorphism

$$[(Z_2\,,\,2n + 1)^F, L(Z_2\,,\,n)^F] \xrightarrow{\theta} \pi_{2n+2}L((Z_2\,,\,n) + (Z_2\,,\,n + 1)).$$

Let $\bar{w}_{n+1} = \mathrm{cl}\,(\frac{1}{2}[[\bar{x}_{n+1}\,,\,\bar{x}_{n+1}]])$ in $\pi_{2n+2}L((Z_2\,,\,n) + (Z_2\,,\,n + 1))$; then we have

PROPOSITION 2.26. *Let $\theta^{-1}(\bar{w}_n) = w_n$; then unstably*

$$\begin{aligned} w_n\,\beta &= \theta^{-1}\,\mathrm{cl}\,[[\bar{x}_n\,,\,\bar{x}_{n+1}]], && \text{if } n \text{ is odd,} \\ &= \theta^{-1}\,\mathrm{cl}\,[[\bar{x}_n\,,\,\bar{x}_{n+1}]] + \sigma w_{n-1}\,, && \text{if } n \text{ is even.} \end{aligned}$$

Proof. Just check the definitions.

Passing to the limit algebra, this yields

COROLLARY 2.27. $w_n \beta = 0$, *if n is odd,*
$= w_{n-1}$, *if n is even.*

PROPOSITION 2.28. *The following diagram*

$$\begin{array}{ccc}
\pi_{k+1} L(Z_2, n+1) \xrightarrow{i_2} \pi_{k+1} L((Z_2, n) + (Z_2, n+1)) & \xrightarrow{\theta^{-1}} & [(Z_2, k)^F, L(Z_2, n)^F] \\
\Big\downarrow i_1 & & \Big\downarrow \beta_L \\
\pi_{k+1} L((Z_2, n+1) + (Z_2, n+2)) & \xrightarrow{\theta^{-1}} & [(Z_2, k)^F, L(Z_2, n+1)^F]
\end{array}$$

is commutative where i_1 (i_2) is induced by the inclusion homomorphism into the first (second) factor, and β_L is left multiplication by β.

Proof. Let cl (a) be in $\pi_{k+1} L(Z_2, n+1)$. Then

$$\theta^{-1} i_2(\mathrm{cl}\,(a)) = \mathrm{cl}\begin{pmatrix} \bar{a} \\ \frac{1}{2} d_0 \bar{a} \end{pmatrix},$$

where $\bar{a}$ lies in $\iota^{-1}(a)$ and $(NL(Z_2, n)^F)_{k+1}$ (here we take a to be in $NL(Z_2, n+1)$ and this is always possible). Among $\iota^{-1}(a)$, there is a' such that a' is expressible in terms of y_{n+1} only. Since $\bar{a} = a' + 2b$,

$$\theta(\beta \circ \theta^{-1}(\mathrm{cl}\,(a))) = \mathrm{cl}\left(\iota L \begin{pmatrix} x_{n+1} \\ 0 \end{pmatrix} (a' + 2b)\right) = \mathrm{cl}\left(\iota L \begin{pmatrix} x_{n+1} \\ 0 \end{pmatrix} (a')\right)$$
$$= i_1(\mathrm{cl}\,(\iota a')) = i_1(\mathrm{cl}\,(a)).$$

COROLLARY 2.29. $M = \beta \circ \pi L(Z_2) + \pi L(Z_2)$ *and* $\dim(\beta \circ \alpha) = \dim \alpha - 1$.

Proof. It suffices to verify that the identification given by Proposition 2.28 goes stably. But this is trivial.

It is very easy to show that the multiplication in $\pi L(Z_2)$ goes nicely into M. Summarizing the above results, we then get the following

THEOREM 2.30. *There is a spectral sequence $\{E^s_{r,q}, d^s\}$ such that $E^1_{r,q} = M_{r,q}$, $\deg d^s = (s, -1)$ and $\sum_r E^\infty_{r,q}$ is the graded group associated with a certain filtration to the group $\pi^s_{q+n}(Z_2; X_2, n)$, and $M = \beta \circ \pi L(Z_2) + \pi L(Z_2)$, the multiplication of M is determined by $w_{2n}\beta = w_{2n-1}$, $\beta^2 = 0$ and that of $\pi L(Z_2)$, and d^1 is a derivation.*

Proof. The first part is just the stable version of the Curtis spectral sequence for Z_2-Moore spaces and the second part is the summary of the preceding results. That d^1 is a derivation follows from the definitions of d^1 and the multiplication of M.

THEOREM 2.31. $w_i w_{2i+1} = 0$ *for all positive integers i.*

Proof. Let w_i and w_{2i+1} be considered in $\pi_{2i+1} L^2(Z_2, i+1)$ and $\pi_{4i+2} L^2(Z_2, 2i+1)$ respectively, then $w_i w_{2i+1}$ is defined and sits in $\pi_{4i+2} L^2(Z_2, i+1)$. But $\pi_* L(Z_2, i+1)$ is generated by W_I where I is admissible and with $i_1 \leq i+1$. Hence $w_i w_{2i+1} = n w_{i+1} w_{2i}$ where n is either i or 0. Applying β on both sides, we have $n w_{i+1} w_{2i-1} = 0$. Since $w_{i+1} w_{2i-1}$ is not 0, n has to be 0, i.e., $w_i w_{2i+1} = 0$.

2.32. *The* mod-2 *binomial relations generated by* $w_i w_{2i+1} = 0$. From [5, 2.4.iii], we know that all the defining relations among w_i's are obtained by applying an operator D and its powers on $w_i w_{2i+1} = 0$, $i > 0$, where D acts like a derivation sending each w_j to w_{j+1}.

Let $w_i w_{2i+1+n} = \sum_{n>j>0} a_{n-j,j} w_{i+n-j} w_{2i+1+j}$ be expressed in admissible form, i.e., $a_{n-j,j} = 1$ or 0 and $a_{n-j,j} = 0$ for $2(i+n-j) < 2i+1+j$. Apply D on both sides, we get the following recursive formula.

PROPOSITION 2.33.

$$a_{n,i} + a_{n-1,i+1} + a_{n-1,i-1} = a_{n,i+1} \qquad \textit{for } i \geq 1$$

$$a_{n-1,0} + a_{n,0} = 0 \qquad \textit{for } n \geq 2$$

$$a_{n,0} + a_{n-1,1} = a_{n,1} \qquad \textit{for } n \geq 2$$

$$a_{0,0} = 0, \qquad a_{1,0} = 1, \qquad a_{1,1} = 0 \pmod 2.$$

Set $F(X, Y) = \sum_{n,j\geq 0} a_{n,j} X^n Y^j$. Then Proposition 2.33 yields the identity $F(X, Y) = F(X, Y)(X + Y + XY^2) + X + XY \pmod 2$. Hence $F(X, Y) = X/(1 - (1+Y)X) = X + \sum_{n=1}^{\infty} X^{n+1}(1-Y)^n \pmod 2$ and

$$a_{n,i} = \binom{n-1}{i} \pmod 2.$$

THEOREM 2.34.

$$w_i w_{2i+1+n} = \sum_{j\geq 0} \binom{n-j-1}{j} w_{i+n-j} w_{2i+1+j},$$

the binomial coefficients are, of course, taken mod 2 *and with the usual convention* $\binom{r}{s} = 0$ *for* $r < s$.

3. The derivations of the algebra M

Algebraically M is a graded Z_2-algebra generated by the symbols β and w_n for each positive integer n with the relations $\dim \beta = -1$, $\dim w_n = n$, $\beta^2 = 0$, $w_{2n}\beta = w_{2n-1}$ and

$$w_i w_{2i+1+m} = \sum_{j\geq 0} \binom{m-1-j}{j} w_{i+m-j} w_{2i+1+j}$$

for $m \geq 0$ (here dim = degree). Then $M_{r,n}$ is just the subgroup of M generated by w_I and βw_J such that length I = length J = r and dim I =

$\dim J - 1 = n$. Clearly $M = \sum_{r,n} M_{r,n}$ and we call r the filtration degree. Let $M_r = \sum_n M_{r,n}$ and

$$M \xrightarrow{P_r} M_r$$

be the natural projection. Given any derivation D, let D_i be defined by $D_i x = P_{i+r} Dx$, if x lies in M_r. Then the following proposition is immediate.

PROPOSITION 3.1. *Let D be a derivation; then each D_i is also a derivation.*

PROPOSITION 3.2. *Let D be a derivation which lowers the dimension at least by one and $Dw_i = 0$, $1 \leq i \leq 8$; then $D = 0$.*

Proof. If $Dw_i = 0$ for $1 \leq i \leq 2m$, $m \geq 4$, the defining relations always imply $Dw_{2m+1} = 0$ and $Dw_{2m+2} = 0$. Hence by induction the proposition follows easily.

THEOREM 3.3. *Let $\bar{D}$ be the group of derivations of M which lower the dimension by one and $\bar{D}_i$ be the subgroup of derivations which raise the filtration by i; then $\bar{D} = \sum_i \bar{D}_i$.*

Proof. Given any derivation D, by Proposition 3.2, all but a finite of D_i are trivial, hence $D = \sum_i D_i$. Since D_i is in $\bar{D}_i$, there is a canonical homomorphism from $\bar{D}$ to $\sum_i \bar{D}_i$ which maps D into $\sum_i D_i$. Clearly this homomorphism is an isomorphism.

Remark. The theorem is still true when $\bar{D}$ is replaced by the group of derivations which lower the dimension at least by 1.

PROPOSITION 3.4. *$\bar{D}_0$ is generated by $\bar{\beta}$ where $\bar{\beta}x = \beta x + x\beta$, for x in M.*

Proof. It is trivial that $\bar{\beta}$ is a derivation. Let D be any element in $\bar{D}_0$. Then it is not difficult to verify that $D'w_i = 0, 1 \leq i \leq 8$ where $D' = D - a_1 \bar{\beta}$ and $Dw_1 = a_1 \beta w_1$. Therefore by Proposition 3.2, $D' = 0$, i.e., $D = a_1 \bar{\beta}$.

PROPOSITION 3.5. *$\bar{D}_i = 0$ for $i \geq 2$.*

Proof. Let D be any element in $\bar{D}_i$, $i \geq 2$. The defining relations of M imply $Dw_i = 0$ for $1 \leq i \leq 8$, hence by Proposition 3.2, imply $D = 0$.

COROLLARY 3.6. $\bar{D} = \bar{D}_0 + \bar{D}_1$.

Since $\deg d^1 = (1, -1)$, d^1 lies in $\bar{D}_1$. Our next task is to determined d^1. Let D be any element in $\bar{D}_1$, we express

$$Dw_n = \sum_{0<j<n-1} a_{n-j,j}\, w_{n-j-1}\, w_j + \sum_{0<j<n} b_{n-j,j}\, \beta w_{n-j}\, w_j$$

in admissible form, i.e., $a_{n-j,j}$ and $b_{n-j,j} = 1$ or 0, $a_{n-j,j} = 0$ for $2(n-j-1) < j$ and $b_{n-j,j} = 0$ for $2(n-j) < j$.

Since $w_n w_{2n+1} = 0$ and $w_n w_{2n+2} = w_{n+1} w_{2n+1}$, we have

$$(Dw_n)w_{2n+1} + w_n Dw_{2n+1} = 0,$$

and

$$w_n\, Dw_{2n+2} + (Dw_n)w_{2n+2} = (Dw_{n+1})w_{2n+1} + w_{n+1}\, Dw_{2n+1}\,.$$

This yields

$$\begin{aligned}
&\textstyle\sum_j a_{n-j,j}\, w_{n-j-1}\, w_j\, w_{2n+1} + \sum_j b_{n-j,j}\, \beta w_{n-j}\, w_j\, w_{2n+1} \\
&\qquad = \textstyle\sum_i a_{2n+1-i,i}\, w_n\, w_{2n-i}\, w_i \\
&\qquad\qquad + \textstyle\sum_i (n-1) b_{2n+1-i,i}\, w_{n-1}\, w_{2n+1-i} w_i\,,
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
&\textstyle\sum_j a_{n-j,j}\, w_{n-j-1}\, w_j\, w_{2n+2} + \sum_j b_{n-j,j}\, \beta w_{n-j}\, w_j\, w_{2n+2} \\
&\quad + \textstyle\sum_i a_{2n+2-i,i}\, w_n\, w_{2n+1-i}\, w_i + \sum_i (n-1) b_{2n+2-i,i}\, w_{n-1} w_{2n+2-i} w_i \\
&= \textstyle\sum_s a_{n+1-s,s}\, w_{n-s}\, w_s\, w_{2n+1} + \sum_s b_{n+1-s,s}\, \beta w_{n+1-s}\, w_s\, w_{2n+1} \\
&\quad + \textstyle\sum_t a_{2n+1-t,t}\, w_{n+1}\, w_{2n-t}\, w_t + \sum_t n b_{2n+1-t,t}\, w_n\, w_{2n+1-t}\, w_t\,.
\end{aligned} \tag{3.8}$$

Notice that (3.7) and (3.8) are not in admissible form, i.e., not expressed in terms of w_I, βw_J with I and J admissible. However through the defining relations of M, we can always render them with both sides in admissible forms and we assume that this is done. Then equating the coefficients of those admissible terms $w_{(n-1,\ ,\ )}$ in (3.7), we have the following

Proposition 3.9. *$a_{n-j,j} = (n-1) b_{2n-2j,2j+1}$ for even j.*

Similarly equating the coefficients of those admissible terms $\beta w_{(n,\ ,\ )}$ in (3.8), we have the following

Proposition 3.10.

$$b_{n-j,j} = \binom{n-j-1}{j-1} b_{n,1}\,.$$

Again, equating the coefficients of those terms $w_{(n,\ ,\ )}$ in (3.8), we obtain the following

Proposition 3.11. $a_{n-j,j} = a_{2n-2j,2j+2} + n b_{2n-2j-1,2j+2}\,.$

Theorem 3.12.

$$b_{n-j,j} = \binom{n-j-1}{j-1} b_{1,1}\,, \qquad a_{n-j,j} = \binom{n-j-1}{j+1} b_{1,1}\,.$$

Proof. From Proposition 3.10, $b_{n,1} = b_{1,1}$. Hence

$$b_{n-j,j} = \binom{n-j-1}{j-1} b_{1,1}\,.$$

From Propositions 3.11 and 3.9, we have

$$\begin{aligned}
a_{n-j,j} &= a_{2n-2j,2j+2} + n b_{2n-2j-1,2j+2} \\
&= (2n+1) b_{4n-4j,4j+5} + n b_{2n-2j-1,2j+2}
\end{aligned}$$

$$(3.13)\qquad \begin{aligned} &= \left(\binom{4n-4j-1}{4j+1} + n\binom{2n-2j-2}{2j+1}\right) b_{1,1} \\ &= \binom{2n-2j-1}{2j+2} b_{1,1} = \binom{n-j-1}{j+1} b_{1,1} \pmod 2, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for } n \geq 2. \end{aligned}$$

COROLLARY 3.14. *$\bar{D}_1$ is generated by d^1 and $d^1\beta = 0$, $d^1 w_1 = 0$,*

$$d^1 w_n = \sum_{j\geq 1} \binom{n-j-1}{j+1} w_{n-j-1} w_j + \sum_{j \geq 1} \binom{n-j-1}{j-1} \beta w_{n-j} w_j, \qquad n \geq 2;$$

of course, the binomial coefficients are taken mod 2 *and with the usual convention $\binom{r}{s} = 0$ for $r < s$.*

Proof. d^1 is nontrivial, otherwise we would have that $\pi^s_{n+1}(Z_2 ; Z_2 , n)$ is of order 8 which contradicts the fact that $\pi^s_{n+1}(Z_2 ; Z_2 , n) = Z_2 + Z_2$.

LEMMA 3.15.

$$\binom{n-s}{s} = 0 \pmod 2$$

for $s > 0$, if and only if $n + 1$ is a power of 2.

PROPOSITION 3.16.

$$d^1(\beta w_n) = 0 \quad \textit{iff } n = 2, 2^r - 1 \textit{ and}$$

$$d^1(w_n + \beta w_{n+1}) = 0 \quad \textit{iff } n = 2^r - 1.$$

Proof. Immediate from Lemma 3.15.

4. Some computations of homotopy groups

Recall that M is just the E^1 term for the groups $\pi^s_{q+n}(Z_2 ; Z_2 , n)$. In Table 4.a a table of $E^2 = H_*(M, d^1)$ in low dimensions is given.

Column 8 is incomplete. However all terms with dimension ≤ 7 are retained in E^∞. Using this, we are able to determine $\pi^s_{q+r}(Z_2 ; Z_2 , q)$ and, through the universal coefficient theorem, $\pi^s_{q+r}K'(Z_2 , q)$ with $r \leq 7$.

Table 4.a. (d = dimension, f = filtration degree, $v = \langle \beta w_2 , w_1^2 , w_1 w_2 \rangle$)

4									v	
3					w_1^3	$w_1^2 w_2$			$w_3^2 w_1$	
				$\beta w_1 w_2$				$(w_3 - \beta w_4)^2$	$w_1 w_6$	$\beta w_7(\beta w_2 - w_1)$
2					$w_1 w_2$		βw_3^2			
				w_1^2				$\beta w_1 w_6$	$\beta w_7 w_1$	$(w_7 - \beta w_8) w_1$
			βw_2							
1		βw_1		βw_3	$w_3 - \beta w_4$			βw_7	$w_7 - \beta w_8$	
			w_1							
0	β	1								
f/d	-1	0	1	2	3	4	5	6	7	8

TABLE 4.b

r	$\pi^s_{q+r}(Z_2 ; Z_2 , q)$
-1	Z_2
0	Z_4
1	$Z_2 + Z_2$
2	$Z_2 + Z_2 + Z_2$
3	$Z_2 + Z_4$
4	Z_2
5	Z_2
6	$Z_2 + Z_2 + Z_2$
7	$Z_2 + Z_4 + Z_4$

TABLE 4.c

$r =$	0,	1,	2,	3,	4,	5,	6,	7
$\pi^s_{q+r}(Z_2 , q) =$	Z_2 ,	Z_2 ,	Z_4 ,	$Z_2 + Z_2$,	Z_2 ,	0,	Z_2 ,	$Z_2 + Z_2$

From Table 4.a, we know that $\pi^s_q(Z_2 ; Z_2 , q)$ is a group of order 4. This is not enough, we still have the group extension problem. However Barratt's theorem gives us the following

PROPOSITION 4.1 (Barratt). *$\pi^s_q(Z_2 ; Z_2 , q) = Z_4$, and βw_1 corresponds to $2(\mathrm{cl}\,(\alpha_1))$ where α_1 is the identity map of $K'(Z_2 , q)$.*

Since it is known that βw_1 corresponds to $2(\mathrm{cl}(\alpha_1))$, Table 4.b is immediate from Table 4.a and the relations

$$\mathrm{cl}\,((\beta w_1)w_1 w_2) = \mathrm{cl}\,(w_1^3),$$

$$\mathrm{cl}\,((\beta w_1)(w_7 - \beta w_8)) = \mathrm{cl}\,(\beta w_7 w_1)$$

and

$$\mathrm{cl}\,((\beta w_1)w_1 w_6) = \mathrm{cl}\,(w_3^2 w_1).$$

$\pi^s_{q+r}(Z_2 , q)$ and $\pi^s_{q+r}(Z_2 ; Z_2 , q)$ are connected by the following

PROPOSITION 4.2. *There is an exact sequence*

(4.3) $\mathrm{Ext}\,(Z_2 , \pi^s_{q+r+1}(Z_2 , q)) \to \pi^s_{q+r}(Z_2 ; Z_2 , q) \to \mathrm{Hom}\,(Z_2 , \pi^s_{q+r}(Z_2 , q))$.

Proof. This is immediate from the universal coefficient theorem.

Consider the cofibration

$$S^q \xrightarrow{i} S^q U_2 e^{q+1} \xrightarrow{p} S^{q+1};$$

then the stable version of the Blakers-Massey theorem gives

PROPOSITION 4.4. *There exists an exact sequence*

(4.5) $\to \pi^s_{q+r}(S^q) \xrightarrow{i_*} \pi^s_{q+r}(Z_2 , q) \xrightarrow{p_*} \pi^s_{q+r}(S^{q+1}) \xrightarrow{2} \pi^s_{q+r-1}(S^q) \to$.

Proof. The only nontrivial fact is that the boundary map is 2. But the boundary homomorphism is induced by the attaching map for e^{q+1} in $S^q U_2 e^{q+1}$, clearly it is 2.[4]

From Propositions 4.2, 4.4, Table 4.b, and the knowledge of stable homotopy groups of spheres, we immediately have Table 4.c.

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[4] This was pointed out to the author by P. J. Hilton.