TRIVIAL LOOPS IN HOMOTOPY 3-SPHERES[1]

BY
WOLFGANG HAKEN

In this paper we show that every homotopy 3-sphere possesses a cell-decomposition Γ which is in some respect especially simple:

THEOREM. *If M^3 is a homotopy 3-sphere then there exists a cell-decomposition Γ of M^3 with the following properties:*

(i) *Γ consists of one vertex E^0, r open 1-cells, $E_1^1, \cdots, E_r^1$, r open 2-cells, $E_1^2, \cdots, E_r^2$, and one open 3-cell E^3.*

(ii) *There exist (nonsingular, polyhedral) disks $V_1^2, \cdots, V_r^2$ in M^3 such that*[2] $\dot{V}_i^2 = \bar{E}_i^1$ *for all $i = 1, \cdots, r$.*

(iii) *The disks $V_1^2, \cdots, V_r^2$ may be chosen such that the connected components of $V_i^2 \cap V_j^2 - E^0$ ($i \neq j$, between 1 and r) are normal double arcs in which V_i^2 and V_j^2 pierce each other such that the interior of each double arc lies in ${}^\circ V_i^2 \cap {}^\circ V_j^2$, one of its boundary points lies in E_i^1, and the other one lies in E_j^1 (see Fig. 1), and such that $V_i^2 \cap V_j^2 \cap V_k^2 = E^0$ (if i, j, k are pairwise different, between 1 and r).*

It is a known fact that every closed 3-manifold M^3 possesses a cell-decomposition Γ with property (i) (this follows easily from results in Seifert-Threlfall [4], see [2, Sec. 5]). If M^3 is a homotopy 3-sphere, i.e., simply connected, then this is equivalent to the fact that the 1-skeleton $G^1 = \bigcup_{i=1}^r \bar{E}_i^1$ of Γ bounds a "singular fan" in M^3 (see [2, Sec. 6]). Now property (ii) of Γ means that G^1 is a wedge of trivial loops in M^3, and (iii) means that G^1 bounds a singular fan $\bigcup_{i=1}^r V_i^2$ which is especially simple in the sense that its single leaves V_i^2 are nonsingular.

As Bing has shown in [1] it would be sufficient for a proof of the Poincaré conjecture if one could show that every polyhedral, simple closed curve in M^3 lies in a 3-cell in M^3, or that the 1-skeleton G^1 of some cell-decomposition Γ of M^3 lies in a 3-cell in M^3. The property (ii) of Γ means that every single closed curve $\bar{E}_i^1 \subset G^1$ lies not only in a 3-cell V_i^3 (which may be obtained as a small neighborhood of V_i^2) in M^3 but moreover is unknotted in that 3-cell V_i^3. So one may hope that the above theorem could be used as a tool for proving the Poincaré conjecture or for deriving further partial results on homotopy 3-spheres.

Proof of the theorem

1. Preliminaries. We choose the semilinear standpoint as described in [3, Sec. 3], i.e., we assume for convenience that M^3 is a piecewise rectilinear

Received July 15, 1966.
[1] This research was supported by grants from the National Science Foundation.

[2] $\dot{X}$ denotes the *boundary*, $\bar{X}$ and ^-X the *closure*, and $^\circ X$ the *interior* of X.

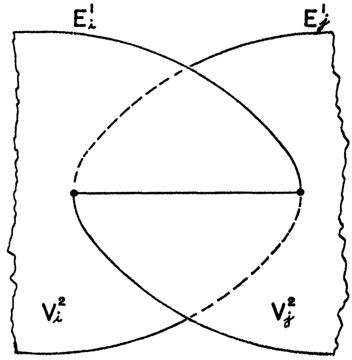

FIG. 1

polyhedron in a euclidean space $\mathfrak{E}^n$; and all point sets denoted by capital roman letters are supposed to be piecewise rectilinear polyhedral point sets in $\mathfrak{E}^n$, etc.

2. Decomposing a singular fan V_{I}^2 by arcs B_k^1. We start with a cell-decomposition Γ_{I} of M^3 into one vertex E_{I}^0, r_{I} elements, $E_{\mathrm{I}1}^1, \cdots, E_{\mathrm{I}r_{\mathrm{I}}}^1$, of dimension 1, r_{I} elements, $E_{\mathrm{I}1}^2, \cdots, E_{\mathrm{I}r_{\mathrm{I}}}^2$, of dimension 2, and one open 3-cell E_{I}^3; (the existence of Γ_{I} has been proved in [2, Sec. 5]). We consider a singular fan, defined by a map $\zeta : V_{\mathrm{I}}'^2 \to M^3$ with $\zeta(V_{\mathrm{I}}'^2)$ denoted by V_{I}^2, such that the following holds (the existence of ζ has been proved in [2, Sec. 6]):

(i) $V_{\mathrm{I}}'^2$ consists of r_{I} disks $V_{\mathrm{I}1}'^2, \cdots, V_{\mathrm{I}r_{\mathrm{I}}}'^2$ (see Fig. 2), possessing one common boundary point $E_{\mathrm{I}}'^0$ and otherwise being pairwise disjoint; $V_{\mathrm{I}}'^2$ is disjoint from M^3.

(ii) ${}^{\cdot}V_{\mathrm{I}}^2$ is the 1-skeleton $G_{\mathrm{I}}^1 = \bigcup_{i=1}^{r_{\mathrm{I}}} \bar{E}_{r_{\mathrm{I}}}^1$ of Γ_{I}.

(iii) The only singularities of V_{I}^2 (with respect to ζ) are pairwise disjoint, normal, double arcs $A_1^1, \cdots, A_s^1$ such that each of the two connected components $A_j'^1$, $A_j''^1$ of $\zeta^{-1}(A_j^1)$ (see Fig. 2) possesses just one boundary point in ${}^{\cdot}V_{\mathrm{I}}'^2 - E_{\mathrm{I}}'^0$ and otherwise lies in ${}^{\circ}V_{\mathrm{I}}'^2$ (for all $j = 1, \cdots, s$).

If $s = 0$ then we may take Γ_{I} for Γ and the theorem is proved. So we may assume that $s > 0$.

We choose a small neighborhood T_{I}^3 of G_{I}^1 in M^3. There is a connected component $V_{\mathrm{T}}'^2$ of $\zeta^{-1}(V_{\mathrm{I}}^2 \cap T_{\mathrm{I}}^3)$ (see Fig. 2) that is a neighborhood of ${}^{\cdot}V_{\mathrm{I}}'^2$ in $V_{\mathrm{I}}'^2$; the other connected components of $\zeta^{-1}(V_{\mathrm{I}}^2 \cap T_{\mathrm{I}}^3)$ are neighborhoods of the points ${}^{\cdot}A_j'^1 \cap {}^{\circ}V_{\mathrm{I}}'^2$ and ${}^{\cdot}A_j''^1 \cap {}^{\circ}V_{\mathrm{I}}'^2$ in $V_{\mathrm{I}}'^2$. Obviously, T_{I}^3 is a Heegaard-handlebody in M^3 (compare [3, Sec. 2]). For brevity we denote ${}^{-}(V_{\mathrm{I}}'^2 - V_{\mathrm{T}}'^2)$ by $V_{\mathrm{I}*}'^2$, and ${}^{-}(V_{\mathrm{I}i}'^2 - V_{\mathrm{T}}'^2)$ by $V_{\mathrm{I}*i}'^2$.

Now we choose pairwise disjoint arcs $B_1'^1, \cdots, B_t'^1$ in $V_{\mathrm{I}*}'^2$ such that (see Fig. 2):

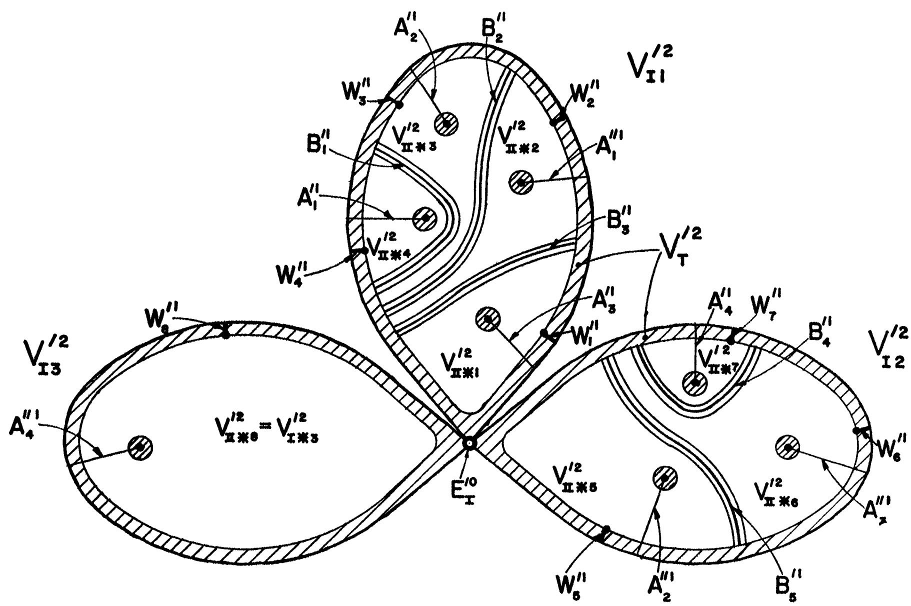

FIG. 2 $\zeta^{-1}(V_I^2 \cap T_I^3)$ is indicated by hatching
$\zeta^{-1}(W_m^2 \cap V_I^2)$ is for brevity denoted by W_m''

(a) ${}^{\cdot}B_k'^1 = B_k'^1 \cap {}^{\cdot}V_{I*}'^2$ (for all $k = 1, \cdots, t$);

(b) $B_k'^1$ is disjoint from the $A_j'^1$'s, $A_j''^1$'s, and ${}^{\circ}B_k'^1$ is disjoint from $\zeta^{-1}(V_I^2 \cap T_I^3)$;

(c) each connected component of $V_{I*}'^2 - \bigcup_{k=1}^t B_k'^1$ contains at most two of the $B_k'^1$'s in its boundary;

(d) each connected component of $V_{I*}'^2 - \bigcup_{k=1}^t B_k'^1$ contains at most one of the points ${}^{\cdot}A_j'^1 \cap V_{I*}'^2$, ${}^{\cdot}A_j''^1 \cap V_{I*}'^2$ ($j = 1, \cdots, s$).

We denote $\zeta(B_k'^1)$, $\zeta(V_{I*i}'^2)$, $\zeta(V_{I*}'^2)$ by B_k^1, V_{I*i}^2, V_{I*}^2, respectively, and $\bigcup_{k=1}^t B_k^1$ by B^1.

3. Projecting the arcs B_k^1 into the Heegaard-surface ${}^{\cdot}T_I^3$. The arcs B_k^1 decompose V_{I*}^2 into nonsingular disks. Hence, if we add small neighborhoods B_k^3 of the B_k^1's to the handlebody T_I^3, then we get a handlebody with t more handles such that "each handle spans a nonsingular disk"; (i.e., we can find a complete system of meridian circles and a corresponding "canonical" system of longitude circles in the boundary of the new handlebody such that each longitude bounds a nonsingular disk in M^3 and intersects just one of the meridians, and that in just one point). But the new handlebody $T^3 + \bigcup_{k=1}^t B_k^3$ is not necessarily a Heegaard-handlebody in M^3. In order to overcome this difficulty

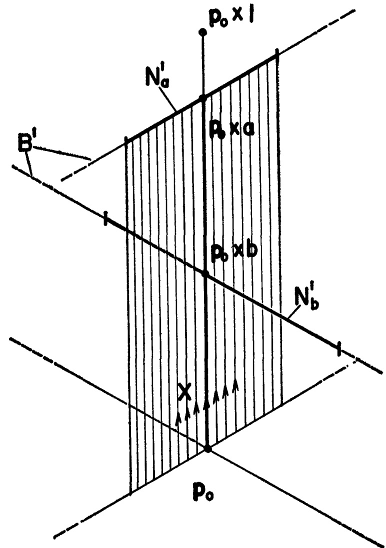

FIG.3

we shall add some more handles to the handlebody in such a way that we obtain a Heegaard-handlebody with the desired properties.

We choose a cell-decomposition $\Gamma_{\#}$ of M^3 which is dual to Γ such that the 1-skeleton $G^1_{\#}$ of $\Gamma_{\#}$ is disjoint from T^3_{I} and from the arcs B^1_k. Let $T^3_{\#}$ be a small neighborhood of $G^1_{\#}$ in M^3. Now $M^3 - {}^{\circ}(T^3_{\mathrm{I}} + T^3_{\#})$, denoted by H^3, may be represented as cartesian product ${}^{\cdot}T^3_{\mathrm{I}} \times I^1$, where I^1 means an interval $0 \leqq x \leqq 1$ such that $p \times 0 = p$ for all $p \in {}^{\cdot}T^3_{\mathrm{I}}$ and such that ${}^{\cdot}T^3_{\mathrm{I}} \times 1 = {}^{\cdot}T^3_{\#}$.

We may assume that the product representation of H^3 is chosen such that B^1 "*projects normally into* ${}^{\cdot}T^3_{\mathrm{I}}$", i.e., such that the following holds:

(A) if p is a point in ${}^{\cdot}T^3_{\mathrm{I}}$ then $p \times I^1$ intersects B^1 at most in two points;
(B) if p is a point in ${}^{\cdot}B^1$ then $p \times {}^{\circ}I^1$ is disjoint from B^1;
(C) if $p_0 \times I^1$ $(p_0 \in {}^{\cdot}T^3_{\mathrm{I}})$ intersects B^1 in two points $p_0 \times a$, $p_0 \times b$ (see Fig. 3), where $1 > a > b > 0$, and if N^1_{a}, N^1_{b} are small neighborhoods of $p_0 \times a$ and $p_0 \times b$, respectively, in B^1, then N^1_{a} "*overcrosses*" N^1_{b}, i.e., N^1_{b} pierces the "projection cylinder" of N^1_{a} (which is the union of all those intervals $p \times [0, c]$ with $p \in {}^{\cdot}T^3_{\mathrm{I}}$ and $p \times c \in N^1_{\mathrm{a}}$).

We consider the *projection cylinder* K^2 of B^1, i.e., the union of all those intervals $p \times [0, c]$ with $p \in {}^{\cdot}T^3_{\mathrm{I}}$ and $p \times c \in B^1$ (where c may be zero such that the interval degenerates to a point in ${}^{\cdot}B^1$). Correspondingly we denote by K^2_k the projection cylinder of B^1_k $(k = 1, \cdots, t)$. Let $p_1, \cdots, p_u$ be those points in ${}^{\cdot}T^3_{\mathrm{I}}$ for which $p_l \times I^1$ intersects B^1 in two points, say $p_l \times a_l$, $p_l \times b_l$ with $1 > a_l > b_l > 0$. We call the points $p_l \times a_l$ the *overcrossings points, and* $p_l \times b_l$ the *undercrossing points* of B^1, and the intervals $p_l \times [0, b_l]$ the *double arcs* of the projection cylinder K^2. We may further assume that

(D) $p_1, \cdots, p_u$ do not lie in $V^2_{\mathrm{I}*}$.

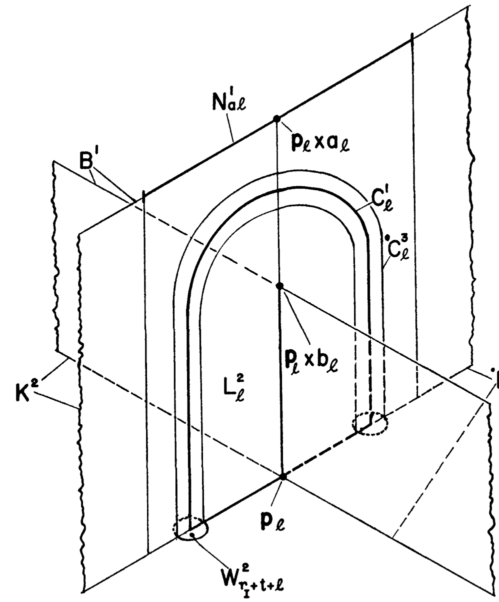

FIG. 4

4. Decomposing the projection cylinder K^2 by arcs C_l^1. We choose pairwise disjoint, small neighborhoods N_{al}^1 of the points $p_l \times a_l$ ($l = 1, \cdots, u$) in B^1 (see Fig. 4); then we choose small neighborhoods L_l^2 of the double arcs $p_l \times [0, b_l]$ in the projection cylinders of the arcs N_{al}^1. Now $^-(^{\cdot}L_l^2 \cap {}^{\circ}K^2)$ is an arc C_l^1 (and $^{\cdot}L_l^2 - {}^{\circ}C_l^1$ is an arc in $^{\cdot}T_{\mathrm{I}}^3$). Moreover, $^-(K^2 - \bigcup_{l=1}^u L_l^2)$ consists of t pairwise disjoint disks J_k^2 ($k = 1, \cdots, t$) where $J_k^2 = {}^-(K_k^2 - \bigcup_{l=1}^u L_l^2)$.

5. Adding handles B_k^3 and C_l^3 to the handlebody T_{I}^3. We choose small, pairwise disjoint neighborhoods B_k^3 ($k = 1, \cdots, t$) of the arcs B_k^1 and C_l^3 ($l = 1, \cdots, u$; see Fig. 4) of the arcs C_l^1 in $M^3 - {}^{\circ}T_{\mathrm{I}}^3$. Then we consider the handlebody $T_{\mathrm{I}}^3 + \bigcup_{k=1}^t B_k^3 + \bigcup_{l=1}^u C_l^3$, denoted by T^3. The genus r of T^3 is $r = r_{\mathrm{I}} + t + u$.

We denote the $r_{\mathrm{I}} + t$ connected components of $^-[V_{\mathrm{I}*}'^2 - \bigcup_{k=1}^t \zeta^{-1}(B_k^3 \cap V_{\mathrm{I}*}^2)]$ (see Fig. 2) by $V_{\mathrm{II}*1}'^2, \cdots, V_{\mathrm{II}*r_{\mathrm{I}}+t}'^2$; their images under ζ, denoted by $V_{\mathrm{II}*1}^2, \cdots, V_{\mathrm{II}*r_{\mathrm{I}}+t}^2$, are nonsingular disks. Further we denote the disks $^-(L_l^2 - C_l^3)$ ($l = 1, \cdots, u$) by $V_{\mathrm{II}*r_{\mathrm{I}}+t+l}^2$. The boundaries $^{\cdot}V_{\mathrm{II}*i}^2$ ($i = 1, \cdots, r$) of the disks $V_{\mathrm{II}*i}^2$ are pairwise disjoint (because of (D) in Sec. 3).

6. Choosing suitable meridian disks in T^3. Now we choose $r_{\mathrm{I}} + t$ pairwise disjoint meridian disks $W_1^2, \cdots, W_{r_{\mathrm{I}}+t}^2$ in T_{I}^3 (compare Fig. 2) such that for all $m = 1, \cdots, r_{\mathrm{I}} + t$

(α) $^{\cdot}W_m^2$ intersects $^{\cdot}V_{\mathrm{II}*m}^2$ in just one piercing point and is disjoint from $^{\cdot}V_{\mathrm{II}*i}^2$ if $i \neq m$, $i = 1, \cdots, r$;

(β) $\dot{W}^2_m$ is disjoint from the $\dot{B}^3_k$'s ($k = 1, \cdots, t$) and from the $\dot{C}^3_l$'s ($l = 1, \cdots, u$) and intersects $\dot{K}^2 \cap \dot{T}^3_{\mathrm{I}}$ at most in isolated piercing points.

Further we denote one of the two connected components of $\dot{C}^3_l \cap \dot{T}^3_{\mathrm{I}}$ by $W^2_{r_{\mathrm{I}}+t+l}$ (for all $l = 1, \cdots, u$; see Fig. 4). Then the disks $W^2_1, \cdots, W^2_r$ form a complete system of meridian disks of T^3, i.e., $\circ T^3 - \bigcup_{i=1}^{r} W^2_i$ is an open 3-cell C^3; moreover, the $\dot{W}^2_i$'s and the $\dot{V}^2_{\mathrm{II}*i}$'s are two "canonical" systems of 1-spheres in $\dot{T}^3$, i.e., we have

$$(*)\qquad \begin{aligned} \dot{W}^2_i \cap \dot{V}^2_{\mathrm{II}*j} &= \text{one piercing point} \quad \text{if } j = i \qquad (i, j = 1, \cdots, r). \\ &= \emptyset \qquad\qquad\qquad\qquad \text{if } j \neq i \end{aligned}$$

7. T^3 is a Heegaard-handlebody. We prove that $M^3 - \circ T^3$ is a handlebody by constructing a complete system of meridian disks in $M^3 - \circ T^3$.

We choose a complete system, $F^2_1, \cdots, F^2_{r_{\mathrm{I}}}$, of meridian disks in the handlebody $M^3 - \circ T^3_{\mathrm{I}}$ such that for all $i = 1, \cdots, r_{\mathrm{I}}$ the following holds:

(1) $F^2_i \cap H^3 = \dot{F}^2_i \times I^1$;
(2) $\dot{F}^2_i$ is disjoint from the arcs $\dot{L}^2_l \cap \dot{T}^3_l$ ($l = 1, \cdots, u$) and from $\dot{B}^1$;
(3) $\dot{F}^2_i$ intersects $\dot{K}^2 \cap \dot{T}^3_{\mathrm{I}}$ and the $\dot{W}^2_j$'s ($j = 1, \cdots, r$) at most in isolated piercing points;
(4) the neighborhoods B^3_k, C^3_l of B^1_k, C^1_l, respectively, are small with respect to F^2_i.

Now

$$M^3 - (T^3 + K^2 + \bigcup_{i=1}^{r_{\mathrm{I}}} F^2_i)$$

is an open 3-cell, since $T^3 + K^2 + \bigcup_{i=1}^{r_{\mathrm{I}}} F^2_i$ collapses to $T^3_{\mathrm{I}} + \bigcup_{i=1}^{r_{\mathrm{I}}} F^2_i$ (definition see [5, p. 201]).

The disks $F^2_1, \cdots, F^2_{r_{\mathrm{I}}}, V^2_{\mathrm{II}*r_{\mathrm{I}}+t+1}, \cdots, V^2_{\mathrm{II}*r_{\mathrm{I}}+t+u}$ are pairwise disjoint and disjoint from the $\circ C^3_l$'s; we denote their union by F^2. Further we denote the disks $J^2_k - \circ T^3$ ($k = 1, \cdots, t$) by E^2_{*k}. Obviously $T^3 + K^2 + \bigcup_{i=1}^{r_{\mathrm{I}}} F^2_i = T^3 + F^2 + \bigcup_{k=1}^{t} E^2_{*k}$.

We remove, step by step, the intersections of F^2 with the E^2_{*k}'s and with the $\circ B^3_k$'s in the following way:

If D^1 is a connected component of $F^2 \cap E^2_{*k}$ (see Fig. 5) then $D^1 = q \times [0, c]$ for some point $q \in \dot{E}^2_{*k} \cap \dot{T}^3_{\mathrm{I}}$ where $q \times c \in \dot{E}^2_{*k} \cap \dot{B}^3_k$. Then we may find a connected component $D^1_1 = q_1 \times [0, c_1]$ of $F^2 \cap E^2_{*k}$ such that a connected component, say Q^2, of $E^2_{*k} - D^1_1$ is disjoint from F^2. Then we choose a small neighborhood Q^3 of $\bar{Q}^2$ in $M^3 - \circ T^3$ (see Fig. 5); $Q^3 \cap F^2$ is a disk D^2, containing D^1_1, such that $^-(\dot{D}^2 - T^3)$ consists of two disjoint arcs $D^1_{\#}$, $D^1_{\#\#}$, "parallel" to D^1_1. Now $\dot{Q}^3 - (\dot{T}^3 + \dot{D}^2)$ consists of three disjoint open disks, such that one of them, denoted by $Q^2_{\#}$, has a boundary which is the union of $D^1_{\#}$ and an open arc in $\dot{T}^3$, and such that a second one, denoted by $Q^2_{\#\#}$, has a boundary which is the union of $D^1_{\#\#}$ and an open arc in $\dot{T}^3$ (see Fig. 5). Finally let R^2 be that connected component of $F^2 \cap B^3_k$ that contains $q_1 \times c_1$. Now we replace F^2 by

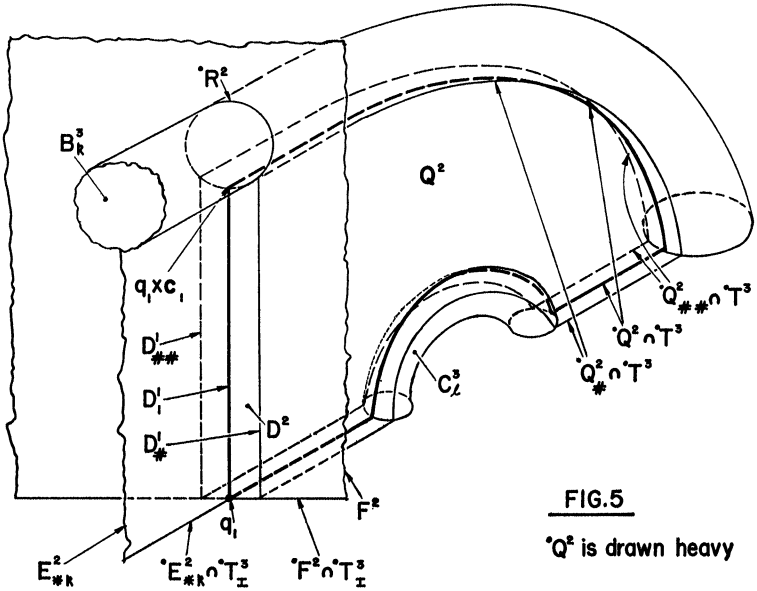

FIG. 5

·Q² is drawn heavy

$$F^2_{(1)} = [F^2 - (D^2 + {}^\circ R^2)] + \bar{Q}^2_{\#} + \bar{Q}^2_{\#\#}\,.$$

Obviously $F^2_{(1)}$ is the union of $r_{\rm I} + u$ pairwise disjoint disks such that $M^3 - (T^3 + F^2_{(1)} + \bigcup_{k=1}^{t} E^2_{*k})$ is an open 3-cell; but the number of intersection arcs in $F^2_{(1)} \cap \bigcup_{k=1}^{t} E^2_{*k}$ is one less than the corresponding number of F^2.

We repeat the procedure described in the above paragraph as often as possible, and by this we obtain a union $F^2_{(*)}$ of $r_{\rm I} + u$ pairwise disjoint disks, denoted by $E^2_{*t+1}, \cdots, E^2_{*r}$, which are disjoint from the disks E^2_{*k} $(k = 1, \cdots, t)$ such that $M^3 - (T^3 + \bigcup_{i=1}^{r} E^2_{*i})$ is an open 3-cell, and $\dot{E}^2_{*i} = E^2_{*i} \cap T^3 = E^2_{*i} \cap \dot{T}^3$. That means that $M^3 - {}^\circ T^3$ is a handlebody and that the E^2_{*i}'s form a complete system of meridian disks of $M^3 - {}^\circ T^3$; moreover, the meridian circles $\dot{E}^2_{*i}$ of $M^3 - {}^\circ T^3$ intersect the meridian circles $\dot{W}^2_i$ of T^3 at most in isolated piercing points.

8. Constructing Γ. We take for Γ a cell-decomposition of M^3, corresponding to the Heegaard-diagram defined by $\dot{T}^3$ and by the $\dot{E}^2_{*i}$'s and the $\dot{W}^2_i$'s:

For the only vertex of Γ we choose a point E^0 in the open 3-cell ${}^\circ T^3 - \bigcup_{i=1}^{r} W^2_i$. For the 1-dimensional elements of Γ we choose open arcs $E^1_1, \cdots, E^1_r$ in ${}^\circ T^3$ such that $\dot{E}^1_i = E^0$,

$$\begin{aligned} E^1_i \cap W^2_j &= \text{one piercing point} \quad \text{if } i = j \qquad (\text{for all } i, j = 1, \cdots, r) \\ &= \emptyset \qquad\qquad\qquad\quad\; \text{if } i \neq j \end{aligned}$$

and T^3 may be regarded as a neighborhood of $\bigcup_{i=1}^{r} \bar{E}^1_i$ in M^3. For the 2-dimensional elements of Γ we choose open disks $E^2_1, \cdots, E^2_r$ in $M^3 - \bigcup_{j=1}^{r} \bar{E}^1_j$

such that $E_i^2 \cap (M^3 - {}^\circ T^3) = E_{*i}^2$ (as constructed in the last section), and such that $E_i^2 \cap {}^\circ T^3$ is an open annulus E_{Ti}^2 with ${}^\cdot E_{Ti}^2 \cap {}^\cdot T^3 = {}^\cdot E_{*i}^2$, ${}^\cdot E_{Ti}^2 \cap {}^\circ T^3 \subset \bigcup_{j=1}^r \bar{E}_j^1$ where E_j^1 lies as often in ${}^\cdot E_{Ti}^2$ as ${}^\cdot E_{*i}^2$ intersects ${}^\cdot W_j^2$ (if ${}^\cdot E_{*i}^2$ does not intersect any ${}^\cdot W_j^2$, then ${}^\cdot E_{Ti}^2 \cap {}^\circ T^3$ is just the vertex E^0). For the only 3-dimensional element of Γ we choose the open 3-cell $M^3 - \bigcup_{i=1}^r \bar{E}_i^2$.

Now Γ fulfills condition (i) of the theorem.

9. Constructing the V_i^2's. It remains to show that the $\bar{E}_i^1$'s bound nonsingular disks V_i^2 in M^3 as demanded.

First we choose annuli $V_{\mathrm{IIT}i}^2$ in T^3 such that ${}^\cdot V_{\mathrm{IIT}i}^2 = {}^\cdot V_{\mathrm{II}*i}^2 + \bar{E}_i^1$ (this is possible because of (*) in Sec. 6); we may choose the $V_{\mathrm{IIT}i}^2$'s such that ${}^\circ V_{\mathrm{IIT}i}^2 \subset {}^\circ T^3$, and $V_{\mathrm{IIT}i}^2 \cap V_{\mathrm{IIT}j}^2 = E^0$ if $j \neq i$ (for all $i, j = 1, \cdots, r$).

Next we deform $V_{\mathrm{II}*i}^2$ isotopically into a disk $V_{\mathrm{III}*i}^2$, in such a way that $V_{\mathrm{II}*i}^2 - {}^\circ T^3$ remains fixed and ${}^-(V_{\mathrm{II}*i}^2 \cap {}^\circ T^3)$ is deformed within T^3, such that ${}^\circ V_{\mathrm{III}*i}^2 \cap V_{\mathrm{IIT}i}^2 = \emptyset$; (this is possible since ${}^-(V_{\mathrm{II}*i}^2 \cap {}^\circ T^3)$ is disjoint from one of the boundary curves, namely ${}^\cdot V_{\mathrm{IIT}i}^2 \cap {}^\cdot T^3 = {}^\cdot V_{\mathrm{II}*i}^2$, of $V_{\mathrm{IIT}i}^2$). We do this deformation for all $i = 1, \cdots, r$ (where it is permissible to introduce new intersections between different $V_{\mathrm{IIT}*i}^2$'s).

Then we denote the nonsingular disks $V_{\mathrm{III}*i}^2 + V_{\mathrm{IIT}i}^2$ by $V_{\mathrm{III}i}^2$ ($i = 1, \cdots, r$). The $V_{\mathrm{III}i}^2$'s fulfill condition (ii) of the theorem.

In order to fulfill condition (iii) of the theorem we normalize the intersections $V_{\mathrm{III}i}^2 \cap V_{\mathrm{III}j}^2$ ($j \neq i$) by a procedure as described in [2, Sec. 6, Steps 1 to 4]. This does not destroy the nonsingularity of the single $V_{\mathrm{III}i}^2$'s, and we obtain in this way the demanded V_i^2's.

This finishes the proof of the theorem.

BIBLIOGRAPHY

1. R. H. BING, *Necessary and sufficient conditions that a 3-manifold be S^3*, Ann. of Math. (2), vol. 68 (1958), pp. 17–37.
2. W. HAKEN, *On homotopy 3-spheres*, Illinois J. Math., vol. 10 (1966), pp. 159–180.
3. ———, *Some results on surfaces in 3-manifolds*, Studies in modern topology, Math. Assoc. Amer. Studies, vol 5.
4. H. SEIFERT AND W. THRELFALL, *Lehrbuch der Topologie*, Leipzig, B. G. Teubner, 1934.
5. E. C. ZEEMAN, *The Poincaré conjecture for $n \geq 5$*, Topology of 3-manifolds and related topics, Englewood Cliffs, Prentice Hall, 1962, pp. 198–204.

INSTITUTE FOR ADVANCED STUDY
PRINCETON, NEW JERSEY
UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS