ALGEBRAICALLY TRIVIAL DECOMPOSITIONS OF HOMOTOPI
3-SPHERES

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Every compact 3-manifold $M^3$ without boundary possesses a cell-decomposition $\Psi$ that contains just one vertex, say 0, (see for instance [3, Sec. 5]). From $\Psi$ we may read by a well-known procedure (see [7, 62]) a "corresponding" presentation

$$\mathcal{P}(\Psi) = \langle \{g_1, \ldots, g_a\}, \{v_1, \ldots, v_b\} \rangle$$

of the fundamental group $\pi_1(M^3)$ where the generators $g_1, \ldots, g_a$ are in 1-1 correspondence with the (oriented) 1-dimensional elements $E_i^1, \ldots, E_i^a$ of $\Psi$ and the relations $v_1, \ldots, v_b$ are in 1-1 correspondence with the 2-dimensional elements $E_i^2, \ldots, E_i^b$ of $\Psi$, i.e., $v_i$ is a word in the $g_i$'s obtained by running once around the boundary of $E_i^2$. In this way the relations $r_i$ are uniquely defined up to cyclic permutations and inversions, i.e., if we denote by $\langle r_i \rangle$ the set of all cyclic permutations of $r_i$ and of $r_i^{-1}$ then the $\langle r_i \rangle$'s are uniquely defined.

In the special case that $M^3$ is a homotopy 3-sphere, $\mathcal{P}(\Psi)$ is a presentation of the trivial group. However, it is—in general—an unsolved problem to recognize whether or not a given presentation $\mathcal{P}(\Psi)$ presents the trivial group; this problem seems to be extremely difficult and it may be unsolvable, since the triviality problem of group theory is unsolvable (see [1], [6]). One might expect that these group theoretic difficulties are also the reason for the difficulties of the Poincaré problem. But the result of this paper shows that this is not so: We shall prove that every homotopy 3-sphere $M^3$ possesses a cell-decomposition $\Psi$ such that the corresponding presentation

$$\mathcal{P}(\Psi) = \langle \{g_1, \ldots, g_a\}, \{v_1, \ldots, v_b\} \rangle$$

is obviously trivial, i.e., such that $\mathcal{P}(\Psi)$ can be transformed by simple cancellation operations (without changing the generators $g_i$ and the number $b$ of relations) into the "standard trivial presentation"

$$\mathcal{O} = \langle \{g_1, \ldots, g_a\}, \{g_1, \ldots, g_a, \ast^{b-a} \} \rangle$$

where $\ast^{b-a}$ means that $\mathcal{O}$ contains $b - a$ times the empty relator (i.e., the relations of $\mathcal{O}$ are $^2 g_1 = 1, \ldots, g_a = 1$, and $b - a$ times the trivial relation $1 = 1$). To make this precise we say that a presentation $\mathcal{P}$" is obtained from

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2 Here the equality sign means that both sides of the equation represent the same group element; but in general, if not stated otherwise, we call two words equal if and only if they read, letter by letter, in the same way.
a presentation $\mathcal{P} = (\{g_1, \cdots, g_a\}, \{r'_1, \cdots, r'_b\})$ by a cancellation operation of Type 1 or 2, respectively, if the following holds:

**Type 1.** (Cancelling a syllable $g_i$.) For some $i, j$ ($i = 1, \cdots, a$; $j = 1, \cdots, b$) there is a word $r''$ such that $g_i^{-1} g_i r'' \in \langle r'_j \rangle$ and $\mathcal{P}''$ is obtained by replacing $r'_j$ by $r''$. (Note that this operation does not in general allow cancelling a syllable $g_i g_i^{-1}$.)

**Type 2.** (Cancelling a syllable which is itself a relator.) For some $j, k$ ($j, k = 1, \cdots, b$; $j \neq k$) there are words $r_k^*$ such that $r_k^* r_k r'' \in \langle r'_j \rangle$, and $\mathcal{P}''$ is obtained by replacing $r'_j$ by $r''$; the length $l$ of $r'_k$ is called the length of the cancellation operation.

We shall prove the following

**Theorem.** If $M^3$ is a homotopy 3-sphere then there exists a cell-decomposition $\Psi$ of $M^3$, containing just one vertex $O$, such that a corresponding presentation

$$\mathcal{P}(\Psi) = (\{g_1, \cdots, g_a\}, \{r_1, \cdots, r_b\})$$

of the fundamental group $\pi_1(M^3)$ with generators $g$ and relators $r$ can be transformed into the standard trivial presentation

$$\Omega = (\{g_1, \cdots, g_a\}, g_1, \cdots, g_a, \ast^{b-a})$$

by means of a finite sequence of cancellation operations of Type 1 and a subsequent finite sequence of cancellation operations of Type 2 with lengths not exceeding 3.

One might call a cell-decomposition $\Psi$ with the above properties "algebraically trivial". I hope that the above theorem will be useful for deriving a proof of the Poincaré conjecture. However, this remains a difficult problem. A reason for the difficulty is the lack of correspondence between Tietze transformations of the group presentation $\mathcal{P}(\Psi)$ and transformations of the cell-decomposition $\Psi$. If a presentation $\Omega$ is derived from $\mathcal{P}(\Psi)$ by a Tietze transformation then we may ask the question: does there exist a cell-decomposition of $M^3$ such that $\Omega$ corresponds to $\Omega$? Let us call the Tietze transformation good if the answer to the question is "yes", and bad if the answer is "no". Unfortunately, most Tietze transformations are bad from this point of view. The only large class of good and simple Tietze transformations I know are the eliminations: If $\mathcal{P}(\Psi) = (\{g_1, \cdots, g_a\}, \{r_1, \cdots, r_b\})$ contains a relator, say $r_l$, such that for some $k, g_k w^{-1} \in \langle r_l \rangle$ where $w$ is a word in the $g_i$'s not containing $g_k$, and $w$ is obtained from $\mathcal{P}(\Psi)$ by deleting $g_k$ and $r_l$ and replacing in all relators $r_j$ ($j \neq l$) the letter $g_k$ by the word $w^{-1}$, then the Tietze transformation $\mathcal{P}(\Psi) \rightarrow \Omega$ is good. Moreover, I would like to remark without proof: If it were possible to restrict the lengths of the cancellation operations in our theorem to 2 instead of 3 then the sequence of cancellation operations could be changed into a sequence of good Tietze transformations. This would mean a proof of the Poincaré conjecture since it is easy
to show that $M^5$ is a 3-sphere if it possesses a cell-decomposition $\Omega$ such that the standard trivial presentation $\partial$ corresponds to $\Omega$.

**Proof of the theorem**

1. **Preliminaries.** Let $M^5$ be a homotopy 3-sphere, i.e., a compact, simply connected 3-manifold without boundary.

   We choose the semilinear standpoint as described in [4, Sec. 3]; i.e., we assume that all point sets, denoted by capital roman letters, are piecewise rectilinear polyhedral point sets in a euclidean space $E^n$ of sufficiently large dimension $n$, etc. We denote the closure, boundary, and interior of a point set $X$ by $\overline{X}$, $\partial X$, $X$, respectively.

2. **The idea of the proof.** First we consider (as in [3], [5]) a cell-decomposition $\Gamma$ of $M^5$ into one vertex $E^0$, $r$ open arcs $E^1_i$, $r$ open disks $E^2_i$, and one open 3-cell, and a singular fan $V^2$ corresponding to $\Gamma$ (i.e., a wedge of $r$ singular disks $V^2_i$ with $V^2_i = E^2_i$, where the $V^2_i$'s may intersect themselves and each other in double arcs; for details see [3, Sec. 5, 6]). Let $T^3$ be a small neighborhood of $\partial V^2$ in $M^5$. Now we consider the "middle parts," $A^3_{i,j}$, of the double arcs $A^1_i$ ($i = 1, \ldots, r$) of $V^2$ that lie outside of $T^3$ (see Fig. 1) and the "middle part" $V^3_*$ of $V^2$ obtained from $V^2$ by removing its boundary $\partial V^2$ and the open annuli that lie in the $V^2_i \cap T^3$'s between $V^2_i$ and $V^2_i \cap T^3$. Since $T^3$ is a Heegaard-handlebody in $M^5$ we can "project" the $A^3_{i,j}$'s into $T^3$ (in the same way we projected the arcs $B$ in [5, Sec. 3]) so that we obtain a projection cylinder $K^3_j$ for each arc $A^3_{i,j}$. Now we "thicken" $V^3_*$ and we obtain by this a 3-dimensional polyhedron $V^3_*$ where $V^3_* + T^3$ is obviously a handlebody with $s$ handles corresponding to the $A^3_{i,j}$'s. Moreover, one can show that $V^3_* + T^3$ is a Heegaard-handlebody in $M^5$, and that those parts, say $K^3_{i,b} (h = 1, \ldots, b)$, of the projection cylinders $K^3_j$ that lie outside of $\partial (V^3_* + T^3)$ contain a complete system of meridian disks of $M^5 - \partial (V^3_* + T^3)$.

   Now one may expect to obtain an especially simple Heegaard-diagram of $M^5$ (and a corresponding cell-decomposition $\Psi$; compare [5, Sec. 8]) from the handlebody $V^3_* + T^3$ and the outer meridian disks $K^2_{i,b}$. It remains to select inner meridian disks $X^3_j$ ($j = 1, \ldots, s$) of $V^3_* + T^3$ in a suitable way. This can be done as indicated in Fig. 1: The polyhedron $X_j$ in Fig. 1 consists of two disks in $T^3$, parallel to the disk $V^2_{j,x} \subset V^3_* \cap T^3$, and one arc outside of $\partial T^3$ joining these disks in $V^3_*$ (encircling $A^3_{i,j}$ and the disk $V^2_{j,x} \subset V^3_* + T^3$). If $V^3_*$ is thickened to $V^3_*$ then the joining arc of $X_j$ may be thickened to a disk which (together with the two disks in $T^3$) yields a meridian disk $X^3_j$ of $V^3_* + T^3$.

   First let us discuss the pleasant case that the arcs $A^3_{i,j}$ are unknotted and unlinked over $\partial T^3$, i.e., that there can be found projection cylinders $K^3_j$ which are nonsingular and pairwise disjoint. In this case we obtain a Heegaard-diagram $^3$ which is so simple that it is fairly easy to show that $M^5$ is a 3-sphere:

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$^3$ Here we admit the case that the number of "outer" meridian circles $\partial K^2_{i,b}$ is greater than the genus $s$ of the Heegaard-surface $\partial (V^3_* + T^3)$.
The "projection arc" $K_j^2 \cap T^3$ of $A^1_{*j}$ (see Fig. 2) intersects the meridian circle $X_j^2$ in a point near the point $A^1_{*j} \cap V^2_{x_j^2}$; it may have further intersections with the circles $V^2_{x_k^2}$, $V^2_{x_{*k}}$ ($k = 1, \cdots, s$) and with $V^2_*$ where the intersections with the $V^2_{x_k^2}$'s and $V^2_{x_{*k}}$'s lie close to intersections of $K_j^2$ with the $X_j^2$'s. (In Fig. 2 it is assumed that the projection arc, from left to right, intersects the circles $V^2_{x_k^2}$, $V^2_{x_{*k}}$, $V^2_*$.) We may easily achieve that no connected component of $V_+^2 \cap K_j^2$ is a closed curve but that all these connected components are open arcs with end points in $(K_j^2 \cap T^3)$. Now, $(K_j^2 - V_+^2)$ contains at least one "inner" component, say $K_{*1}^2$, that borders on just one connected component of $V_+^2 \cap K_j^2$. (In Fig. 2 the components $K_{*1}^2$ and $K_{*2}^2$ are inner ones.) This disk $K_{*1}^2$ corresponds to a relator $r_1$ of the group presentation $\mathfrak{B}$ corresponding to our Heegaard-diagram (the generators $g_k$ corresponding to the oriented meridian circles $X_k^2$) where the

By this we mean that the generator $g_k$ may be represented by an oriented simple closed curve in $(T^3 + V^2_*)$ that pierces $X_k^2$ in just one point, in the positive sense, and is disjoint from $X_1^2$, $\cdots$, $X_{i-1}^2$, $X_{i+1}^2$, $\cdots$, $X_s^2$. Each intersection point in $K_{*1}^2 \cap X_k^2$ corresponds to a letter $g_k^{b_1}$ in the word $r_1$. 

**FIG. I**
FIG. 2  The points on $K_j \cap T^3$ marked by x mean from left to right intersections with $V_{x_p}^2, V_{x_f}^2, V_{x_2}^2, V_x^2, V_{x_m}^2, V_{x_q}^2$.
length of \( r_1 \) is at most 2. Moreover, if \( K_{s1}^2 \neq -(K_j^2 - V_s^3) \), then there is another connected component, say \( K_{s2}^2 \), of \( -(K_j^2 - V_s^3) \) that corresponds to a relator \( r_1 \) which contains \( r_1 \) as a syllable. That means that we can simplify \( \mathcal{B} \) by a cancellation operation of Type 2 whose length is not greater than 2. Then, if

\[
K_{s1}^2 + K_{s2}^2 \neq -(K_j^2 - V_s^3),
\]

we can carry out another cancellation operation of that type, and so on, until we obtain a relator equal to \( g_j \). This can be done for all \( j = 1, \cdots, s \), yielding a standard trivial presentation \( \mathcal{D} \). Now it is not difficult to show that this sequence of cancellation operations can be replaced by a sequence of good Tietze transformations since no cancellation operation is of length greater than 2. (A cancellation operation of length 2 can be replaced by an elimination such that in all relators a certain generator \( g_i \) is replaced by another generator \( g_i^{-1} \), and by certain subsequent operations which can be arranged to be good Tietze transformations; the essential point is that the lengths of the relators do not increase under these eliminations.) Hence there is a cell-decomposition \( \mathcal{D}_0 \) of \( M^3 \) that is obviously a cell-decomposition of a 3-sphere.

Of course, one may try to find \( F \) and \( V \) so that the \( A_i \)’s are unknotted and unlinked over \( T^3 \). This would prove the Poincaré conjecture. But my attempts in this direction failed. (It was possible to achieve the unknottedness but not the unlinkedness.)

Now we are left with the general case, namely that the arcs \( A_{s1}^1 \) may be knotted and linked over \( T^3 \). We may apply a cheap trick: We consider the double arcs, say \( C_1^1, \cdots, C_u^1 \), of the projection cylinder \( K^3 = \bigcup_{j=1}^r K_j^3 \) (compare Fig. 5, Case 2, in Sec. 4) and we add small neighborhoods \( C_1^1, \cdots, C_u^1 \) of them (in \( M^3 - T^3 \)) to \( T \), obtaining an expanded handlebody \( T_a^3 \). Now we have enforced that those pieces, say \( A_{js}^1 \), of the \( A_s^1 \)’s that lie outside of \( T_a^3 \) are unknotted and unlinked over \( T_a^3 \), where we simply take \( K^3 \) for the projection cylinder. Then the connected components of the projection cylinder (i.e., the projection cylinders of the \( A_{js}^1 \)’s into \( T_a^3 \)) yield diagrams very similar to Fig. 2. But the essential difference is that the handlebody \( V_s^3 + T_a^3 \) has more handles than \( V_s^3 + T_a^3 \) (corresponding to the connected components of the \( C_m \)’s in \( V_s^3 \)’s); therefore we need additional meridian disks in \( V_s^3 + T_a^3 \) (which we shall construct in detail in Sec. 7). These additional meridian disks intersect the projection cylinder, with the result that the “inner disks” correspond to relators which may contain “cancellation syllables” \( g_i^{-1} g_i \) and which may remain of length 3 even after the cancellation syllables are deleted. That is the reason why this attempt yields a proof of our theorem but not a proof of the Poincaré conjecture.

3. Projecting the 1-skeleton of a cell-decomposition \( \Delta \) of the singular fan \( V_s^3 \) into the Heegaard-surface \( T^3 \). We consider a cell-decomposition \( \Gamma \) of \( M^3 \) that contains just one vertex \( E^0 \), just \( r \) elements \( E_1^1, \cdots, E_r^1, E_1^3, \cdots, \)
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Er, of each dimension 1 and 2 (r 0), and just one 3-dimensional element $E^3$ (see [3, Sec. 5]). Further we consider a singular fan $V^2$, defined by a map $\xi: V^2 \to M^3$, (as in [3, Sec. 6]), such that the only singularities of $V^2$ are double arcs $A^1_1$, $\cdots$, $A^1_s$ (s \neq 0) with inverse images $A^1_j$, $A^1_j$ (j = 1, $\cdots$, s) as in Fig. 3, and such that $V^2 = \bigcup_{i=1}^r E^3_i$, where $V^2$ consists of disks $V^2_i$ (i = 1, $\cdots$, r) with just one common vertex $E^0$ in their boundaries and with $\xi(V^2_i) = E^3_i$. We choose a small neighborhood $T^3$ of $\bigcup_{i=1}^r E^3_i$ in $M^3$ (as in [5, Sec. 2]) which is a Heegaard-handlebody in $M^3$.

Notation. (See Fig. 3.) We denote the connected components of $\xi^{-1}(V^2 \cap T^3)$ by $V^2_1$, $V^2_2$, $V^2_s$ (j = 1, $\cdots$, s) such that $V^2_1$, $V^2_2$, $V^2_s$ are neighborhoods of $A^1_1$, $A^1_j$, $A^1_s$ respectively, in $V^2$. Further we denote 

\[ -(V^2_1 - V^2_2), -(V^2_2 - V^2_1), -(A^1_j - (V^2_1 + V^2_2)), \]

by $V^2_1$, $V^2_2$, $A^1_j$, $A^1_s$, respectively. We denote the images under $\xi$ by omitting upper primes, i.e., $\xi(V^2_1)$, $\xi(V^2_2)$, $\xi(A^1_j)$, etc., are denoted by $V^3_1$, $V^3_2$, $A^3_j$, etc., respectively. Finally we denote $\bigcup_{i=1}^r A^3_i$ by $A^3$. We choose a coherent orientation $\omega^3$ of $M^3$ and an orientation $\omega^2$ of $V^2$ that is carried over by $\xi$ from a coherent orientation $\omega^3$ of $V^3$; now, if an oriented arc $O^1$ intersects $V^3$ in a piercing point, not in $A^3$, then we call this intersection positive or negative according to whether the corresponding intersection number (see [7, 73]) is positive or negative.

We choose a cell-decomposition $\Gamma^3$ of $M^3$ which is dual to $\Gamma$ (compare [5, Sec. 3]) such that the 1-skeleton $G^1$ of $\Gamma^3$ is disjoint from $T^3 + A^3$, such that the vertex of $\Gamma^3$ does not lie in $V^3$, and such that the 1-dimensional elements of $\Gamma^3$ intersect $V^3$ at most in isolated piercing points. Then we choose a small neighborhood $T^3$ of $G^1$ in $M^3$ and we denote the “handle-shell” $M^3 - \partial (T^3 + T^3)$ by $F^3$ and $V^2 \cap F^3$ by $V^2_\Gamma$.

Now we can project $V^2_\Gamma$ nicely” into the handle-surface $T^3$:

Our main objective is, of course, to nicely project the double arcs $A^1_j$ of $V^2$. But the double arcs of the projection cylinders, corresponding to the overcrossings $^5$ of $A^1_j$, will pierce $V^2_\Gamma$ in points that do not lie in $A^3$. These piercing points will correspond to certain handles of the Heegaard-handlebody we shall construct. Therefore we shall also need arcs in $V^2$ which join the piercing points to points in $V^2$, and we shall have to consider projection cylinders of these arcs; the additional projection cylinders so obtained will contain additional double arcs, yielding additional piercing points with $V^2$, and so on. For this reason it seems convenient to consider a cell-decomposition $\Delta$ of $V^2$ and to demand that all its elements project in a nice way:

**Lemma 1.** $F^3$ can be represented as cartesian product $T^3 \times I^1_F$, where $I^1_F$

\[ ^5 \text{We use the expressions “over”- and “under-crossingpoint”, “projection cylinder”, etc., as in [5].} \]
means a unit interval $0 \leq x_r \leq 1$, such that $p \times 0 = p$ for all $p \in T^3$, and such that there exists a cell-decomposition $\Delta$ of $V^2_\Gamma$ with the following properties:

1. $\Delta$ projects normally into $T^3$, i.e.,
   - (1a) if $N$ is an element of $\Delta$ and if $p \in T^3$ then $p \times I^1_\Gamma$ intersects $N$ in at most one point;
   - (1b) if $N_1, N_2$ are elements of $\Delta$ and if $D$ is the union of all points $p \in T^3$ such that $p \times I^1_\Gamma$ intersects both $N_1$ and $N_2$ then $D$ is a cell or is empty;
   - (1c) if in (1b) the dimensions of $N_1, N_2$ are $d_1, d_2$, respectively, and if $D$ is not empty then the dimension of $D$ is $d_1 + d_2 - 2$;
   - (1d) if $p \in T^3$ then $p \times I^1_\Gamma$ intersects the 1-skeleton of $\Delta$ in at most two points;
   - (1e) if $p \times I^1_\Gamma (p \in T^3)$ intersects two edges $N_1^1, N_2^1$ of $\Delta$ in the points $p \times a_1, p \times a_2$, respectively, ($0 \leq a_2 < a_1 \leq 1$) then $N_1^1$ overcrosses $N_2^1$ (i.e., $N_1^1$ pierces the projection cylinder of $N_2^1$ in $p \times a_2$, see [5, Sec. 3]; if $a_1 = 0$ then $N_2^1$ pierces the projection arc of $N_1^1$ in $T^3$).

2. $\Delta$ is sufficiently fine, i.e.,
   - (2a) if $q$ is a vertex of $\Delta$ that lies in $V^2_\Gamma - A^1_\ast$ then $q$ can be joined to a point $q_0$ in $V^2_* - A^1_\ast$ by an arc $Q^1$ that lies in the 1-skeleton of $\Delta$ so that $0Q^1$ lies in $V^2_\ast - A^1_\ast$;
   - (2b) if $N^1$ is an edge of $\Delta$ then $N^1$ overcrosses $A^1_\ast + V^2_\Gamma$ at most once.

3. $V^2_\Gamma$ is not folded and not twisted along $A^1_\ast$, i.e., there exists a small neighborhood $U^3_\ast$ of $A^1_\ast$ in $F^3$ such that the following holds:
   - if $O^1$ is an oriented interval in $U^3_\ast$, in the $x_\Gamma$-direction (i.e., an arc in $U^3_\ast$ that projects into one point in $T^3$ and that is oriented in the direction of increasing $x_\Gamma$) then $\zeta^{-1}(O^1 \cap V^2_\ast)$ consists of at most two points and all piercings of $O^1$ through $V^2_\ast - A^1_\ast$ are positive.

Proof. I. Let $\mathbb{E}^4$ be a euclidean 4-space and let us denote one of its coordinates by $x_\Gamma$ and the unit interval of the $x_\Gamma$-axis by $I^1_\Gamma$. We choose a (semilinear) homeomorphism $\eta_\ast$ of $T^3$ into the 3-dimensional subspace $x_\Gamma = 0$ of $\mathbb{E}^4$ and we denote $\eta_\ast(T^3)$ by $T^3$. We denote the handle shell $T^3 \times I^1_\Gamma$ by $F^3$ and we associate with any point $q \in F^3$ the coordinates $(p, a)$ so that $p$ is the projection of $q$ into $T^3$ in the $x_\Gamma$-direction and $a$ is the $x$-coordinate of $q$. We can extend $\eta_\ast$ to a (semilinear) homeomorphism $\eta$ of $T^3 + F^3$ onto $T^3 + F^3$. We denote $\eta(V^2_\Gamma)$ by $V^2_1$ and $\eta(A^1_\ast)$ by $A^1_\ast$. We choose a rectilinear triangulation $\Delta_T$ of $T^3$ and a corresponding "prismatical" decomposition $\Delta$ of $F^3$. For the elements of $\Delta$ we take $W \times 0, W \times I^1_\Gamma$, and $W \times 1$ for all $W \in \Lambda_T$.

II. We can transform $V^2_1$ by a "small isotopic deformation" into a polyhedron $V^2_\Pi$ such that the transform $A^1_* \ast$ of $A^1_\ast$ projects normally into $T^3$ and is in "normal position" with respect to $\Lambda$; by this we mean: There exists a self-homeomorphism $\varphi_1$ of $F^3$ which is the identity outside of a small neighborhood of $A^1_\ast$ in $F^3$ such that $\varphi_1(F^3) = F^3$, and such that with the notation $\varphi_1(V^2_\Pi) = V^2_\Pi$, $\varphi_1(A^1_\ast) = A^1_\ast$ the following holds: (II.1) if $p \in T^3$
then $p \times I_p$ intersects $A_1^* \setminus A_1^*$ in at most two points; (II.ii) if $p \in A_1^*$ then $(p \times I_p) \cap A_1^*$ is empty; (II.iii) if $p \not\in I_p$ intersects $A_1^*$ in two points $p \times a_1, p \times a_2 (p \in T^{\infty}, 0 < a_2 < a_1 < 1)$ then $p$ lies in an open triangle of $\Lambda_T$ and there are small neighborhoods $N_1^1, N_2^1$ of $p \times a_1, p \times a_2$, respectively, in $A_1^*$ which are straight line segments such that $N_1^1$ overcrosses $N_2^1$; (II.iv) $A_1^*$ is disjoint from the 1-skeleton of $\Lambda$, and $A_1^*$ intersects the 2-dimensional elements of $\Lambda$ at most in isolated piercing points.

III. We choose a small neighborhood $U_3^3$ of $A_1^*$ in $F^{\infty}$, and we can find a small neighborhood $U_3$ of $A_1^*$ in $U_3^3$ such that (see Fig. 4) the following holds:

(0) if $0$ is an interval in $U_3^3$, in the $x_p$-direction, then $0 \cap U_3^3$ is connected (or empty) and $0 \setminus U_3$ consists of at most two points.

To obtain $U_3$ we may choose a rectilinear triangulation $\Sigma$ of $A_1^*$ which contains all intersection points of $A_1^*$ with 2-elements of $\Lambda$ as vertices, but
does not contain any over- or under-crossing point of \( A^{*1} \) as a vertex. Then we can find a rectilinear sub-triangulation \( \Delta^{*}_T \) of \( \Delta_T \) and a corresponding prismatical subdivision \( \Delta^{*} \) of \( \Delta \) such that every vertex of \( Z \) lies in a 2-element of \( \Delta^{*} \) and such that \( \Delta^{*}_T, A^{*1} \) have the properties (II.iii) and (II.iv). We denote the open, rectilinear intervals in which \( \Delta^{*1} \) intersects the 3-elements of \( \Delta^{*} \) by \( D^1_1, \ldots, D^1_w \) such that the \( D^1_i \)'s lie in the order of the enumeration in \( A^{*1} \), and we assume that the neighborhood \( U^{*}_a \) is small with respect to \( \Delta^{*} \); further we denote the 3-element of \( \Delta^{*} \) that contains \( D^1_i \) by \( P^i_a \). Now we can find small, cylindrical neighborhoods \( U^a_i, \ldots, U^a_w \) of \( D^1_1, \ldots, D^1_w \), respectively, in \( U^*_a \cap \bar{P}^a_1, \ldots, U^*_a \cap \bar{P}^a_w \), respectively, that have the property (0) such that \( U^a_i \cap U^a_j \) is either empty (if \( D^1_i \) and \( D^1_j \) are "end pieces" of connected components of \( A^{*1} \)) or is a connected component of \( U^*_a \cap \bar{P}^a_i \) and also a connected component of \( U^*_a \cap \bar{P}^a_j \) (and such that \( U^*_a \cap U^a_i \) is empty whenever \( |i - j| > 1 \)). Then \( U^a = U^a_{i=1} U^a_{j=1} \) has the demanded properties. Now we can "isotopically smooth out" \( V^{*}_T \) in the neighborhood \( U^a \) of \( A^{*1} \) and "wind it about \( A^{*1} \) so that it is pierced by the intervals \( o^1 \) in the demanded way. By this we mean: We can find a self-homeomorphism \( \sigma_{II} \) of \( F^{*3} \) with \( \sigma_{II}(U^a_i) = U^a_i \) and \( \sigma_{II}(P^a_i) = P^a_i \) (for all \( i = 1, \ldots, w \)) which is the identity on \( \sigma_{II}(F^{*3} - U^a) \) and on \( A^{*1} \) such that the image \( V^{**}_T \) of \( V^{*}_T \) under \( \sigma_{II} \) has the following properties: (III.i) if \( U^a_i \) is a connected component of \( U^a \) then \( V^{**}_T \cap U^a_i \) consists of two disks \( D^a_i, D^a_j \), piercing each other in \( A^{*1} \cap U^a_j \), such that every interval \( O^a_j \subset U^a_j \) in \( x^a \)-direction intersects each disk \( D^a_i \), \( D^a_j \) in at most one point; (III.ii) if an interval \( O^a_j \) in \( x^a \)-direction pierces \( D^a_i \) or \( D^a_j \) then the intersection number is positive when \( O^a_j \) is oriented in the direction of increasing \( x^a \) and \( D^a_i, D^a_j \) in \( F^{*3} \) are oriented according to \( \omega^a, \omega^a \), respectively, carried over by \( \sigma_{II} \sigma_{II} \); (III.iii) there exists a rectilinear triangulation \( \Delta^*_0 \) of \( V^{**}_T \) such that no vertex of \( \Delta^*_0 \) is an over- or under-crossing point of \( A^{*1} \).

To obtain \( \sigma_{II} \) we first deform \( V^{**}_T \cap U^a_i \) in a suitable way, i.e. we can find a self-homeomorphism \( \sigma_1 \) of \( F^{*3} \) with \( \sigma_1(P^a_1) = P^a_1 \) which is the identity outside of a small neighborhood of \( U^a_1 \) in \( F^{*3} \) and on \( A^{*1} \) such that the conditions (III.i, ii, iii) hold with \( U^a \) replaced by \( U^a_1 \) and \( V^{**}_T \) replaced by \( \sigma_1(V^{**}_T) \). Then we can find, step by step, self-homeomorphisms \( \sigma_2, \ldots, \sigma_w \) of \( F^{*3} \) with \( \sigma_i(P^a_i) = P^a_i \) such that \( \sigma_1 \) is the identity on \( A^{*1} \), on \( U^a_{i-1} \), and outside of a small neighborhood of \( U^a_i \), and such that (III.i, ii, iii) hold with \( U^a \) replaced by \( U^a_1 + \cdots + U^a_i \) and \( V^{**}_T \) replaced by \( \sigma_i, \sigma_{i-1} \cdots \sigma_1(V^{**}_T) \). Then we may take \( \sigma_w \cdots \sigma_1 \) for \( \sigma_{II} \).

\( V^{**}_T \) and \( \Delta^*_0 \) fulfill the conditions corresponding to (3) and (1a), (1b). Moreover, each connected component of \( V^{**}_T - A^{*1} \) contains in its boundary arcs of \( \sigma_{II} \sigma_{II} \eta(\bar{V}^*_a - \bar{A}^*_a) \). So if \( \Delta_{III} \) is a regular subdivision of \( \Delta^*_0 \) (obtained by starring each edge and each triangle of \( \Delta^*_0 \)) then each vertex \( q \) of \( \Delta_{III} \) can be joined to a vertex in \( \sigma_{II} \sigma_{II} \eta(\bar{V}^*_a - \bar{A}^*_a) \) by an edge path in the 1-skeleton of \( \Delta_{III} \) whose interior lies in \( \bar{V}^{**}_a - \bar{A}^{*1} \), i.e., the condition corresponding to (2a) is fulfilled by \( V^{**}_T \) and \( \Delta_{III} \). We choose \( \Delta_{III} \) so that it fulfills condition
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(III.iii) (i.e., none of the starring points is an over- or under-crossing point of \( A^* \)).

IV. Now we can deform \( V_{III}^2 \) and \( \Delta_{III} \) by a small isotopic deformation, leaving \( A^* \) pointwise fixed, into a polyhedron \( V^{*2} \) and a triangulation \( \Delta_{IV} \) of \( V^{*2} \), respectively, such that the rectilinearity of the triangulation may be destroyed, but (IV.i) the conditions (1a), (1b), (3) are preserved, and (IV.ii) the conditions corresponding to (1c, d, e) are also fulfilled. We denote the corresponding self-homeomorphism of \( F^{*3} \) by \( \partial_{III} \).

V. We can subdivide the edges of \( \Delta_{IV} \) by new vertices in such a way that the condition corresponding to (2b) is fulfilled and such that all the other conditions are preserved. We call the cell-decomposition so obtained \( \Delta^* \).

VI. Now we carry over the product representation of \( F^{*3} \) and the decomposition \( \Delta^* \) from \( F^{*3} \) to \( F^3 \) by means of the homeomorphism \(( F)| F^{*3} \rightarrow F^3\), denoted by \( \kappa \). In other words, we associate with each point \( q \in F^3 \) the coordinates \((\kappa(q), a)\) where \((q, a)\) are the coordinates of \( K(q) \) in \( F^{*3} \); by this we define the product representation \( F^3 = T^3 \times I^3 \) of \( F^3 \); further we denote by \( \Delta \) the cell-decomposition of \( V_{F^3}^3 = \kappa(V^{*2}) \) whose elements are the images under \( \kappa \) of the elements of \( \Delta^* \). Then all conditions of Lemma 1 are fulfilled (where we choose for \( U_{\Delta} \) a small neighborhood of \( A^* \) in \( \kappa(U^3) \)). This proves Lemma 1.

4. Expanding the handlebody \( T^3 \) into \( T^*_3 \). By \( \Delta( V_{F^3}^2 ) \) we mean the set of all points that are either vertices of \( \Delta( V_{F^3}^2 ) \) or undercrossing points of the 1-skeleton of \( \Delta( V_{F^3}^2 ) \). Let \( C^1_1, \ldots, C^3_u \) be the projection intervals of \( p_1, \ldots, p_u \), respectively (i.e., the arcs in \( F^3 \), in \( x_F \)-direction, joining the \( p_i \)'s to points in \( F^3 \)). Then we choose small, pairwise disjoint neighborhoods \( C^1_1, \ldots, C^3_u \) of \( C^1_1, \ldots, C^3_u \), respectively, in \( F^3 \) (see Fig. 5 which shows the two most complicated cases) and we denote the handlebody \( T^3 + U_{\Delta} C^3_1 \) by \( T^*_3 \). We choose the \( C^3_i \)'s so that each interval \( p \times I^3 \) with \( p \in C^3_i \cap T^3 \) intersects \( ( C^3_i \cap T^3 ) \) in just one point (which is a piercing point if \( p \in ( C^3_i \cap T^3 ) \)).

Notation. (See Fig. 6.) We choose an orientation \( \omega_\Delta \) of \( A^* \) such that the arc \( A^*_{j} \) \((j = 1, \ldots, s)\) is oriented from its boundary point in \( V_{X,F}^2 \) to its boundary point in \( V_{X,F}^2 \). We arrange the enumeration of the points \( p_1, \ldots, p_u \) so that \( p_1, \ldots, p_u \) lie in \( A^* \) and that \( p_{u_A + 1}, \ldots, p_u \) do not lie in \( A^* \); moreover, if a point \( p \) runs through \( A^*_{1}, \ldots, A^*_{s} \) in the order of the enumeration and in the direction of \( \omega_\Delta \) then we assume that \( p \) meets the points \( p_1, \ldots, p_u \) in the order of the enumeration. For convenience we denote the points \( p_1, \ldots, p_u \) also by \( p_1, \ldots, p_{u_A} \).
FIG. 6. The arrows correspond to $\omega_A$; the arcs and intervals map into the 1-skeleton of $\Delta$; the points $\circ$ map into $\bigcup_{k=1}^u C_k \cap V_\ast^2$; $\zeta^{-1}\left(\bigcup_{k=1}^u C_k \cap V_\ast^2\right)$ is indicated by hatching, $\zeta^{-1}(T_\ast^3 \cap V_\ast^2)$ by double hatching.
so that $p_{j_1}, \ldots, p_{j_t}$ lie in $A_{j_t}$ in the order of the second index, between $\overline{V_{j_t}^2}$ and $\overline{V_{j_{t-1}}^2} (u_A = \sum_{j=1}^t t_j)$. We denote the points in $\bigcup_{i=1}^n C_i \cap V_F^2$ by $p_{u_A+1}, \ldots, p_u$. The inverse images $\xi^{-1}(p_{j_i})$ of $p_{j_i}$ $(j = 1, \ldots, s; k = 1, \ldots, t_j)$ are denoted by $p_{j_i}^*$ and $p_{j_i}^*$ so that $p_{j_i}^* \in A_{j_i}^*$ and $p_{j_i}^* \in A_{j_i}$; the point $\xi^{-1}(p_m) (m = u_A + 1, \ldots, v)$ is denoted by $p_m$. Further we denote that connected component of $\xi^{-1}(\bigcup_{i=1}^n C_i \cap V_F^2)$ that is a neighborhood of $p_{j_i}, p_{j_i}^*, p_m$, respectively $(j = 1, \ldots, s; k = 1, \ldots, t_j; m = u_A + 1, \ldots, v)$ by $V_{j_i}^{2*, j_i}, V_{j_i}^{2*}, V_{j_i}^{2}, V_{j_i}^{2}, V_{j_i}^{2}, V_{j_i}^{2}, V_{j_i}$, respectively. Finally we denote the connected components of $A_{j_i}^* \setminus \bigcup_{i=1}^n C_i^0$ by $A_{j_i}^1, \ldots, A_{j_i}^{n+1}$ so that the $A_{j_i}^k$'s lie in $A_{j_i}$ in the order of the index $k$ (in the sense of the orientation $\omega_A$); and we denote the connected components of $\xi^{-1}(A_{j_i}^k)$ by $A_{j_i}^1$ and $A_{j_i}^{n+1}$ so that $A_{j_i}^1 \subseteq A_{j_i}^k$ and $A_{j_i}^{n+1} \subseteq A_{j_i}^k$.

5. Trees in $V_F^2$. The intersections $V_{j_i}^{2*} (l = u_A + 1, \ldots, v)$ of $\bigcup_{i=1}^n C_i^0$ with $V_F^2$ correspond to certain handles of the handlebody $H^3$ composed of $T_s^3$ and a polyhedron $V_F^*$ obtained from $V_F^0$ by thickening. We shall need disks in $M^3$ with boundaries in $H^3$ that correspond to these handles in the following way: The boundary of the first disk, say $K_{u_A+1}^0$, runs just once over the handle corresponding to $V_{j_i}^{2*} (l = u_A+1)$ (under proper notation) and over no other handles that correspond to $V_{j_i}^{2*}$'s. The boundary of the $(m - u_A)^{th}$ disk $(u_A < m \leq v)$, $K_m^0$, runs just once over the handle corresponding to $V_{j_i}^{2*}$ but not over handles that correspond to $V_{j_i}^{2*}$'s with $l > m$. We can find such disks $K_m^0$ in a convenient way in the projection cylinder of some polyhedron $J^1$ (see Fig. 7) in the 1-skeleton of $\Delta$ that contains all the points $p_{u_A+1}, \ldots, p_u$ and that consists of trees each of which contains just one point in $\overline{V_F^2} - \overline{A_*}$.

**LEMMA 2.** In the 1-skeleton of $\Delta$ there exists a 1-dimensional polyhedron $J^1$ with the following properties:

(i) every connected component of $J^1$ is a tree (i.e., simply connected) that contains just one point in $\overline{V_F^2} - \overline{A_*}$, the so-called base point, and otherwise lies in $\overline{V_F^2} - \overline{A_*}$;

(ii) $J^1$ contains all the points $p_{u_A+1}, \ldots, p_u$;

(iii) if $p$ is an end point of $J^1$ (i.e., a point in $J^1 \cap \overline{V_F^2}$ from which just one edge of $J^1$ originates) then $p$ is one of the points $p_{u_A+1}, \ldots, p_u$.

**Proof.** Let $p_l$ be an arbitrary point with $u_A < l \leq u$. Then, because of property (2a) of $\Delta$ (in Lemma 1), there is an arc, say $Q_l^1$, that lies in the 1-skeleton of $\Delta$ so that $Q_l^1 \subset \overline{V_F^2} - \overline{A_*}$, and $Q_l^1 = p_l + q_l$ where $q_l$ is a point in $\overline{V_F^2} - \overline{A_*}$.

Now we consider the following sequence $J_{h+1}^1, \ldots, J_{u-u_A}^1$ of 1-dimensional polyhedra: $J_1^1 = Q_{u_A+1}^1$. If $p_{u_A+h+1} \in J^1_h (1 \leq h < u - u_A)$ then we take $J_{h+1}^1 = J_h^1$. If $p_{u_A+h+1} \not\in J_h^1$ then $Q_{u_A+h+1}^1$ contains an arc, say $Q_{u_A+h+1}^1$, such that $Q_{u_A+h+1}^1 = p_{u_A+h+1} + q_{u_A+h+1} - q_{u_A+h+1}$ where $q_{u_A+h+1} \in J_{h+1}^1$ or $q_{u_A+h+1} = q_{u_A+h+1}$ (in which case $Q_{u_A+h+1}^1 = Q_{u_A+h+1}^1$) and such that $Q_{u_A+h+1}^1 \cap J_{h+1}^1 = \emptyset$; then we take $J_{h+1}^1 = J_{h+1}^1 + Q_{u_A+h+1}^1$. 

Each $J^1(u)$ has properties (i) and (iii); the last element, $J^1(u-u_A)$, has all three properties demanded for $J^1$ which proves Lemma 2.

*Notation.* We denote the projection cylinder of $J^1 + A^1_\#$ by $K^2$. (We assume that the neighborhoods $C^4_i$ of Sec. 4 are small also with respect to $K^2$.)

**FIG. 7** The heavy segments mean $\xi^{-1}(J^1 + A^1_\#)$; the dotted segments mean $\xi^{-1}(\{V^e_x \cap K^e\})$; the heavy dotted segments mean $\xi^{-1}(J^1_\#)$. 
LEMMA 2. In \( (V^3_F \cap \mathcal{K}^3) \) there exists a 1-dimensional polyhedron \( J^1 \) (see Fig. 7) with the following properties:

(i) every connected component of \( J^1 \) is a tree that contains just one point in the 1-skeleton of \( \Delta \), the "base point" (it follows that a base point of \( J^1 \) is either one of the points \( p_1, \ldots, p_u \) or lies in \( V^3_F \cap T^3 \) - \( A^1 \));

(ii) \( J^1 \) contains all the points \( p_{u+1}, \ldots, p_v \);

(iii) if \( p \) is an end point of \( J^1 \) \((p \in J^1; \) \( p \) is not a base point) then \( p \) is one of the points \( p_{u+1}, \ldots, p_v \).

Proof. Let \( p_m \) be an arbitrary point with \( u < m \leq v \). Then \( p_m \) lies in a 2-dimensional element, say \( N^2 \), of \( \Delta \) (see Fig. 8); moreover, \( p_m \) lies under a point \( p_{\mu(m)} \) with \( \mu < m \leq \mu(m) \leq u \). Now let \( J^1 \) be that connected component of \( J^1 \) that contains \( p_{\mu(m)} \) and let \( q_1 \) be its base point. Then \( q_1 \) does not lie over or in \( N^2 \), and hence the projection cylinder \( K^1 \) of \( J^1 \) intersects \( N^2 \) in a 1-dimensional polyhedron that contains an arc, say \( Q^1 \), so that \( Q^1 \subset N^2 \) and \( Q^1 = p_m + q_1 \) where \( q_1 \) is a point in \( N^2 \) (see Fig. 8). Now we may continue as in the second paragraph of the proof of Lemma 2 (replacing \( l \) by \( m \), \( J^1_{(b)} \) by \( J^1_{(h)} \), \( u \) by \( v \), and \( u_A \) by \( u \)). This proves Lemma 2.

Notation. We arrange the enumeration of the points \( p_{u_A+1}, \ldots, p_v \) so that for each \( m = u_A + 1, \ldots, v \), \( J^1 \) contains an arc, denoted by \( J^1 \), that joins \( p_m \) either to a point in \( (V^3_F \cap T^3 - A^1) \) or to a point \( p_{\lambda(m)} \) with \( u_A < \lambda(m) < m \) so that \( J^1 \) does not contain any point \( p_l \) \((l = 1, \ldots, v)\).

6. A prismatical neighborhood \( V^3_f \) of \( V^3_f \). We "thicken" \( V^3_f \). First we choose a "prismatic neighborhood" \( V^3_f \) of \( V^3_f \), i.e., a polyhedron containing \( V^3_f \) (and consisting of \( r \) pairwise disjoint 3-cells, disjoint from \( M^3 \)) that can be represented as cartesian product \( V^3_f \times I^1 \), where \( I^1 \) means an interval \(-1 \leq x \leq 1\), with \( p' \times 0 = p' \) for all \( p' \in V^3_f \). Then we extend the map \( \xi \) on \( V^3_f \) to a map \( \xi: V^3_f \to M^3 \) such that the following holds:

Notation. \( V^3_f \) means \( V^3_f \times I^1 \); \( V^3_f \) \( \xi(V^3_f) \), \( \xi(V^3_f) \) respectively.

(1) \( V^3_f \) is a small neighborhood of \( V^3_f \) in \( M^3 \) in \( *T^3 \).

Notation. (See Fig. 9.) Let \( K^1 \) be the projection of \( A^1 \) \((j = 1, \ldots, s; k = 1, \ldots, t_j + 1)\) or of \( J^1 \) \((m = u_A + 1, \ldots, v)\), and let \( K^2 \) be that connected component of \( \neg (K^1 \cup \bigcup_{i=1}^{s} C^1_i) \cap V^3_f \) that contains \( A^1 \) or \( \neg (J^1 \cup \bigcup_{i=1}^{s} C^1_i) \), respectively; then we denote \( \neg (K^1 \cup \bigcup_{i=1}^{s} C^1_i) \) by \( K^2 \) and \( K^2 \) respectively. Further we denote \( \neg (K^1 \cup \bigcup_{i=1}^{s} C^1_i) \) by \( K^2 \) \( K^2 \) \( K^2 \), \( K^2 \) \( K^2 \) \( K^2 \) \( K^2 \) \( K^2 \), respectively, and \( \neg (K^1 \cup \bigcup_{i=1}^{s} C^1_i) \) by \( K^2 \) \( K^2 \) \( K^2 \) \( K^2 \) \( K^2 \), respectively.

6 It is essential to remark that \( K^2 \) is a neighborhood of \( A^1 \) or \( J^1 \) \((m = u_A + 1, \ldots, v)\), respectively, in \( \neg (K^1 \cup \bigcup_{i=1}^{s} C^1_i) \), and that consequently the \( K^2 \) 's and \( K^2 \) 's are disks. This holds since none of those 2-elements of \( \Delta \) that are incident to \( A^1 \) or \( J^1 \), respectively, intersects \( K^1 \) because of (1a) in Lemma 1.
(2) \( T^g, T^h, F^g, V^2, V^2_T, K^2_{jk} - K^1_{vk}, \) and \( K^2_m - K^1_v \) intersect \( V^*_v \) prismatically with respect to \( x_v, \xi \); i.e.,

\[
(\xi^{-1}(T^g \cap V^*_v) = [\xi^{-1}(T^g \cap V^*_v)] \times I_v
\]

and correspondingly for \( T^h, F^g, \) etc.
FIG. 9 $V_3^*$ is indicated by hatching.
Cross section through a neighborhood of $A^*_e$
(3) The singularities of $V^2_*$ are orthogonal with respect to $x_v$, $\bar{z}$; by this we mean the following (see Fig. 10):

(3.1) The set of all singular points of $V^2_*$ with respect to $\bar{z}$ is a neighborhood, denoted by $A^3_*$, of $A^1_*$ in $M^3 = \mathbb{T}^3$ which is small with respect to $V^2_*$ and intersects $V^2_*$ primitively with respect to $x_v$, $\bar{z}$.

(3.2) Let $A^3_{*j} = A^3_{*j}(j = 1, \ldots, s)$ be connected components of $\bar{z}^{-1}(A^3_*)$ such that $A^3_{*j} \subset A^3_{*j}$ and $A^3_{*j} \subset A^3_{*j}$, then $\bar{z} | A^3_{*j}$, $\bar{z} | A^3_{*j}$, and $\bar{z} | [V^2_* - \bar{z}^{-1}(A^3_*)]$ are homeomorphisms.

(3.3) Let $p$ be an arbitrary point of $A^3_*$ and let $p'$, $p''$ be the two points of $\bar{z}^{-1}(p)$, $p' = p_1 \times a_1$, $p'' = p_2 \times a_2$ ($p_1$, $p_2 \in V^2_*; a_1, a_2 \in [-1, 1]$); denote $\bar{z}(p_1)$, $\bar{z}(p_2)$ by $p_1$, $p_2$, respectively; now let $p'_1$, $p'_2$ be that point of $\bar{z}^{-1}(p_1)$, $\bar{z}^{-1}(p_2)$, respectively, that is different from $p_1$, $p_2$, respectively. Then there is a point $q \in A^1_*$ with $\bar{z}^{-1}(q) = q' + q''$ such that $p'_1 = q' \times a_2$ and $p'_2 = q' \times a_1$.

(4) If $p' \in V^2_*$ such that $\bar{z}(p) \in A^1_*$, and if (see Fig. 11) the interval $\bar{z}(p' \times I^1_v)$ is oriented according to increasing $x_v$, then the intersection of $\bar{z}(p' \times I^1_v)$ and $V^2_*$ is positive (with respect to the orientations $\omega_v$ and $\omega_M$ introduced in Sec. 3).

(5) (See Fig. 11.) Let $A^3_{jk} (j = 1, \ldots, s; k = 1, \ldots, t_j + 1)$ be that connected component of $\bar{z}^{-1}(A^3_* - \bigcup_{i=1}^n C_i)$ that contains $A^3_{jk}$; then $K^1_{Vj}$ lies in $\bar{z}^{-1}(A^3_{jk})$ so that

FIG. II (Cross section)
\[ \xi^{-1}(K_{V_J}) \subset V^*_{\gamma} \times -1. \]

Further
\[ \xi^{-1}(K_{v_m}) \subset \xi^{-1}(J^1_m) \times \pm 1 \]
(for all \( m = u_\lambda + 1, \ldots, v \)).

Notation. We denote the connected components of \( \xi^{-1}(\langle V^a \rangle) \) by \( V_{x_1}^a, V_{x_2}^a, V_{y_1}^a, V_{y_2}^a, V_{y_3}^a, V_{x_2}^{a_m} \) so that
\[
V_{x_i}^a = V_{x_i}^a \times I_V, \ldots, V_{x_2}^{a_m} = V_{x_2}^{a_m} \times I_V;
\]
correspondingly we denote \( \xi(V_{x_1}^a), \ldots, \xi(V_{x_2}^{a_m}) \) by \( V_{x_1}^a, \ldots, V_{x_2}^{a_m} \), respectively. Let \( A^3_\lambda, A^3_{x_1}, A^3_{x_2}, A^3_{x_3}, A^3_{y_1}, A^3_{y_2}, A^3_{y_3} \) be as introduced in (3.1), (3.2), (5), respectively, and correspondingly
\[
A^n_{x_i} = \xi^{-1}(A^3_{x_i}) \cap A^n_{x_i}, \quad A^n_{y_i} = \xi^{-1}(A^3_{y_i}) \cap A^n_{y_i};
\]
finally we denote \( A^n_{x_1} \cap V_{x_1}^a, A^n_{y_1} \cap V_{y_1}^a, \) etc., by \( A^n_{x_1}, A^n_{y_1}, \) etc. Finally we denote \( V_{x_1}^a \times +1, V_{x_2}^a \times -1 \) by \( V_{x_1}^a, V_{x_2}^a \), respectively, and correspondingly \( V_{y_1}^a \) and \( V_{y_2}^a \), etc., by \( V_{y_1}^a, V_{y_2}^a, V_{y_3}^a, V_{x_1}^a, \) etc., and \( \xi(V_{x_1}^a), \xi(V_{x_2}^a), \) etc., by \( V_{x_1}^a, V_{x_2}^a, \) etc.

7. Constructing meridian disks \( X^2_1, Y^2_{j_k}, Z^3_\lambda \) in \( H^3 = T^3 + V^*_\gamma \). We denote the handlebody \( T^3 + V^*_\gamma \) by \( H^3 \). For the following construction see Fig. 12.

We choose pairwise disjoint, small neighborhoods \( U^2_{x_1} (j = 1, \ldots, s) \) of \( A^3_{x_1} \cap V_{x_1}^a \). \( U^2_{x_2} \) is an arc, denoted by \( x^2_{V_J} \), with boundary points in \( V_{x_1}^a \). (We denote \( X^1_{V_J} \times I_V, \xi(X^1_{V_J}), \xi(X^1_{V_J} \times I_V) \) by \( X^1_{V_J}, X^1_{V_J}, X^2_{V_J}, \) respectively.)

Then we choose pairwise disjoint disks \( X^2_{x_1}, X^2_{x_2} \) which are topologically parallel to \( V_{x_1}^a, V_{x_2}^a \), respectively, in \( T^3 \), such that
\[
(\cdot X^2_{x_1} + \cdot X^2_{x_2}) \cap V^*_\gamma = \cdot X^2_{V_J} \cap \cdot T^3
\]
and such that the parallelism is with respect to \( V^2, V^*_\gamma, \langle K^2, T^3 \rangle \) (as defined in [4, Sec. 3]).

Now we denote the disks \( X^2_{y_1} \times X^2_{y_2} \times X^2_{y_3} \) by \( X^2_{y_1} \).

Each \( \cdot (V^2_{y_1} - A^3_{y_1}) \) and \( \cdot (V^2_{y_2} - A^3_{y_2}) \) consists of two connected components \( V^2_{y_1}, V^2_{y_2} \), \( V^2_{y_1} \), \( V^2_{y_2} \), \( V^2_{y_3} \), \( V^2_{y_4} \), respectively; we arrange the notation so that \( V^2_{y_1}, V^2_{y_2}, \) \( V^2_{y_3} \) intersect \( V^*_\gamma \) (in one arc each) and that \( \cdot V^2_{y_1} \), \( \cdot V^2_{y_2} \), \( \cdot V^2_{y_3} \) intersect \( V^*_\gamma \) (in one arc each).

We choose pairwise disjoint arcs \( B^1_{y_1}(j_1, k_1), B^1_{y_2}(j_1, k_1), B^1_{y_3}(j_1, k_1), B^1_{y_4}(j_1, k_1), B^1_{y_5}(j_1, k_1), B^1_{y_6}(j_1, k_1) \) that join points in \( \cdot V^2_{y_1}, \cdot V^2_{y_2}, \cdot V^2_{y_3}, \cdot V^2_{y_4}, \cdot V^2_{y_5}, \cdot V^2_{y_6} \), respectively, to points in \( V^*_\gamma \), such that the following holds (Fig. 12 shows the inverse images of the B's marked by upper primes):
(1) \( B_{V,jk}^1 \) and \( B_{V,jk}^2 \) lie in the boundary of a small neighborhood, say \( U_{\lambda,jk}^3 \), of
\[
A_{j,k+1}^3 + V_{j,k+1}^2 + \cdots + A_{j,t_j}^3 + V_{j,t_j}^2 + A_{j,t_j+1}^3
\]
in \( M^3 \) (with \( U_{\lambda,jk}^3 \subset U_{\lambda,jk-1}^3 \) if \( k > 1 \));
(2) \( B_{V',jk}^3 \) lies in the boundary of a small neighborhood, say \( U_{\lambda,jk}^3 \), of
\[
A_{j,k}^3 + V_{j,k}^2 + \cdots + A_{j,k-1}^3 + V_{j,k-1}^2 + A_{j,k}^3
\]
in \( M^3 \) (with \( U_{\lambda,jk}^3 \subset U_{\lambda,jk+1}^3 \) if \( k < t_j \));
(3) \( B_{Z_m}^3 \) is disjoint from \( K^2 \) and from the \( X_s^3 \)'s; \( B_{Z_m}^3 \) is disjoint from \( T^3 \);
\( B_{Z_m}^3 \) intersects \( K^2 \) at most in isolated piercing points.

We denote \( \xi^{-1}(B_{Y'(j,k)}^1 \times I^4) \) by \( B_{Y'(j,k)}^2 \); etc.

We choose pairwise disjoint meridian disks \( T_{Y,j,k}^3, T_{Y',j,k}^3, T_{Y',r,j}^3, T_{Z_m}^3 \) of
\( T^3 \) which are disjoint from \( U_{\lambda}^3 \) such that
(a) \( T_{Y,j,k}^3 \cap V_{Y}^3 = T_{Y',j,k}^3 \cap V_{Y}^3 = T_{Y',r,j}^3 \cap V_{Y}^3 \cap T^3 \), etc.,
(b) \( T_{Y,j,k}^3, T_{Y',j,k}^3, T_{Y',r,j}^3, T_{Z_m}^3 \) are topologically parallel to \( V_{x',r}^2, V_{x',r}^2, V_{x',r}^2, V_{x',r}^2 \),
\( V_{x',r}^2, V_{x',r}^2, V_{x',r}^2, V_{x',r}^2 \), respectively, in \( T^3 \), with respect to \( V^2, K^2, T^3 \);
(c) \( T_{Z_m}^3 \) is disjoint from the \( C_j^3 \)'s (i.e., \( T_{Z_m}^3 \subset (T^3 \cap \cdot T^3) \)) and intersects \( K^2 \cap T^3 \) at most in isolated piercing points;
(d) \( T_{Z_m}^3 \) intersects \( V^2 \) in just one point, different from \( E_0 \), and intersects \( V^2 \) in just one arc which is a piercing arc.

Now we choose pairwise disjoint, small neighborhoods \( U_{Y,j,k}^3 \) and \( U_{Z,m}^3 \) of
\[
V_{Y,j,k}^3 + V_{Y,r,jk}^3 + B_{Y,jk}^3 + B_{Y',j,k}^3 + B_{Y',r,jk}^3 + T_{Y,j,k}^3 + T_{Y',r,jk}^3 + T_{Z,m}^3
\]
and
\[
V_{Z,m}^3 + B_{Z,m}^3 + T_{Z,m}^3
\]
respectively, in \( M^3 \), which intersect \( V_{Y}^3 \) prismatically with respect to \( x_0, \xi \).

Then we denote the disks \( U_{Y,j,k}^3 \cap H^3, U_{Z,m}^3 \cap H^3 \) by \( Y_{j,k}^3, Z_{m}^3 \), respectively.

We have \( X_j^3, Y_{j,k}^3, Z_{m}^3 \subset H^3 \) and \( X_j^3, Y_{j,k}^3, Z_{m}^3 \subset H^3 \); hence the disks \( X_j^3, Y_{j,k}^3, Z_{m}^3 \) are meridian disks of \( H^3 \).

**Thickening the meridian disks.** Let \( X_j^3, Y_{j,k}^3, Z_{m}^3 \) be pairwise disjoint, small neighborhoods of \( X_j^3, Y_{j,k}^3, Z_{m}^3 \), respectively, in \( H^3 \) which intersect \( V_{Y}^3 \) prismatically with respect to \( x_0, \xi \); we can represent them as cartesian products \( X_j^3 \times I^1_z, Y_{j,k}^3 \times I^1_z, Z_{m}^3 \times I^1_z \), respectively, where \( I^1_z \) is an interval \(-1 \leq x_z \leq +1 \), such that the following holds:

(a) \( p \times 0 = p \) for all \( p \in X_j^3, Y_{j,k}^3, Z_{m}^3 \);
(b) the top and bottom disks
\[
X_j^3 \times \pm 1, Y_{j,k}^3 \times \pm 1, Z_{m}^3 \times \pm 1,
\]

\(^7\) This is possible because of the orthogonality condition (3) in Sec. 6.
denoted by $X_{\pm j}$, $Y_{\pm jk}$, $Z_{\pm m}$, respectively, are the connected components of 
\(-\langle X^3_j \cap \cdot H^3 \rangle, -\langle Y^3_{jk} \cap \cdot H^3 \rangle, -\langle Z^3_m \cap \cdot H^3 \rangle\), respectively; (these disks are not indicated in Fig. 12, but the $xz$-direction is indicated by small arrows);

(γ) $\xi^{-1}(X_{\pm j} \cap \cdot V^3_*)$ separates $\xi^{-1}(X_{\pm j} \cap \cdot V^3_*)$ from $V^3_x$ in $V^3_x$,
\[ Y_{\pm jk} \cap \cdot V^3_* \]
separates $Y_{\pm jk} \cap \cdot V^3_*$ from $V^3_y$ in $V^3_y$, $Z_{\pm m} \cap \cdot V^3_*$
separates $Z_{\pm m} \cap \cdot V^3_*$ from $V^3_z$ in $V^3_z$;

(δ) $T^3$, $V^3$, $A^3_*$, $K^2$ intersect $X^3_j$, $Y_{jk}$, $Z^3_m$ prismatically with respect to $x_z$;

(c) the intersections $X^3_j \cap \cdot V^3_*$, $Y^3_{jk} \cap \cdot V^3_*$, $Z^3_m \cap \cdot V^3_*$ are orthogonal with respect to $x_z$, $x_v$, $\xi$, i.e., the following condition is fulfilled which is completely analogous to (3.3) in Sec. 6 (compare Fig. 10):

Let $p$ be an arbitrary point of $X^3_j \cap \cdot V^3_*$, $Y^3_{jk} \cap \cdot V^3_*$, or $Z^3_m \cap \cdot V^3_*$ and let $p''$ be a point in $\xi^{-1}(p)$ where $p = p_1 \times x_z a_1$ and $p'' = p_2 \times x_v a_2$ (we use the symbols $X_z$ and $X_v$ to distinguish the product representation of the $X_j$, $Y_{jk}$, $Z_m$'s from that of $V^3_*$); denote $\xi(p''_1)$ by $p_1$; now, if $p \notin A^3_*$, let $p''_1 = \xi^{-1}(p_1)$, and if $p \in A^3_*$ let $p''_1$ be that point in $\xi^{-1}(p_1)$ that lies in the same connected component of $\xi^{-1}(A^3_*)$ as $p''$. Then there is a point $q$ in $X^3_j \cap \cdot V^3_*$, $Y^3_{jk} \cap \cdot V^3_*$, or $Z^3_m \cap \cdot V^3_*$, respectively, and there is a point $q''$ in $\xi^{-1}(q)$ such that $p''_1 = q'' \times x_v a_2$ and $p_2 = q \times x_z a_1$.

8. $H^3$ is a Heegaard-handlebody in $M^3$. We denote the connected components of $K^2 - \cdot H^3$ by $K^2_{a1}$, $\cdots$, $K^2_{ab}$. Note that these are disks (because of (1) in Lemma 1).

**Lemma 3.** $H^3$ is a Heegaard-handlebody in $M^3$, and more in detail we have:

(a) $\langle H^3 - (U_{j=-1} \cdot X^3_j + U_{j=1} \cdot Y^3_{jk} + U_{m=-a} \cdot Z^3_m) \rangle$ is a 3-cell, say $W^3$, i.e., $H^3$ is a handlebody, the disks $X^3_j$, $Y^3_{jk}$, $Z^3_m$ form a complete system of meridian disks of $H^3$, and the genus $a$ of $H^3$ is equal to

\[ s + \sum_{j=-1}^{j=1} t_j + (v - u_\lambda) = v + s; \]

(b) the connected components of $M^3 - \cdot H^3$ are open 3-cells, i.e., $M^3 - \cdot H^3$ is a handlebody, the disks $K_{a1}$, $\cdots$, $K_{ab}$ contain a complete system of meridian disks of $M^3 - \cdot H^3$, and $b \geq a$.

**Proof of (a).** Let $T^3_\lambda$ be a handlebody of genus $r$, disjoint from $M^3$, such that $T^3_\lambda \cap \cdot V^3_\lambda = \cdot T^3_\lambda \cap \cdot V^3_\lambda = \cdot V^3_\lambda \times I^1_v$, such that $T^3_\lambda + V^3_\lambda$ is a 3-cell, denoted by $H^3$, and such that there is a map $\xi : H^3 \rightarrow H^1_\lambda$ onto $H^3$ with $\xi | V^3_\lambda = \xi$ and with $\xi | T^3_\lambda$ a homeomorphism of $T^3_\lambda$ onto $T^3_\xi$. The connected components of

\[ \langle H^3 - \xi^{-1}(U_{j=-1} \cdot X^3_j + U_{j=1} \cdot Y^3_{jk} + U_{m=-a} \cdot Z^3_m) \rangle \]
are 3-cells; we may denote them by $H^3_0, H^3_{X_{ij}}, H^3_{Y_{jk}}, H^3_{Z_{k}}$ ($j = 1, \ldots, s; k = 1, \ldots, t_j; l = 1, \ldots, t_j + 1; m = u_A + 1, \ldots, v$) such that (compare Fig. 12) $H^3_{X_{ij}} \cap A_{ij}$ is an arc in $A_{ij}$, $H^3_{Y_{jk}}$ contains $V^3_{Y_{jk}}$, $H^3_{Y_{jk}}$ contains $V^3_{Y_{jk}}$, and $H^3_{Z_{k}}$ contains $V^3_{Z_{k}}$. The restrictions

$$\xi \mid H^3_0, \xi \mid \bigcup_{j=1}^{t_j} H^3_{X_{ij}}, \xi \mid \bigcup_{j=1}^{t_j} H^3_{Y_{jk}},$$

$$\xi \mid \bigcup_{j=1}^{t_j} H^3_{A_{ij}} + \bigcup_{m=1}^{u_A+1} H^3_{Z_{k}}$$

are homeomorphisms.

Now $\xi(H^3_{X_{ij}}) + \xi(H^3_{Y_{jk}})$ ($j = 1, \ldots, s; k = 1, \ldots, t_j$) are pairwise disjoint 3-cells, say $H^3_{X_{ij}}$ and $H^3_{Y_{jk}}$, such that

$$\xi(H^3_{X_{ij}}) + \xi(H^3_{Y_{jk}}) + \xi(H^3_{A_{ij}}) = \xi(H^3_{Z_{k}})$$

is a 3-cell, say $H^3_{00}$, where

$$H^3_{00} \cap H^3_{X_{ij}} = V^3_{Y_{jk}} + V^3_{Z_{k}}.$$

Now $H^3_{00} + \bigcup_{j=1}^{t_j} H^3_{X_{ij}}$ is a 3-cell and is equal to

$$-[H^3 - (\bigcup_{j=1}^{t_j} X^3_j + \bigcup_{j=1}^{t_j} Y^3_{jk} + \bigcup_{m=1}^{u_A+1} Z^3_{k})]$$

which proves (a).

**Proof of (b).** First we prove that the first homology group $\mathcal{H}_1(H^3 + K^2)$ is trivial: We denote by $a_{X_{ij}}, a_{Y_{jk}}, a_{Z_{k}}$ ($j = 1, \ldots, s; k = 1, \ldots, t_j; m = u_A + 1, \ldots, v$) those elements of $\mathcal{H}_1(H^3)$ which correspond to piercings of $X^3_j, Y^3_{jk}, Z^3_{k}$, respectively (i.e., $a_{X_{ij}}$ may be represented by an oriented simple closed curve in $H^3$ that intersects $X^3_{ij}$ in just one prismatical arc with induced orientation in the direction of increasing $x^3_{ij}$, and that is disjoint from $X^3_j, Y^3_{jk}, Z^3_{k}$; etc.). The $a$'s form a basis of $\mathcal{H}_1(H^3)$. Let $\alpha$ be the inclusion map $H^3 \subset H^3 + K^2$ and let

$$\alpha_* : \mathcal{H}_1(H^3) \rightarrow \mathcal{H}_1(H^3 + K^2)$$

be induced by $\alpha$, then the $\alpha_*(a)$'s form a basis of $\mathcal{H}_1(H^3 + K^2)$. Now the properly oriented boundary $K^2_{ji}$ of $K_{ji}$ (Sec. 6) belongs to $a_{Y_{ji}}$, further $K^2_{ji}$ belongs to $a_{Y_{jk}} - a_{Y_{jk-1}}$ for all $k = 2, \ldots, t_j$, and finally $K^2_{ji}$ belongs to $a_{X_{ij}} - a_{X_{ij-1}}$ (compare the more detailed discussion of the $K^2_{ji}$'s in Sec. 10.1); hence

$$\alpha_* (a_{Y_{ji}}) = \cdots = a_*(a_{Y_{ji}}) = \alpha_* (a_{X_{ij}}) = 0$$

where $0$ means the zero-element of $\mathcal{H}_1(H^3 + K^2)$. Similarly, $K^2_{m}$ ($m = u_A + 1, \ldots, v$) belongs (compare Fig. 9) either to $a_{Z_{m}} - a_{Z_{m-1}}$ (where $u_A < \lambda(m) < m$, see Sec. 5) or to $a_{Z_{m}} + b$ with $\alpha_*(b) = 0$; hence

$$\alpha_* (a_{Z_{m+1}}) = \cdots = \alpha_* (a_{Z_{v}}) = 0,$$

i.e., $\mathcal{H}_1(H^3 + K^2)$ is trivial, Q.E.D.

Let $K^3, K^3_{s_1}, \ldots, K^3_{s_6}$ be pairwise disjoint, small neighborhoods of $K^3_{s_1}, \ldots,$
K_{s_k}, \text{ respectively, in } M^3 - \partial H^3. \text{ Then } H^3 + \bigcup_{s_1} K_{s_1} \text{ is a 3-manifold with trivial first homology group (since } H^3 + \bigcup_{s_1} K_{s_1} \text{ collapses to } H^3 + K^3, \text{ hence } (H^3 + \bigcup_{s_1} K_{s_1}) \text{ consists of 2-spheres only (see [7, §64]); but these 2-spheres lie in the handlebody } M^3 - \partial H^3, \text{ and therefore, as a consequence of the Alexander theorem [2], bound 3-cells in } M^3 - \partial T^3. \text{ Therefore } M^3 - (H^3 + K^3) \text{ consists of open 3-cells, and hence } M^3 - (H^3 + K^3) \text{ consists of open 3-cells. This proves (b).}

9. Constructing the cell-decomposition } \Psi \text{ of } M^3. \text{ We take for } \Psi \text{ a cell-decomposition of } M^3, \text{ corresponding to the Heegaard-handlebody } H^3 \text{ with the two systems }

\{X_j, Y_{jk}, Z_m\}, \{K_{s_k}\}

\text{of meridian disks (compare [5, Sec. 8]):}

For the only vertex of } \Psi \text{ we choose a point } O \text{ in } W^3. \text{ For the 1-dimensional elements of } \Psi \text{ we choose pairwise disjoint, open arcs } E_{X_j}, E_{Y_{jk}}, E_{Z_m} \text{ in } H^3 \text{ with common boundary } O \text{ such that } H^3 \text{ is a neighborhood of the 1-skeleton } G^1 \text{ of } \Psi, \text{ and such that } E_{X_j} \text{ intersects } X_j^3 \text{ in just one prismatical arc and is disjoint from } X_j^3, \text{ from the } Y_{jk}'s, \text{ and from the } Z_m^3; \text{ etc. For the 2-dimensional elements of } \Psi \text{ we choose pairwise disjoint, open disks } E_{X_j}^2, \cdots, E_{Z_m}^2 \text{ in } M^3 - G^1 \text{ such that }

E_{X_j}^2 \cap (M^3 - \partial H^3) = K_{s_k}, \quad E_{X_j}^2 \subset G^1,

\text{and such that } E_{X_j}^2 \cap \partial H^3 \text{ is an open annulus, say } E_{X_j}^2, \text{ which intersects } X_j^3, Y_{jk}^3, Z_m^3 \text{ prismatically with respect to } x_k \text{ so that } X_j^3, Y_{jk}^3, Z_m^3 \text{ are intersected (at most) in open arcs each of which joins } G^1 \text{ to } H^3. \text{ For the 3-dimensional elements of } \Psi \text{ we take the connected components of } M^3 - \bigcup_{s_1} E_{s_1}^2. \text{ We choose a coherent orientation of } G^1 \text{ so that in } E_{X_j}^1 \cap X_j^3, E_{Y_{jk}}^1 \cap Y_{jk}^3, E_{Z_m}^1 \cap Z_m^3 \text{ the direction of } E_{X_j}^1, E_{Y_{jk}}^1, E_{Z_m}^1, \text{ respectively, coincides with the direction of increasing } x_k \text{; then we associate generators } g_{X_j}, g_{Y_{jk}}, g_{Z_m} \text{ of } \pi_1(M^3) \text{ with the so oriented 1-spheres } E_{X_j}^1, E_{Y_{jk}}^1, E_{Z_m}^1, \text{ respectively, (with base point } O). \text{ Now we may read relators } r_1, \cdots, r_s \text{ from the 2-elements } E_{X_j}^2, \cdots, E_{Z_m}^2, \text{ respectively, and we denote the presentation }

\{(g_{X_j}, g_{Y_{jk}}, g_{Z_m}), \{r_i\}\}

\text{of } \pi_1(M^3) \text{ by } \Psi(\Psi).

10. Relator-diagrams corresponding to the presentation } \Psi(\Psi) \text{ of } \pi_1(M^3). \text{ We map the disks } K_{s_k}^2, K_m^2 (j = 1, \cdots, s; k = 1, \cdots, t_j; m = u + 1, \cdots, v); \text{ see Sec. 5 and Fig. 9) onto pairwise disjoint disks } R_{s_k}^2, R_m^2, \text{ respectively, (see Fig. 13 which corresponds to Fig. 9 if one assumes that } m > u, p_i = p_j, \text{ and } t_j = 2, \text{ compare Fig. 12), by means of maps }

\kappa_{s_k} : K_{s_k}^2 \rightarrow R_{s_k}^2, \quad \kappa_m : K_m^2 \rightarrow R_m^2,
respectively, such that:

(i) the restrictions of $\kappa_{jk}$, $\kappa_m$ to the open disks $K^2_{s_i}$ ($i = 1, \ldots, b$) are homeomorphisms;
(ii) $\kappa_{jk}$, $\kappa_m$ map each connected component of $K^2_{jk} \cap W^3$, $K^2_m \cap W^3$, respectively, into a single point;
(iii) if $L$ is a connected component of the intersection of $K^2_{jk}$ or $K^2_m$ with $X^3_k - \cdot W^3$, $Y^3_{ki} - \cdot W^3$, or $Z^3_q - \cdot W^3$, then $\kappa_{jk}$ or $\kappa_m$, respectively, maps $L$ onto an open arc in such a way that all points with the same $xz$-coordinate have the same image point (but points with different $xz$-coordinates map always into different points).

If $L$ as in (iii) then we orientate the image $\kappa_{jk}(L)$ or $\kappa_m(L)$, respectively, according to the direction of increasing $xz$, and we associate it with the generator $g_{xh}, g_{yh}, g_{za}, \ldots$, respectively.

We consider the cell-decompositions $\Theta_{jk}$, $\Theta_m$ of $R^2_{jk}$, $R^2_m$, respectively, into the connected components of the images of $K^2_{jk} \cap \cdot W^3$, $K^2_{jk} \cap \cdot (X^3_k - W^3)$, etc., etc. From each 2-dimensional element of $\Theta_{jk}$ or $\Theta_m$ we may read the relator $r_i$ that is associated with the inverse image disk $K^2_{s_i}$. We call the decomposition $\Theta_{jk}$ or $\Theta_m$, together with the association of its oriented edges to the generators $g$ and of its 2-elements to the relators $r$ (see Fig. 13) a relator-diagram corresponding to $\Psi(\Psi)$ and we denote it by $\mathcal{R}_{jk}$ or $\mathcal{R}_m$, respectively.

From the boundary of $R^2_{jk}$ or $R^2_m$ we may read a word $r^*_{jk}$ or $r^*_m$, respectively, in the generators $g$ (where all members of the cyclic class $\langle r^*_m \rangle$ or $\langle r^*_m \rangle$, respectively, are equivalent). Now the relator-diagram $\mathcal{R}_{jk}$ shows that $r^*_{jk} = 1$ is a true relation in the group $\pi_n(M^3)$; etc. Diagrams like these have been used by E. R. Van Kampen and other authors; see for instance [8].

For the proof of the theorem we shall need some special properties of our relator-diagrams $\mathcal{R}$:

(10.1) By inspection of the curves $\cdot K^2_j$ and $\cdot K^2_m$ we see that we can write for all $j = 1, \ldots, s$; $m = u_A + 1, \ldots, v$ (compare Fig. 12):

$$
\begin{align*}
\ r^*_{jk} &= g_{yh} e_{jk} \\
\ r^*_{jk} &= g_{yh} g^{-1}_{yj} e_{jk} \quad (\text{for all } k = 2, \ldots, l_j) \\
\ r^*_{jk} &= g_{xj} e_{jt} e_{jt+1} = g_{yx} e_{jt} e_{jt+1} \\
\ r^*_{jk} &= g_{ym} e_{m} \quad \text{if } J^1_m \text{ joins } p_m \text{ to } \cdot V^*_{x^m} \\
\ r^*_{jk} &= g_{ym} g^{-1}_{xj} e_{m} \quad \text{if } J^1_m \text{ joins } p_m \text{ to } \cdot V^*_{x^m} \\
\ r^*_{jk} &= g_{ym} g_{ym} g^{-1}_{xj} e_{m} \quad \text{if } J^1_m \text{ joins } p_m \text{ to } \cdot V^*_{x^m} \\
\ r^*_{jk} &= g_{ym} g_{ym} g^{-1}_{xj} e_{m} \quad \text{if } J^1_m \text{ joins } p_m \text{ to } \cdot V^*_{x^m} \text{ and meets } \cdot V^*_{x^m} \\
\end{align*}
$$

* These maps are not semilinear, but can be taken piecewise algebraic.
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\[ g_{zm} e_m g_{Y_j m k_m} e_m \quad \text{if } J^m \text{ joins } p_m \text{ to } p_{j m k_m} \text{ and meets } V^2_{Y_j m k_m} \]

\[ g_{zm} e_m g_{Z(m)} e_m \quad \text{if } J^m \text{ joins } p_m \text{ to } p_{\lambda(m)} \text{ (case of Fig. 13)}, \]

where the \( e \)'s are words in the \( g \)'s that are either empty or cancel to the empty word by repeated deleting of syllables \( g^{-1} g \) (where the subscript \( \gamma \) stands for \( Xc, Ycd, \) or \( Zc; c = 1, \ldots, s; d = 1, \ldots, t; e = u_A + 1, \ldots, v \)).

More in detail: The \( e \)'s are products of syllables of

**Type a.**

\[ g^{-1}_{x_1} g^{-1}_{x_2} g_{x_A} \cdots g^{-1}_{x_t} g_{x_A} \cdots g^{-1}_{x_1} g_{x_A} g_{x_A} \]

which occur in \( e_{-} \) (\( \sim \) stands for a pair of indices \( jk \) or a single index \( m \)), corresponding to the intersections \( K_{T^-}^1 \cap V^2_{x_{e_1}} \) (compare Fig. 12) and of

**Type b.**

\( g^{-1}_{x_1} g_{x_A} \), but not \( g^{-1}_{x_c} g_{x_c} \) for any \( c = 1, \ldots, s \), which occur in \( e_{-} \), \( e_{\#} \), \( e_{\#} \) corresponding to the intersections \( K_{T^-}^1 \cap T^2_{x_c} \) and \( K_{V^-}^1 \cap B^2_{x_e} \)

\( T^2_{Y_c} B^2_{Y_c} \) stand for

\[ T^2_{Y_c} + T^2_{Y_c} + T^2_{Y_c} + T^2_{Y_c}, \quad B^2_{Y_c} + B^2_{Y_c} + B^2_{Y_c}, \]

respectively.

We always have

\[ e_{j t + 1} = g^{-1}_{y_{j t}} g_{y_{j t}} \cdots g^{-1}_{y_{A}} g_{y_{A}} \]

and

\[ e_{\#} = g^{-1}_{y_{j m}} g_{y_{j m}} \cdots g^{-1}_{y_{j m k_m}} g_{y_{j m k_m}} \cdot \]

The relations \( \tau^2 \) show obviously that \( \pi_1(M^3) \) is the trivial group.

(10.2) It is essential that the decompositions \( \Theta \) are especially simple: The 1-skeleton of \( \Theta \) intersects \( R^2 \) in pairwise disjoint open arcs with boundaries in \( R^2 \) (see Fig. 13); we denote these open arcs (in all the \( R^2 \)'s) by \( Q^1_1, \ldots, Q^1_w \); the \( Q^j_f \)'s \( (f = 1, \ldots, w) \) are the images of the connected components of \( K^2 \cap V^2_{x_e} \) under \( x_{-} \) (where these components are open arcs, say \( P^1_1, \ldots, P^1_w \), with boundaries in \( K^2_{x_{-}} \) such that either

\[ \bar{P}^1_1 = P^1_{j_1} \text{ or } P^1_{j_1} \cap P^1_{j_2} = \emptyset \]

if \( f_1, f_2 \in \{1, \ldots, w\} \). We denote the words corresponding to \( Q^1_1, \ldots, Q^1_w \) by \( q_1, \ldots, q_w \), respectively. In detail, we have the following five types of words \( q_f \) (corresponding to six types of arcs \( P^1_f \); \( f = 1, \ldots, w \)):

- **(Type 1)** \( q_f = g_{x f} e_{Q_f} g^{-1}_{z m} \)

  \[ \text{if } P^1_f \text{ joins } V^2_{z m} \text{ to } V^2_{z m} \text{ (} m \neq 1 \). \]

- **(Type 2)** \( q_f = g_{x f} e_{Q_f} g^{-1}_{z m} \)

  \[ \text{if } P^1_f \text{ joins } V^2_{x f} \text{ to } V^2_{z m} \]


**FIG. 13** $K_m^2 \cap W^3$ is drawn heavy; the arrows indicate $x_z$;

\[ a_1 = g_{yiz} g_{zm_6} (e_{ai} \text{ empty}) \]
\[ a_2 = g_{yiz} g_{vjt} g_{yiz} g_{xj} g_{zm_2} \]
\[ a_3 = g_{yiz} g_{vjt} g_{yjt} g_{xj} g_{zm_5}^{-1} \]
\[ a_4 = g_{yiz} g_{zm_4} \]

\[ r_1 = g_{zm_6} g_{zme} g_{zla(m)} g_{zm_4} g_{yiz}^{-1} g_{yi2} g_{yjt} g_{yi2} g_{yjt} g_{xj} g_{zm_6} \]
\[ r_2 = g_{zm_2} g_{vjt} g_{yjt} g_{yjt}^{-1} g_{yi2} g_{yjt} g_{yi2} g_{zmi} \]
\[ r_3 = g_{yi2} g_{vjt} g_{yjt} g_{xj} g_{zm_2}^{-1} g_{zm_5} g_{zm_3} \]
\[ r_4 = g_{zm_6} g_{xj} g_{yjt} g_{yi2} g_{yi2} g_{yjt} g_{zm_4}^{-1} \]
\[ r_6 = g_{zm_4} g_{yjt} \]
\[ r_m = g_{zm} g_{zm_6} g_{zm_6} g_{zla(m)} g_{zm_3}^{-1} g_{zm_3}^{-1} ]
(Type 3) \( q_f = \epsilon_{Qf} \gamma_{zm}^{-1} \) if \( P_f \) joins \( V_{xj}^2 \) to \( V_{zm}^2 \) (for some \( j = 1, \ldots, s \))

or if \( P_f \) joins \( V_*^2 \) to \( V_{zm}^2 \)

(Type 4) \( q_f = \gamma_{jk} \epsilon_{Qf} \gamma_{zm}^{-1} \) if \( P_f \) joins \( V_{xk}^2 \) to \( V_{zm}^2 \) (in Fig. 13)

(Type 5) \( q_f = \gamma_{jk} \epsilon_{Qf} \gamma_{xj} \epsilon_{Qf} \gamma_{zm}^{-1} \) if \( P_f \) joins \( V_{xk}^2 \) to \( V_{zm}^2 \) (in Fig. 13)

where \( \epsilon_{Qf} \) and \( \epsilon_{Qf}' \) are products of syllables \( g_i^{-1} g_j \) which correspond to the intersections of \( P_f \) with \( B_1 \). We do not have more than these six types of arcs \( P_f \) since it follows from the property (2b) of \( \Delta \) (Lemma 1, Sec. 3) that at least one boundary point of \( P_f \) lies in the boundary of a disk \( V_{zm}^2 \). It is remarkable that the word \( q_f \) cancels down to a word of length either 1 (Type 3), or 2 (Types 1, 2, 4), or 3 (Type 5).

(10.3) If there are two (or three) edges of \( \Theta \) in \( R_2 \) that do not correspond to parts of \( \epsilon_{e_1}, \epsilon_{e_2}^2 \), or \( \epsilon_{e_3}^2 \) in (2b) then (any two of) these edges are not separated by the \( Q_1 \)'s in \( R_2 \). Similarly, if two edges in \( R_2 \) correspond to a syllable \( g_i^{-1} g_j \) of Type b in \( \epsilon_{e_1}, \epsilon_{e_2}^2 \), or \( \epsilon_{e_3}^2 \) then these edges are not separated by the \( Q_1 \)'s in \( R_2 \).

(10.4) Another essential property of the relator-diagrams \( \Psi_{m} \) is the following: if \( i \) is a fixed integer, \( 1 \leq i \leq b \), then the decompositions \( \Theta_{jk}, \Theta_t \) (\( j = 1, \ldots, s; k = 1, \ldots, t_j + 1; l = u_1 + 1, \ldots, u \)) contain all together just one 2-dimensional element that is associated with the relator \( r_i \). However, the decompositions \( \Theta_{u+1}, \ldots, \Theta_s \) may contain some more 2-dimensional elements associated with \( r_i \); but in this case, if \( \Theta_{m} \) (where \( m \) stands for two fixed indices \( j_0k_0 \) or for one fixed index \( l_0 \) with \( u_1 < l_0 \leq u \)) each contain an element associated with \( r_i \), then \( \Theta_{m} \) is isomorphic to a “part” of \( \Theta_m \), i.e., we have \( K_m^2 \subset K_m^2 \) and there exists a homeomorphism

\[
\alpha_m : R_m^2 \rightarrow R_m^2
\]

of \( R_m^2 \) into \( R_m^2 \) such that \( \{ \chi_m = \alpha_m^{-1} \mid K_m^2 \} \), where \( \alpha_m \) carries elements of \( \Theta_{m} \) onto elements of \( \Theta_{m} \), preserving the association of these elements to the \( g_i \)'s and \( r_i \)'s. Moreover, \( \alpha_m \) (\( R_m^2 \)) intersects \( R_m^2 \) in just one of the open arcs \( Q_1^1 \).

11. Conclusion. It remains to show that the presentation \( \Psi(\Psi) \) can be transformed into \( \{ g_1, \{ g_2, \}^{b-a} \} \) by cancellation operations as asserted in the theorem. Guided by the relator-diagrams we first transform \( \Psi(\Psi) \) into a presentation whose relators are derived from the \( r_i \)'s by deleting all the cancellation words \( e, e^2, e^{2*} \). The rest is obvious.

Step i. Removing the cancellation syllables of Type a from the \( r_i \)'s (see Fig.

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9 Of course, there may be other 2-elements associated with relators, say \( r_{i_1}, r_{i_2}, \ldots \), such that \( r_{i_1} = r_{i_2} = \cdots = r_i \) (letter by letter), but then \( i_1, i_2, \cdots, i \) are pairwise distinct.
We transform the relator-diagrams $\mathcal{R}_m$ in the following way:

Let $P_1$ be an open arc in $K^2_n \cap V^2_1$, where $j_0 = 1, \cdots, s$, $k_0 = 1, \cdots, t_0 + 1$, $l_0 = u_A + 1, \cdots, w$ that joins a point, say $p_1$, in $V_{x_1}$ to a point in $V_{x_2}$. Then $\kappa_m(P_1)$ is an arc $Q_1$ in $R_1^m$, corresponding to a word $q_d = \epsilon_{Q_d} g_{z_{d_1}}$ of Type 3 as considered in (10.2). From $\kappa_m(p_1)$ there originate two arcs, say $N_1^+, N_1^-$, in $R_1^m$, that correspond (if oriented towards $\kappa_m(p_1)$) to the same word

$$n = g_{x_{1_1}}^{-1} g_{y_{1_1}} g_{y_{1_2}} \cdots g_{y_{1_t}} g_{x_{1_t}}$$

where $n^{-1}$ is a syllable of Type $a$ in $e_m$ as discussed in (10.1)). Now we replace $R_1^m$ by a disk $R_1^*$ corresponding to an identification of $N_1^+$ to $N_1^-$, i.e., so that there is a map $\beta_m$ of $R_1^m$ onto $R_1^*$ which is one-to-one on $R_1^m - (N_1^+ + N_1^-)$, on $N_1^+$, and on $N_1^-$, which maps $N_1^+$ and $N_1^-$ onto the same arc $N$, and which maps the elements of $\Theta_m$ onto elements of a cell-decomposition $\Theta^*_m$ of $R_1^*$ (where $R_1^*$ is disjoint from $M^*$ and from the $R_1^{i_1}$'s). We replace $\mathcal{R}_m$ by the relator-diagram $\mathcal{R}_m^*$ (consisting of $\Theta_m^*$ and the association of its 1- and 2-elements to the $g$'s and $t$'s as carried over by $\beta_m$). We denote the open arc $[\beta_m(Q_1) + N]$ by $Q_1^*$ and $\beta_m(Q_1)$ by $Q_1^*$ for all $Q_1 \subset R_1^m$ ($f = 1, \cdots, w; f \neq d$); then to $Q_1^*$ there corresponds the word

$$q_d = nq_d = g_{x_{1_1}}^{-1} g_{y_{1_1}} g_{y_{1_2}} \cdots g_{y_{1_t}} g_{x_{1_t}}$$

let us call this “of Type 3”. To $R_1^*$ there corresponds a word $r_1^*$ that is obtained from $r_1^*$ by deleting a syllable $n^{-1}$. We remark that $\mathcal{R}_m^*$ is a relator-diagram corresponding to $\mathcal{R}(\Psi)$ and that $\mathcal{R}_m^*$ has also the properties stated for $\mathcal{R}_m$ in (10.3).

We carry out the above procedure for all those disks $K^2_n$ that contain the open arc $P_1$; (these disks $K^2_n$ lie in $K_m$, and $m > u$). We denote the corresponding maps by $\beta_m : R_1^m \to R_1^*$, and the relator-diagrams and decompositions so obtained by $\mathcal{R}_m^*$ and $\Theta_m^*$, respectively. If $K^2_n$ does not contain $P_1$ then we simply denote the identity map on $R_1^m$ by $\beta_m$, and $\mathcal{R}_m^*$, $\Theta_n^*$, respectively; etc. Now the relator-diagrams $\mathcal{R}_m^*$ have again the property stated in (10.4): We obtain the required homeomorphisms $\alpha_m^*(m = u + 1, \cdots, v)$ by taking

$$\alpha_m^* = \alpha_m \text{ if } K_m \subset K_n^m$$

$$= \beta_m \alpha_m \beta_m^{-1} \text{ if } K_m \subset K_n^m$$

(see Fig. 14), where we assume that $\beta_m$ has been chosen in such a way that if two points $p_1, p_2$ of $\cdot R_1^m$ have the same image point under $\beta_m$, then $\alpha_m^{-1}(p_1)$ and $\alpha_m^{-1}(p_2)$ have the same image point under $\beta_m$.

We carry out the procedure described in the above two paragraphs for all $P_1$'s of the type considered and we obtain in this way relator-diagrams $\mathcal{R}^1$ corresponding to $\mathcal{R}(\Psi)$. (We use the notation $R_1^{1_1}, \Theta_1^1, Q_1^{1_1}, r_1^{1_1}, a_1^1$, etc., for the disks, decompositions, etc., of the $\mathcal{R}_m^*$'s.) The words $r_1^{1_1}$ are still of
the form \( (\cdot) \), but we have the following essential simplification: all the cancellation words \( \epsilon^1, \epsilon^2, \epsilon^{1\#} \) are products of syllables \( g^{-1}g \) of Type \( b \) in such a way that both edges in \( R^{12} \) that correspond to such a syllable lie in the boundary of the same 2-dimensional element of \( \Theta^1 \) (since all syllables of Type \( a \) have been deleted, but (10.3) has been preserved). The \( R^{1\#} \)'s have also the property (10.2), modified by admitting \( q's \) of Type 3*, and the properties (10.3) and (10.4).

**Step ii.** "Outer" cancellations (see Fig. 15). We consider an arc, say \( L^1 \), in \( R^{1\#} \) (\( \sim \) stands again for \( j_{0k} \) or \( l_0 \)) that corresponds to a syllable \( g^{-1}g \) of Type \( b \), \( e_0 \) or \( e \). Because of (10.3) all of \( L^1 \) lies in the boundary of just one 2-dimensional element of \( \Theta^1 \) corresponding to a relator, say \( r_{i_0} \) \((i_0 = 1, \ldots, b)\). So we may cancel the corresponding syllable \( g^{-1}g \) in \( r_{i_0} \) (cancellation operation of Type 1) which yields a new relator, say \( v^1 \) and a new presentation

\[
\Psi^{1\#} = (\{g_{xj}, g_{yj}, g_{zm}\}, \{r_{i}^{1\#}\})
\]

where \( r_{i}^{1\#} = r_i \) if \( i \neq i_0 \).

Now we replace \( R^{1\#} \) by a disk \( R^{1\#2} \) corresponding to shrinking \( L^1 \) to one point, i.e., so that there is a map \( \beta^{1\#} \) of \( R^{1\#2} \) onto \( R^{1\#} \), which\(^\text{10} \) is one-to-one on \( R^{1\#2} - L^1 \), which maps \( L^1 \) into one point, and which maps the elements of \( \Theta^1 \) onto elements of a cell-decomposition \( \Theta^{1\#} \) of \( R^{1\#2} \). Now \( \eta^{1\#} \) (consisting of \( \Theta^{1\#} \) and the association of its 1- and 2-elements to the \( g's \) and \( t^{1\#} \)'s as induced by \( \beta^{1\#} \)) is a relator-diagram corresponding to \( \Psi^{1\#} \) (since by (10.4) \( r_i \)'s occurs just once in \( \Theta^1 \)). In the same way we replace all those relator-diagrams \( \eta^{1\#} \) whose decompositions \( \Theta^1 \) contain a 2-element associated with \( r_{i_0} \) by relator-diagrams \( \eta^{1\#*} \) (defined by maps \( \beta^{1\#} : R^{1\#} \to R^{1\#2} \) that map the arcs \((\alpha_{m}^{1\#})^{-1}(L^1) \) into single points). For the remaining \( \eta^{1\#} \)'s we take \( \beta^{1\#} \) to be the identity on \( R^{1\#} \), and we take \( \eta^{1\#} = \eta^{1\#*} \), etc. Then the \( \eta^{1\#*} \)'s are relator-diagrams corresponding to \( \Psi^{1\#} \) and possess the properties (10.1), (10.2, modified by admitting \( q^{1\#} \)'s of Type 3*), (10.3), and (10.4).

We carry out the above procedure for all arcs of the considered type, and we obtain in this way (by cancellation operations of Type 1) a presentation

\[
\Psi^{1\#} = (\{g_{xj}, g_{yj}, g_{zm}\}, \{r_{i}^{1\#}\})
\]

and corresponding relator-diagrams \( \eta^{1\#} \). (We use the notation \( Q^{1\#}, r_{i}^{1\#*}, \alpha_{m}^{1\#*} \), etc., in the obvious way.) Now the cancellation words \( \epsilon^{1\#}, \epsilon^{1\#*}, \epsilon^{1\#*} \) are empty, except, may be, if \( \sim \) stands for \( m \) with \( m > u \). Furthermore, the boundaries of all those open arcs \( Q^{1\#} \) that lie in some \( R^{1\#2} \) are equal to just one point in \( R^{1\#2} \) (compare Fig. 16).

**Step iii.** "Inner" cancellations. Now we consider those arcs \( L^1 \) in the \( Q^{1\#} \)'s that lie in disks \( R^{1\#2} \) and that correspond to syllables \( g^{-1}g \). As in Step ii we cancel, step by step, all the corresponding syllables in the \( r_{i}^{1\#} \)'s (cancellation operations of Type 1), and we obtain in this way a presentation

\[
\Psi^{1\#} = (\{g_{xj}, g_{yj}, g_{zm}\}, \{r_{i}^{1\#}\})
\]

\(^{10}\) These maps are not semilinear, but can be taken piecewise algebraic.
Then, again as in Step ii, we construct relator-diagrams $R_{m}^{\Pi}$ and $R_{m}^{\Pi}$ that correspond to $R_{m}^{\Pi}$ by shrinking arcs to points. (Note that the arcs $(\alpha_{m}^{\Pi})^{-1}(L)$ may lie in $R_{m}^{\Pi}$ as well as in $R_{m}^{\Pi}(m > u)$.) The $R_{m}^{\Pi}$'s possess again the properties (10.1), (10.2, modified), (10.3), (10.4).

The words $r_{m}^{\Pi}$ read from the $R_{m}^{\Pi}$'s are of the form

\[ r_{j_{1}}^{\Pi} = g_{\gamma_{j_{1}}}; \quad r_{j_{k}}^{\Pi} = g_{\gamma_{j_{k}}} g_{\gamma_{j_{k}-1}}^{-1} \quad (k = 2, \ldots, t_{j}); \]
\[ r_{Y_{j} t+1}^{\Pi} = g_{x_{j}} g_{x_{j_{t+1}}}; \]
(III § ) \( r_m^{III} \) = either \( g_2 m \), or \( g_2 m g_2 x_{ij} \), or \( g_2 m g_2 x_{ij} m \),
or \( g_2 m g_2 x_{ij} m \), or \( g_2 m g_2 x_{ij} m \), \( \lambda(m) < m \),
and the words \( q_j^{III} \) (\( f = 1, \ldots, w \)) read from the \( Q_j^{III} \)'s are of the form
\[ q_j^{III} = \begin{cases} 
\text{either } g_2 z_1 g_2 m, \\
\text{or } g_2 x_{ij} g_2 m, \\
\text{or } g_2 m g_2 x_{ij} m, \\
\text{or } g_2 x_{ij} m, \\
\text{or } g_2 x_{ij} m, \\
\text{or } g_2 x_{ij} m. 
\end{cases} \]

**Step iv. Deleting \( Q \)'s (see Fig. 16).** Provided that \( w \neq 0 \) there exists a disk \( R_m^{III} \) that contains at least one of the open arcs \( Q_j^{III} \). Then there is at least one open arc, say \( Q_d^{III} \) that lies in \( R_m^{III} \) in such a way that \( Q_d^{III} \) is the boundary of a 2-dimensional element, say \( C^d \), of \( \Theta_m^{III} \). There is just one other element, say \( D^2 \), of \( \Theta_m^{III} \) whose boundary contains \( Q_d^{III} \). Let \( r_h^{III} \) and \( r_h^{III} \) \((h \neq i_0)\) be the relators associated with \( C^d \) and \( D^2 \), respectively; then \( q_d^{III} \) is equal to a member of \( \langle r_h^{III} \rangle \), and some member of \( \langle r_{i_0}^{III} \rangle \) can be written as \( (q_d^{III})^{-1} r_{i_0}^{III} \) (where \( r_{i_0}^{III} \) is some word in the \( g \)'s). Now we replace \( r_{i_0}^{III} \) by \( r_{i_0}^{III} \) (cancellation operation of Type 2 and of length 1, 2, or 3) and we obtain in this way a presentation \( \Psi^V \) from \( \Psi^V \).

Then we construct relator-diagrams \( R_{-V} \) that correspond to \( \Psi^V \) as follows: First we delete from \( \Theta_m^{III} \) those elements that lie in \( Q_j^{III} \), and we replace the elements \( C^2, D^2 \) by the open disk \( C^2 + Q_d^{III} + D^2 \); this yields \( \Theta_m^{IV} \) (where \( R_{m}^{IV} = R_{m}^{III} \) and the new 2-dimensional element of \( \Theta_m^{IV} \) is associated with \( r_{i_0}^{IV} \)). If the relator \( r_{i_0}^{IV} \) is associated with a 2-dimensional element, say \( D^2 \), of a decomposition \( \Theta_m^{IV} \), different from \( \Theta_m^{III} \), then [by (10.4)] \( m > u \), and \( \alpha_m^{III}(R_{m}^{III}) \) contains \( Q_d^{III} \) in its interior (since otherwise the closed curve \( Q_d^{III} \) would lie in \( \alpha_m^{III}(R_{m}^{IV}) \), but would not be equal to \( \alpha_m^{III}(R_{m}^{III}) \) in contradiction to the fact that \( \alpha_m^{III}(R_{m}^{III}) \) is a disk). Hence \( R_{m}^{III} \) contains \( \alpha_m^{III}(-1)(Q_d^{III}) \) which is one of the \( Q_j^{III} \)'s, say \( Q_c^{III} \), and \( \Theta_m^{IV} \) possesses an element \( C^2 = (\alpha_m^{III})^{-1}(C^2) \) that is associated with \( r_{i_0}^{IV} \). Then we delete from \( \Theta_m^{III} \) those elements that lie in \( Q_j^{III} \), and we replace the elements \( C^2 \) by \( C^2 + Q_c^{III} + D^2 \) (which we associate with \( r_{i_0}^{IV} \)). This yields \( \Theta_m^{IV} \). For the remaining \( R_{-V} \)'s we take \( R_{-V} = R_{-V} \). We write \( r_i^V = r_i^{IV} \) if \( i \neq i_0 \).

Now the \( R_{-V} \)'s have again the properties (10.1), (10.2, modified), (10.3), (10.4), and \( r_{-V}^V = r_{-V}^{IV} \), but the number \( w^{IV} \) of open arcs \( Q_j^{IV} \) (\( f = 1, \ldots, w^{IV} \)) is smaller than \( w \).

**Step v. Deleting all \( Q \)'s.** We repeat the procedure of Step iv as often as possible and finally obtain in this way (after at most \( w \) steps) a presentation \( \Psi^V = (\{g \}, \{ r_i^V \}) \) and corresponding relator-diagrams \( R_{-V} \) such that each decomposition \( \Theta_{-V} \) possesses just one 2-dimensional element. That means that each of the \( v + s \) words \( r_{-V}^V \) is a cyclic permutation of \( r_{-V}^V \) [see (III § )] is at the same time a member of a class, say \( \langle r_{-V}^V \rangle \), where we may assume that the notation is so arranged that \( 1 \leq i_{-V} \leq v + s \); (we do not specify the remaining relators \( r_{i+i_0}^V \), \( \ldots, r_{-V}^V \)). Further we may assume that \( r_{-V}^V \) is a cyclic permutation of \( r_{-V}^V \) and not one of \( (r_{-V}^V)^{-1} \); (this can be arranged by proper choice of the direction in which \( r_{-V}^V \) is read when the last open arc \( Q^I \) is removed in \( R_{-V}^{III} \)).
Step vi. Obviously we can transform $\mathfrak{B}^V$ by a sequence of cancellation operations of Type 2 and length 1 (a relator out of $\langle g_{Y_{jk}}g_{Y_{jk-1}}^{-1} \rangle$ is replaced by a relator $g_{Y_{jk}}$ where another relator is equal to $g_{Y_{jk-1}}$, $k = 2, \cdots, t_j$; etc.) into a presentation

$$\mathfrak{B}^{VI} = (\{g_v\}, \{g_v, t_{v+s+1}^{VI}, \cdots, t_b^{VI}\})$$

where $t_i^{VI} = t_i^V$ for $i = v + s + 1, \cdots, b$. Now we can transform $\mathfrak{B}^{VI}$ by a sequence of cancellation operations of Type 2 and length 1 (a relator $g_v^{\pm 1}r''$ is replaced by $r''$ where another relator is equal to $g_v$) into the presentation $(\{g_v\}, \{g_v, b^{-(v+s)}\})$. This finishes the proof of the theorem.

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