ASYMPTOTIC EXPANSIONS FOR THE COEFFICIENTS OF ANALYTIC FUNCTIONS[1]

BY

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1. Introduction

The problem of obtaining an asymptotic expansion for the coefficient α_n of $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ arises in many instances. The chief sources of such problems are number theory and the general area of combinatorics.

There are various aspects of this problem depending on how much precision one requires. In the simplest case, one asks only for an asymptotic formula of the kind $\alpha_n \sim c_n$, as $n \to \infty$, where c_n is a relatively simple function of n. On the other hand, one may require that a full asymptotic expansion of the type

$$(1) \qquad \alpha_n = c_n\{1 + F_1(n)/\beta_n + \cdots + F_N(n)/\beta_n^N + o(F_N(n)/\beta_n^N)\}$$

hold for each $N \geqq 0$ as $n \to \infty$; here, one allows $F_k(n)$ to depend on n but wishes to have

$$F_{k+1}(n)/\beta_n^{k+1} = o(F_k(n)/\beta_n^k)$$

for each k, as $n \to \infty$, and one also desires that $\beta_n \to \infty$ as $n \to \infty$. In fact, in most cases, $F_k(n) = o(\beta_n^{\varepsilon})$ for each $\varepsilon > 0$ and each k, as $n \to \infty$.

The literature contains many papers dealing with such problems. Here we mention only the papers of Hayman [4], Grosswald [1], and a previous paper [2] of ours. Hayman deals with the simple formula $\alpha_n \sim c_n$ under relatively weak conditions on $f(z)$. Grosswald, however, assumes more and obtains a result of the type (1). In our earlier paper, we also obtained such a result but for the special function $f_0(z) = \exp(ze^z)$ which does not satisfy Grosswald's hypotheses. In the present work, we generalize our earlier theorem by using some ideas in Grosswald's and Hayman's papers; this yields a result having weaker hypotheses than Grosswald's. Finally, we apply our theorem to $f_0(z)$ which is the exponential generating function of U_n, the number of idempotent elements in the symmetric semigroup on n letters.

2. Statement of the result

We make the following assumptions (A)–(E):

(A) $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ is analytic for $|z| < R$, $0 < R \leqq \infty$, and is real for real z.

(B) There exists an $R_0 \in (0, R)$ and a $d(r)$ defined for all $r \in (R_0, R)$ such

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that we have

$$0 < d(r) < 1, \qquad r\{1 + d(r)\} < R;$$

moreover, $f(z) \neq 0$ for each z such that $|z - r| \leqq r\,d(r)$.

(C) Defining, for $k \geqq 1$,

$$A(z) = f'(z)/f(z), \qquad B_k(z) = (z^k/k!)A^{(k-1)}(z), \qquad B(z) = \tfrac{1}{2}zB_1'(z),$$

we have $B(r) > 0$ for $R_0 < r < R$ and $B_1(r) \to \infty$ as $r \to R-$.

(D) For suitable R_1 and large n, define u_n to be the unique solution of $B_1(r) = n + 1$ which satisfies $R_1 < r < R$. Define

$$C_j(z, r) = -\left\{B_{j+2}(z) + \frac{(-1)^j}{j+2}B_1(r)\right\} \Big/ B(r),$$

and suppose that for a certain fixed $N \geqq 0$ there exist non-negative D_n, E_n and n_0 such that for all $n \geqq n_0$ and for $1 \leqq j \leqq 2N + 1$ we have $|C_j(u_n, u_n)| \leqq E_n D_n^j$. In addition, we have for all $n \geqq n_0$ that either

(D$_1$) $\quad |C_j(u_n, u_n)| \leqq E_n D_n^j \quad$ for all $j \geqq 2N + 2$

or

(D$_2$) $\quad |C_{2N+2}(u_n + iu_n\varphi, u_n)| \leqq E_n D_n^{2N+2}$

for all real φ satisfying $|\varphi| \leqq d(u_n)$.

(E) As $n \to \infty$, we have

$$B(u_n)\{d(u_n)\}^2 \to \infty, \quad D_n E_n B(u_n)\{d(u_n)\}^3 \to 0, \quad D_n d(u_n) \to 0.$$

These assumptions require some comment. First, it follows from (A) and (B) that for each $r \in (R_0, R)$ the function $f(z)$ is nowhere 0 and $A(z)$, $B(z)$, $B_k(z)$ are all defined and analytic on a suitable open disc Ω_r containing the closed disc $\{z : |z - r| \leqq r\,d(r)\}$. Moreover, $A(z)$, $B(z)$ and $B_k(z)$ are real for real z. Next, (C) implies that $B_1'(r) > 0$ for $R_0 < r < R$ so that $B_1(r)$ is strictly increasing for such r. Also, by (C), there exists $R_1 \in (R_0, R)$ such that $n_1 \equiv [B_1(R_1)] \geqq n_0$; for each n, (C) also implies the existence of $\rho_n \in (R_1, R)$ such that $B_1(\rho_n) > n + 1$. Consequently, if $n \geqq n_1$ then

$$B_1(R_1) < n_1 + 1 \leqq n + 1 < B_1(\rho_n);$$

inasmuch as $B_1(r)$ is continuous and strictly increasing on (R_1, R), we see that the equation $B_1(r) = n + 1$ has a unique solution r in (R_1, R) for each $n \geqq n_1$. This remark justifies the definition of u_n given in (D). Clearly, u_n is a strictly increasing function of n and $u_n \to R-$ as $n \to \infty$.

Moreover, $C_j(z, r)$ is defined if $R_1 < r < R$, if $|z - r| \leqq r\,d(r)$, and if $j \geqq 0$ since $B(r) > 0$; and if z is real, then so is $C_j(z, r)$. And since

$$B(r) = \tfrac{1}{2}r\,(d/dr)\{rA(r)\} = \tfrac{1}{2}r\{rA'(r) + A(r)\} = B_2(r) + \tfrac{1}{2}B_1(r),$$

we deduce that $C_0(r, r) = -1$.

We further define:

(2) $$\beta_n = B(u_n).$$

(3) $$\gamma_j(n) = C_j(u_n, u_n).$$

(4) $\lambda(r; d)$ is the maximum value of $|f(z)/f(r)|$ for z on the oriented path $Q(r)$ consisting of the line segment L from $r + ir\,d(r)$ to $r\sqrt{1 - d^2(r)} + ir\,d(r)$ and of the circular arc C from the last point to ir to $-r$.

(5) $$\mu(r; d) = \max\left(\lambda(r; d)\sqrt{B(r)}, \quad \frac{\exp\{-B(r)\,d^2(r)\}}{d(r)\sqrt{B(r)}}\right).$$

(6) $$E'_n = \min(1, E_n), \qquad E''_n = \max(1, E_n).$$

(7) $$\varphi_N(n; d) = \max\{\mu(u_n; d), \quad E'_n(D_n E''_n/\sqrt{\beta_n})^{2N+2}\}.$$

(8) $$F_k(n) = \frac{(-1)^k}{\sqrt{\pi}} \sum_{m=1}^{2k} \frac{\Gamma(m + k + \frac{1}{2})}{m!} \sum_{\substack{j_1+\cdots+j_m=2k \\ j_1,\cdots,j_m \geqq 1}} \gamma_{j_1}(n) \cdots \gamma_{j_m}(n).$$

We can now state our main result, of the kind (1), which will be proved in the next section following which we will compare it with Grosswald's result.

THEOREM 1. *If* (A)–(E) *hold* (*with either* (D_1) *or* (D_2)), *then for the given* N *we have, as* $n \to \infty$,

$$\alpha_n = \frac{f(u_n)}{2u_n^n\sqrt{\pi\beta_n}}\left\{1 + \sum_{k=1}^{N} \frac{F_k(n)}{\beta_n^k} + O(\varphi_N(n; d))\right\}.$$

If (D_1) *holds, then this is valid for all* $N \geqq 0$.

3. Proof of the theorem

If K is an arbitrary, positively-oriented, simple closed path containing the origin in its interior and is such that K lies in the disk $\{z : |z| < R\}$, then Cauchy's theorem gives

$$\alpha_n = \frac{1}{2\pi i}\int_K \frac{f(z)}{z^{n+1}}\,dz \equiv \frac{1}{2\pi i}\int_K M_n(z)dz.$$

Since

(9) $$M'_n(z) = M_n(z)\{B_1(z) - (n + 1)\}/z,$$

the saddle point method suggests that we take K to be a path passing through u_n. Let K_+, the part of K in the upper half-plane, consist of the line segment from u_n to $u_n + iu_n\,d(u_n)$ and of the path $Q(u_n)$, defined in (4), from this last point to $-u_n$. Let K_-, the part of K in the lower half-plane, consist of the reflection of K_+ in the real axis with the orientation reversed; thus, K_- extends from $-u_n$ to u_n. Since (A) implies that $f(z)$ takes conjugate values at conjugate places, it follows that $\lambda(u_n; d)$ is an upper bound of $|f(z)/f(r)|$ for z not merely on $Q(u_n)$ but also for z on the reflection of $Q(u_n)$ in the real axis.

And since $|z| \geqq u_n$ on $Q(u_n)$ which has length not exceeding $u_n + \pi u_n$, we have

$$\left| \alpha_n - \frac{1}{2\pi i} \int_{-d(u_n)}^{d(u_n)} \frac{f(u_n + iu_n \varphi)}{(u_n + iu_n \varphi)^{n+1}} iu_n \, d\varphi \right| \leqq \frac{\lambda(u_n; d) \, |f(u_n)|}{2\pi \, u_n^{n+1}} \cdot 2(u_n + \pi u_n);$$

here the integration is along the real axis. Putting

$$\delta_n = d(u_n), \tag{10}$$

$$G_n(\varphi) = (1 + i\varphi)^{-n-1} \frac{f(u_n + iu_n \varphi)}{f(u_n)}, \qquad J = \tfrac{1}{2} \int_{-\delta_n}^{\delta_n} G_n(\varphi) \, d\varphi, \tag{11}$$

we obtain

$$\left| \frac{\pi}{f(u_n)} u_n^n \alpha_n - J \right| \leqq (\pi + 1) \lambda(u_n; d). \tag{12}$$

It follows from an earlier remark that $f(z)$ has an analytic logarithm $\Lambda(z)$ in Ω_r where it is never 0. Hence, $\Lambda'(z) = f'(z)/f(z) = A(z)$ and $\Lambda^{(m+1)}(z) = A^{(m)}(z)$. So, for all complex φ with $|\varphi| \leqq \delta_n$, we have

$$\begin{aligned} \Lambda(u_n + iu_n \varphi) - \Lambda(u_n) &= \textstyle\sum_{m=0}^{2N+2} \frac{\Lambda^{(m+1)}(u_n)}{(m+1)!} (iu_n \varphi)^{m+1} + Y_1 \\ &= \textstyle\sum_{m=0}^{2N+2} B_{m+1}(u_n)(i\varphi)^{m+1} + Y_1 \end{aligned}$$

where

$$Y_1 = \textstyle\sum_{m=2N+3}^{\infty} B_{m+1}(u_n)(i\varphi)^{m+1}.$$

By using Taylor's theorem with integral remainder, we can also write

$$\begin{aligned} Y_1 &= \frac{1}{(2N+3)!} \int_0^{iu_n\varphi} \Lambda^{(2N+4)}(u_n + \omega)(iu_n \varphi - \omega)^{2N+3} \, d\omega \\ &= \frac{(iu_n)^{2N+4}}{(2N+3)!} \int_0^{\varphi} A^{(2N+3)}(u_n + iu_n \vartheta)(\varphi - \vartheta)^{2N+3} \, d\vartheta \\ &= (-1)^N (2N+4) \int_0^{\varphi} B_{2N+4}(u_n + iu_n \vartheta) \frac{(\varphi - \vartheta)^{2N+3}}{(1 + i\vartheta)^{2N+4}} \, d\vartheta. \end{aligned}$$

Similarly,

$$\log (1 + i\varphi) = \textstyle\sum_{m=0}^{2N+2} \frac{(-1)^m}{m+1} (i\varphi)^{m+1} + Y_2$$

where

$$Y_2 = \textstyle\sum_{m=2N+3}^{\infty} \frac{(-1)^m}{m+1} (i\varphi)^{m+1}$$

and, also,

$$Y_2 = -\int_0^{i\varphi} \frac{(i\varphi - \omega)^{2N+3}}{(1+\omega)^{2N+4}} \, d\omega = (-1)^{N+1} \int_0^{\varphi} \frac{(\varphi - \vartheta)^{2N+3}}{(1 + i\vartheta)^{2N+4}} \, d\vartheta.$$

From (11), we see that a logarithm of $G_n(\varphi)$ is given by

$$\log G_n(\varphi) = \Lambda(u_n + iu_n\varphi) - \Lambda(u_n) - (n+1)\log(1+i\varphi)$$
$$= \{B_1(u_n) - (n+1)\}\, i\varphi$$
$$+ \sum_{m=1}^{2N+2} \left\{B_{m+1}(u_n) - (-1)^m \cdot \frac{n+1}{m+1}\right\}(i\varphi)^{m+1} + Y_3.$$

Then

$$(13) \qquad \log G_n(\varphi) = -\beta_n\varphi^2 + T(\varphi) + Y_3$$

where, by (3) and (2),

$$(14) \qquad T(\varphi) = \beta_n\varphi^2 \sum_{j=1}^{2N+1} \gamma_j(n)(i\varphi)^j$$

and

$$Y_3 = Y_1 - (n+1)Y_2 = \beta_n\varphi^2 \sum_{j=2N+2}^{\infty} \gamma_j(n)(i\varphi)^j$$

$$Y_3 = (-1)^N(2N+4)\int_0^{\varphi}\left\{B_{2N+4}(u_n + iu_n\vartheta) + \frac{n+1}{2N+4}\right\} \cdot \frac{(\varphi-\vartheta)^{2N+3}}{(1+i\vartheta)^{2N+4}}\,d\vartheta$$

$$= (-1)^{N+1}(2N+4)\beta_n \int_0^{\varphi} C_{2N+2}(u_n + iu_n\vartheta, u_n)\,\frac{(\varphi-\vartheta)^{2N+3}}{(1+i\vartheta)^{2N+4}}\,d\vartheta.$$

If (D_1) holds, then the series representation for Y_3 shows that for $n \geqq n_1$ and $-\delta_n \leqq \varphi \leqq \delta_n$

$$|Y_3| \leqq \beta_n\varphi^2 \sum_{j=2N+2}^{\infty} E_n D_n^j |\varphi|^j = E_n\beta_n\varphi^2\,\frac{(D_n\varphi)^{2N+2}}{1 - D_n|\varphi|}$$

provided $D_n|\varphi| < 1$. In fact, by (E) there exists $n_2 \geqq n_1$ such that $n \geqq n_2$ implies $D_n|\varphi| \leqq D_n\delta_n \leqq \frac{1}{2}$ so that

$$(15) \qquad |Y_3| \leqq 2E_n\beta_n\varphi^2(D_n\varphi)^{2N+2}.$$

On the other hand, if (D_2) holds, then the integral expression for Y_3 shows that for $0 \leqq \varphi \leqq \delta_n$

$$|Y_3| \leqq (2N+4)\beta_n \int_0^{\varphi} E_n D_n^{2N+2}(\varphi-\vartheta)^{2N+3}\,d\vartheta$$

from which (15) follows; it also follows if $-\delta_n \leqq \varphi \leqq 0$. Thus, if either (D_1) or (D_2) holds, so does (15); and if $n \geqq n_3 \geqq n_2$ then (E) shows that

$$|Y_3| \leqq 2E_n\beta_n\varphi^2\cdot D_n|\varphi| \leqq 2D_nE_n\beta_n\delta_n^3 \leqq \tfrac{1}{2}$$

so that $e^{Y_3} = 1 + O(Y_3)$. Likewise, (14) gives

$$|T(\varphi)| \leqq \beta_n\varphi^2 \sum_{j=1}^{2N+1} E_n D_n^j |\varphi|^j \leqq 2E_n\beta_n\varphi^2\cdot D_n|\varphi| \leqq \tfrac{1}{2}.$$

Hence, if $n \geqq n_3$ and $-\delta_n \leqq \varphi \leqq \delta_n$, we obtain from (13) that

$$G_n(\varphi) = e^{-\beta_n\varphi^2 + T(\varphi)}e^{Y_3} = e^{-\beta_n\varphi^2}e^{T(\varphi)}\{1 + O(Y_3)\}$$
$$= e^{-\beta_n\varphi^2}\{e^{T(\varphi)} + O(Y_3)\}.$$

Moreover,

$$e^{T(\varphi)} = 1 + \sum_{m=1}^{2N+1} (1/m!)\{T(\varphi)\}^m + Y_4$$

where

$$|Y_4| \leqq \sum_{m=2N+2}^{\infty} (1/m!)|T(\varphi)|^m \leqq \sum_{m=2N+2}^{\infty} (1/m!)(2D_n E_n \beta_n |\varphi|^3)^m$$
$$\leqq 2(2D_n E_n \beta_n \varphi^3)^{2N+2}.$$

Thus, on using (15) and allowing the constant implied by the O-symbol to depend on N, we obtain

(16) $$G_n(\varphi) = e^{-\beta_n \varphi^2}\{1 + \sum_{m=1}^{2N+1} (1/m!)(T(\varphi))^m + O(Y_5)\}$$

where

(17) $$Y_5 = E_n^{2N+2}(D_n \varphi)^{2N+2}(\beta_n \varphi^2)^{2N+2} + E_n(D_n \varphi)^{2N+2}\beta_n \varphi^2.$$

Now, on writing γ_j in place of $\gamma_j(n)$, we obtain

$$(1/m!)\{T(\varphi)\}^m = (1/m!)(\beta_n \varphi^2)^m \sum_{j_1,\cdots,j_m=1}^{2N+1} \gamma_{j_1} \cdots \gamma_{j_m}(i\varphi)^{j_1+\cdots+j_m}$$
$$= (\beta_n \varphi^2)^m \sum_{j=m}^{m(2N+1)} G_{jm}(i\varphi)^j$$

where

(18) $$G_{jm} = (1/m!) \sum_{j_1+\cdots+j_m=j;\ 1 \leqq j_1,\cdots,j_m \leqq 2N+1} \gamma_{j_1} \cdots \gamma_{j_m}.$$

Hence,

$$|G_{jm}| \leqq (1/m!)E_n^m D_n^j \sum_{j_1+\cdots+j_m=j;\ j_1,\cdots,j_m \geqq 1} 1 \leqq (1/m!)E_n^m D_n^j j^{m-1}$$
$$\leqq (1/j)E_n^m D_n^j e^j$$

since each of $j_1, \cdots, j_{m-1}$ can assume at most j values and then j_m has at most one value. Consequently,

$$\sum_{m=1}^{2N+1} (1/m!)\{T(\varphi)\}^m = \sum_{j=1}^{(2N+1)^2} (i\varphi)^j \sum_{1 \leqq m \leqq 2N+1;\ j/(2N+1) \leqq m \leqq j} G_{jm}(\beta_n \varphi^2)^m$$
$$= \sum_{j=1}^{2N+1} (i\varphi)^j \sum_{m=1}^{j} G_{jm}(\beta_n \varphi^2)^m + O(V_{N+1}(\varphi))$$

where

(19) $$V_p(\varphi) = \sum_{j=2p}^{\infty} |\varphi|^j \sum_{m=1}^{j} |G_{jm}|(\beta_n \varphi^2)^m$$
$$\leqq \sum_{j=2p}^{\infty} |\varphi|^j (eD_n)^j(1/j) \sum_{m=1}^{j} (E_n \beta_n \varphi^2)^m.$$

Now for $x \geqq 0$ and $1 \leqq m \leqq j$, we have

$$x^m \leqq \begin{Bmatrix} x & \text{if} & x \leqq 1 \\ x^j & \text{if} & 1 \leqq x \end{Bmatrix} \leqq x + x^j$$

so that

$$V_p(\varphi) \leqq \sum_{j=2p}^{\infty} |\varphi|^j (eD_n)^j\{E_n \beta_n \varphi^2 + (E_n \beta_n \varphi^2)^j\}.$$

By (E), we have for $n \geqq n_4 \geqq n_3$ and $-\delta_n \leqq \varphi \leqq \delta_n$ that

(20) $$V_p(\varphi) \leqq 2E_n \beta_n \varphi^2(eD_n \varphi)^{2p} + 2(eD_n E_n \beta_n \varphi^3)^{2p}.$$

Thus, (17) shows that $V_{N+1}(\varphi) = O(Y_5)$. Hence, (16) becomes

$$G_n(\varphi) = e^{-\beta_n\varphi^2}\{1 + \textstyle\sum_{j=1}^{2N+1} (i\varphi)^j \sum_{m=1}^{j} G_{jm}(\beta_n \varphi^2)^m + O(Y_5)\}$$
$$= e^{-\beta_n\varphi^2}\{1 + S_1(\varphi) + S_2(\varphi) + O(Y_5)\}$$

where $S_1(\varphi)$ consists of the terms with odd j and $S_2(\varphi)$ consists of the terms with even j.

Hence, (11) gives, since $S_1(\varphi)$ is an odd function and $S_2(\varphi)$ is an even function,

$$J = \tfrac{1}{2}\int_{-\delta_n}^{\delta_n} e^{-\beta_n\varphi^2}\{1 + S_1(\varphi) + S_2(\varphi) + O(Y_5)\}\, d\varphi$$
$$= \int_0^{\delta_n} e^{-\beta_n\varphi^2}\, d\varphi + \int_0^{\delta_n} e^{-\beta_n\varphi^2} \textstyle\sum_{k=1}^{N} (i\varphi)^{2k} \sum_{m=1}^{2k} G_{2k,m}(\beta_n \varphi^2)^m\, d\varphi + O(Y_6)$$

where, on setting $t = \beta_n \varphi^2$ and using (17), we find

$$Y_6 = E_n^{2N+2} \int_0^\infty e^{-t} \left(\frac{D_n}{\sqrt{\beta_n}}\right)^{2N+2} \frac{t^{3N+3}}{2\sqrt{\beta_n t}}\, dt + E_n \int_0^\infty e^{-t} \left(\frac{D_n}{\sqrt{\beta_n}}\right)^{2N+2} \frac{t^{N+2}}{2\sqrt{\beta_n t}}\, dt$$
$$= O\left(\frac{E_n}{\sqrt{\beta_n}}\left\{\frac{D_n}{\sqrt{\beta_n}}\right\}^{2N+2} \{E_n^{2N+1} + 1\}\right).$$

On using (6), we have

$$E_n(E_n^{2N+1} + 1) \leqq 2E_n(E_n'')^{2N+1} = 2E_n'(E_n'')^{2N+2}$$

so that $Y_6 = O(\Psi_n)$ where

$$\Psi_n = \frac{E_n'}{\sqrt{\beta_n}}\left(\frac{D_n E_n''}{\sqrt{\beta_n}}\right)^{2N+2}. \tag{21}$$

Putting

$$\Delta_n = \beta_n \delta_n^2 = B(u_n)\{d(u_n)\}^2, \tag{22}$$

we have

$$J = \frac{1}{2\sqrt{\beta_n}}\int_0^{\Delta_n} e^{-t}t^{-1/2}\, dt + \frac{1}{2\sqrt{\beta_n}} \sum_{k=1}^{N} \frac{(-1)^k}{\beta_n^k} \sum_{m=1}^{2k} G_{2k,m} \int_0^{\Delta_n} e^{-t}t^{k+m-1/2}\, dt + O(\Psi_n). \tag{23}$$

Now for $s > -\frac{1}{2}$ we have

$$\int_0^{\Delta_n} e^{-t}t^{s-1/2}\, dt = \Gamma\left(s + \frac{1}{2}\right) - Z_s \tag{24}$$

where, on integrating by parts,

$$(25) \qquad Z_s = \int_{\Delta_n}^{\infty} e^{-t} t^{s-1/2}\, dt = e^{-\Delta_n}\Delta_n^{s-1/2} + \left(s - \frac{1}{2}\right) \int_{\Delta_n}^{\infty} e^{-t} t^{s-3/2}\, dt.$$

If $s \geqq \frac{1}{2}$ and $\Delta_n \geqq 2(s - \frac{1}{2})$, then

$$\begin{aligned} Z_s &\leqq e^{-\Delta_n}\Delta_n^{s-1/2} + \frac{s - 1/2}{\Delta_n} \int_{\Delta_n}^{\infty} e^{-t} t^{s-1/2}\, dt \\ &\leqq e^{-\Delta_n}\Delta_n^{s-1/2} + \frac{1}{2} Z_s \,; \end{aligned}$$

hence, there exists λ_s in [0, 1] such that

$$(26) \qquad Z_s = 2\lambda_s e^{-\Delta_n}\Delta_n^{s-1/2}.$$

Moreover, $\Delta_n \geqq 2(3N - \frac{1}{2})$ by (22) and (E) if $n \geqq n_5 \geqq n_4$; hence (26) holds for $\frac{1}{2} \leqq s \leqq 3N$ if $n \geqq n_5$. And if $-\frac{1}{2} < s < \frac{1}{2}$, then the last term in (25) is negative so that (26) clearly holds in this case also. Thus, if $-\frac{1}{2} < s \leqq 3N$, then we have by (24)

$$\int_0^{\Delta_n} e^{-t} t^{s-1/2}\, dt = \Gamma\left(s + \frac{1}{2}\right) + O(e^{-\Delta_n}\Delta_n^{s-1/2}).$$

From (23), we now obtain

$$\begin{aligned} J = \frac{1}{2\sqrt{\beta_n}} &\left\{\Gamma\left(\frac{1}{2}\right) + \sum_{k=1}^{N} \frac{(-1)^k}{\beta_n^k} \sum_{m=1}^{2k} G_{2k,m}\, \Gamma\left(k + m + \frac{1}{2}\right)\right\} + O(\Psi_n) \\ &+ O\left(\frac{1}{\sqrt{\beta_n}} \cdot \frac{e^{-\Delta_n}}{\sqrt{\Delta_n}} \left\{1 + \sum_{k=1}^{N} \frac{1}{\beta_n^k} \sum_{m=1}^{2k} |G_{2k,m}|\, \Delta_n^{k+m}\right\}\right). \end{aligned}$$

On using (18) and (8), we find

$$J = \frac{\sqrt{\pi}}{2\sqrt{\beta_n}}\left\{1 + \sum_{k=1}^{N} \frac{F_k(n)}{\beta_n^k}\right\} + O(\Psi_n) + O\left(\frac{e^{-\Delta_n}}{\sqrt{\beta_n}\,\Delta_n}\{1 + V_1(\delta_n)\}\right)$$

by (19). By (20) and (E), we have for $n \geqq n_6 \geqq n_5$ that

$$V_1(\delta_n) \leqq 2e^2 \cdot D_n E_n \beta_n \delta_n^3 \{D_n \delta_n + D_n E_n \beta_n \delta_n^3\} \leqq 1.$$

Hence,

$$J = \frac{\sqrt{\pi}}{2\sqrt{\beta_n}}\left\{1 + \sum_{k=1}^{N} \frac{F_k(n)}{\beta_n^k} + O(\Psi_n\sqrt{\beta_n}) + O\left(\frac{e^{-\Delta_n}}{\sqrt{\Delta_n}}\right)\right\}.$$

From (12) and (22), we therefore obtain

$$\begin{aligned} \frac{\pi}{f(u_n)} u_n^n \alpha_n = \frac{\sqrt{\pi}}{2\sqrt{\beta_n}} &\left\{1 + \sum_{k=1}^{N} \frac{F_k(n)}{\beta_n^k} + O(\Psi_n\sqrt{\beta_n}) \right. \\ &\left. + O\left(\frac{e^{-\beta_n\delta_n^2}}{\delta_n\sqrt{\beta_n}}\right) + O(\lambda(u_n\,;\, d)\sqrt{\beta_n})\right\}. \end{aligned}$$

Taking account of (2), (10) and (21), we see that this implies the conclusion of the theorem.

4. Remarks on Theorem 1

In the more usual treatment of this type of problem, one uses a circle for the contour K in place of our path which consists of three line segments joined to a circular arc. One effect of this change is to place our derivation in the exact setting of the saddle point method as we remarked following (9). In Grosswald's treatment, however, the path is the circle, with center at the origin, having radius u_{n-1} so that his path does not quite pass through the saddle point at u_n. In the application given in the next section, our present method produces better numerical results than that based on the circle $|z| = u_{n-1}$; see our paper [3] for a numerical comparison.

A second effect of this change is that we have to work with the $B_k(z)$ of (C) whereas in Grosswald's treatment the corresponding quantities that arise are $[z(d/dz)]^{k-1}\{zA(z)\}$ which are usually more difficult to determine.

In the application made in the next section, we use the hypothesis (D_1) rather than (D_2). Under the assumption of (D_1), the proof can be simplified a bit by replacing the $T(\varphi)$ in (14) by an infinite sum so that Y_3 becomes 0; and in (16) we likewise use an infinite sum so that Y_5 becomes 0. As a consequence, we can proceed directly to the equation preceding (19), and we need only estimate $V_{N+1}(\varphi)$ rather than Y_3 and Y_5 in addition.

Nevertheless, we have treated the case of hypothesis (D_2) because this arises in Grosswald's work where (D_1) does not seem to hold for all $n \geqq n_0$ but only for $n \geqq n_0(j)$, and the proof given under this weaker assumption breaks down. Grosswald essentially uses (D_2) with $D_n(= 1)$ and E_n independent of n; in such cases, the second term in the definition of $\mu(r; d)$ in (5) can be omitted. Thus, our result generalizes that of Grosswald. In our application, we have $D_n = u_n \to \infty$ so that Grosswald's theorem is not applicable.

Several additional remarks are in order. If we define

$$\lambda_0(r; d) = \{d(r)\}^2 \max_{z \in L} |f(z)/f(r)| + \max_{z \in C} |f(z)/f(r)|,$$

then an obvious modification of our argument shows that (12) remains true when λ is replaced by λ_0; the same replacement can therefore be used in (5), and the resulting theorem is still valid. If we apply this form to $f(z) = e^z$ and make the (optimal) choice $d(r) = (2 \log r/r)^{1/2}$, then for all $N \geqq 0$ we merely obtain $\varphi_N(n; d) = O(\log n/\sqrt{n})$ so that only the choice $N = 0$ is of any significance. However, by integrating over the full circle instead of over our path K, we can obtain a full expansion of the type (1) for the coefficient $1/n!$ appearing in the power series for e^z; in fact, this is done in Grosswald's paper. It therefore seems worthwhile to record an alternate form for our general Theorem 1 based on such an integration; the proof adapts an argument given by Hayman in his Lemma 4.

We require a number of modifications in our hypotheses and definitions. In

(B), we now require that $f(z) \neq 0$ for $|z - r| < 2r\,d(r)$ where $0 < d(r) \leqq \frac{5}{4}$ and $r\{1 + 2\,d(r)\} \leqq R$. In (C) and (D), we redefine $B_k(z)$, u_n and $C_j(z, r)$ by:

$$B_k(z) = \frac{1}{k!}\left(z\frac{d}{dz}\right)^{k-1} \{zA(z)\}, \quad B_1(u_n) = n, \quad C_j(z, r) = -\frac{B_{j+2}(z)}{B(r)}.$$

Note that this leaves $B_1(z) = zA(z)$ unchanged in meaning so that now u_n is the old u_{n-1}; likewise, $B(z)$ (which is now $B_2(z)$) is unchanged in meaning so that the new β_n of (2) is the old β_{n-1}. In place of (D_2), we require that

$$|C_{2N+2}(u_n e^{i\varphi}, u_n)| \leqq E_n D_n^{2N+2}$$

for all real φ such that $|\varphi| \leqq d(u_n)$. Finally, we redefine $\lambda(r; d)$ of (4) to be the maximum of $|f(z)/f(r)|$ for z on the circular arc from $re^{id(r)}$ to ir to $-r$. The definitions (3) and (5)–(8) are used with the new meanings for the various quantities. With these new meanings, Theorem 1 still holds.

To see this, we use the full circle $|z| = u_n$ as the path of integration in place of K. Then (12) holds provided we redefine $G_n(\varphi)$ by

$$G_n(\varphi) = e^{-in\varphi}f(u_n e^{i\varphi})/f(u_n).$$

For all complex φ with $|\varphi| \leqq \delta_n$ we have

$$|u_n e^{i\varphi} - u_n| \leqq u_n(e^{|i\varphi|} - 1) \leqq u_n(e^{\delta_n} - 1) < 2u_n\,\delta_n$$

since $0 < \delta_n \leqq \frac{5}{4}$. Hence, $G_n(\varphi)$ has an analytic logarithm for $|\varphi| \leqq \delta_n$; this is given by

$$\begin{aligned} -in\varphi + i\varphi B_1(u_n) + \textstyle\sum_{m=1}^{2N+2} \frac{\varphi^{m+1}}{(m+1)!}\left(\frac{d^m}{d\vartheta^m}\{iu_n e^{i\vartheta}A(u_n e^{i\vartheta})\}\right)_{\vartheta=0} + Y_3 \\ = \textstyle\sum_{m=1}^{2N+2} B_{m+1}(u_n)(i\varphi)^{m+1} + Y_3 \\ = -\beta_n\varphi^2 + \beta_n\varphi^2 \textstyle\sum_{j=1}^{2N+1} C_j(u_n, u_n)(i\varphi)^j + Y_3 \end{aligned}$$

which is formally the same as (13). Here Y_3 has an obvious series representation; in addition, Y_3 has the integral representation

$$\begin{aligned} Y_3 &= \frac{1}{(2N+3)!}\int_0^\varphi \frac{d^{2N+3}}{d\vartheta^{2N+3}}\{iu_n e^{i\vartheta}A(u_n e^{i\vartheta})\}\cdot(\varphi - \vartheta)^{2N+3}\,d\vartheta \\ &= (-1)^N(2N+4)\int_0^\varphi B_{2N+4}(u_n e^{i\vartheta})(\varphi - \vartheta)^{2N+3}\,d\vartheta \\ &= (-1)^{N+1}(2N+4)\beta_n\int_0^\varphi C_{2N+2}(u_n e^{i\vartheta}, u_n)(\varphi - \vartheta)^{2N+3}\,d\vartheta \end{aligned}$$

which corresponds to the last expression for Y_3 a few lines below (14). The rest of the proof of the theorem now proceeds just as before.

As a result of the validity of both forms of Theorem 1, it follows that if both versions of (A)–(E) hold with $N = 0$ and if both terms $\varphi_0(n; d)$ are $o(1)$ as

$n \to \infty$, then as $n \to \infty$

$$\alpha_n \sim f(u_n)/(2u_n^n\sqrt{\pi\beta_n}), \qquad \alpha_n \sim f(u_{n-1})/(2u_{n-1}^n\sqrt{\pi\beta_{n-1}})$$

where u_n and β_n now have their original meanings specified $B_1(u_n) = n + 1$ and $\beta_n = B(u_n)$. Replacing n by $n + 1$ in the second of these asymptotic relations and using the first one, we see that $\alpha_{n+1} \sim \alpha_n/u_n$ as $n \to \infty$.

Finally, we wish to express our gratitude to Professor Grosswald for clarifying a number of points for us.

5. Application to a problem in semigroups

Let U_n be the number of idempotent elements in the symmetric semigroup T_n on n elements; i.e., T_n is the class of functions mapping the set $\{1, 2, \cdots, n\}$ into itself, and multiplication is defined by function composition. In a previous paper [3], we showed that

$$1 + \sum_{n=1}^{\infty} (1/n!)U_n z^n = \exp(ze^z) \equiv f_0(z)$$

and obtained the result for $n \to \infty$,

$$U_n \sim \left\{\frac{u_n + 1}{2\pi(n+1)C_n}\right\}^{1/2} \frac{n!}{u_n^n} e^{(n+1)/(u_n+1)} \equiv I_n \tag{27}$$

where u_n is the positive solution of $u(u + 1)e^u = n + 1$ and $C_n = u_n^2 + 3u_n + 1$. In an earlier paper [2], we obtained the full asymptotic expansion of the type (1).

We now show that Theorem 1 does, indeed, lead to (1) when applied to $f_0(z)$. In this case, $R = \infty$,

$$A(z) = (z + 1)e^z, \quad B_k(z) = (z^k/k!)(z + k)e^z, \quad B(z) = \tfrac{1}{2}z(z^2 + 3z + 1)e^z,$$

and (A)–(C) hold for arbitrary $d(r)$ satisfying $0 < d(r) < 1$. Now

$$C_j(r, r) = -\frac{2}{r(r^2 + 3r + 1)e^r}\left\{\frac{r^{j+2}}{(j+2)!}(r + j + 2)e^r + \frac{(-1)^j}{j+2} r(r+1)e^r\right\}$$

$$C_j(r, r) = -\frac{2r^2}{r^2 + 3r + 1} r^j \left\{\frac{1}{(j+2)!} + \frac{1}{(j+1)!r} + \frac{(-1)^j}{j+2} \cdot \frac{1 + 1/r}{r^{j+1}}\right\} = \frac{W_{j+2}(r)}{V(r)} \tag{28}$$

where $V(r) = r^2 + 3r + 1$ and $W_{j+2}(r)$ is a polynomial of exact degree $j + 2$ with negative leading coefficient. For $r \geqq 1$ and $j \geqq 1$, we have

$$|C_j(r, r)| \leqq 2r^j(\tfrac{1}{6} + \tfrac{1}{2} + \tfrac{2}{3}) < 3r^j$$

so that we can take $E_n = 3$ and $D_n = u_n \sim \log n$ as $n \to \infty$. Also

$$\beta_n \equiv B(u_n) = \frac{u_n^2 + 3u_n + 1}{2(u_n + 1)} u_n(u_n + 1)e^{u_n}$$

$$= \frac{C_n}{2(u_n + 1)} (n + 1) \sim \frac{1}{2} n \log n. \tag{29}$$

On putting $\delta_n = d(u_n)$, we will have all of (A)–(E) satisfied, including (D_1), provided we can determine $d(r)$ so that

$$\delta_n^2 n \log n \to \infty, \quad \delta_n^3 n \log^2 n \to 0, \quad \delta_n \log n \to 0 \tag{30}$$

as $n \to \infty$. We have $E_n' = 1$, $E_n'' = 3$ and, as $n \to \infty$,

$$E_n' \left(\frac{D_n E_n''}{\sqrt{\beta_n}}\right)^{2N+2} = O\left(\frac{\log n}{n}\right)^{N+1}. \tag{31}$$

We will select $d(r)$ so that $\mu(u_n ; d)$ is smaller than the above term which will then provide an estimate for $\varphi_N(n; d)$ of (7).

Since $|f_0(z)| = \exp \mathcal{R}(ze^z)$, we have on setting $z = x + iy = |z| e^{i\vartheta}$ that

$$|f_0(z)| = \exp\{e^x |z| \cos(y + \vartheta)\} = \exp\{e^x (x \cos y - y \sin y)\}. \tag{32}$$

For z on L, we have $y = r\, d(r) \leqq \pi/2$ provided $d(r) \leqq \pi/(2r)$; also $0 < x \leqq r$ so that

$$\begin{aligned} e^x(x \cos y - y \sin y) &\leqq e^x x \cos y \leqq re^r \cos y \\ &\leqq re^r\{1 - 2 \sin^2(\tfrac{1}{2}y)\} \leqq re^r\{1 - 2(y/\pi)^2\} \\ &\leqq re^r - \tfrac{1}{5}re^r y^2 = re^r - \tfrac{1}{5}r^3 e^r d^2(r). \end{aligned}$$

Hence, for z on L,

$$|f_0(z)/f_0(r)| \leqq \exp\{-\tfrac{1}{5}r^3 e^r d^2(r)\}. \tag{33}$$

If z is on C, then

$$\begin{aligned} x &\leqq r\sqrt{1 - d^2(r)} \leqq r\{1 - \tfrac{1}{2} d^2(r)\} = r - \tfrac{1}{2}r\, d^2(r) \\ e^x |z| \cos(y + \vartheta) &\leqq e^x r \leqq re^r e^{-rd^2(r)/2} \\ &\leqq re^r\{1 - \tfrac{1}{2}r\, d^2(r) + \tfrac{1}{8}r^2 d^4(r)\} \\ &\leqq re^r\{1 - \tfrac{1}{2}r\, d^2(r) + \tfrac{1}{8}r^2 d^2(r)(\tfrac{1}{2}\pi/r)^2\} \\ &\leqq re^r - \tfrac{1}{6}r^2 e^r d^2(r) \end{aligned}$$

provided $0 < d(r) \leqq \pi/(2r)$ and $r \geqq 2$. On using (32), we obtain

$$|f_0(z)/f_0(r)| \leqq \exp\{-\tfrac{1}{6}r^2 e^r d^2(r)\}$$

which holds not only for z on C but also for z on L as a result of (33). By (4), $\lambda(r; d)$ does not exceed the preceding quantity so that, since $\frac{1}{2}r^3 e^r \leqq B(r) \leqq$

$\frac{3}{2}r^3e^r$ for $r \geqq 2$, we deduce from (5) that

$$\mu(r; d) = O\left\{ \max\left(r^{3/2}e^{r/2} \exp\left\{-\frac{1}{6} r^2e^rd^2(r)\right\}, \frac{e^{-r/2}}{d(r)r^{3/2}} \exp\left\{-\frac{1}{2} r^3e^rd^2(r)\right\}\right)\right\}$$

$$= O\left(\frac{1}{d(r)} r^{3/2}e^{r/2} \exp\left\{-\frac{1}{6} r^2e^rd^2(r)\right\}\right).$$

Now define $d(r) = \exp(-2r/5)$ so that $d(r) \leqq \pi/(2r) < 1$ if $r > R_2$. Then as $r \to \infty$

$$\mu(r; d) = O(r^{3/2}e^{9r/10} \exp\{-\tfrac{1}{6}r^2e^{r/5}\})$$
$$= O(e^r \exp\{-\tfrac{1}{6}r^2e^{r/5}\}) = O(\exp(-r^2)).$$

Since $u_n \sim \log n$, we have for $n \geqq n_7$

$$\mu(u_n ; d) = O(\exp(-u_n^2)) = O(\exp\{-(N+1)\log n\}) = O(1/n^{N+1}).$$

By (7) and (31), we have for $n \to \infty$

$$\varphi_N(n; d) = O\left(\frac{\log n}{n}\right)^{N+1}.$$

Finally, (30) holds since $u_n \sim \log n$.

By (3) and (28), we have $\gamma_j(n) = W_{j+2}(u_n)/C_n$. For the G_{jm} of (18), we have

$$G_{2k,m} = \frac{X_{k,m}(u_n)}{C_n^m} = \frac{1}{C_n^{2k}} \cdot X_{k,m}(u_n)C_n^{2k-m}$$

where $X_{k,m}(u)$ is a polynomial in u of exact degree $2k + 2m$ with positive leading coefficient. So, (29) and (8) give

$$\frac{F_k(n)}{\beta_n^k} = \frac{1}{(n+1)^k} \cdot \left\{\frac{2(u_n+1)}{C_n}\right\}^k \cdot \frac{(-1)^k}{\sqrt{\pi}} \cdot \frac{1}{C_n^{2k}} \sum_{m=1}^{2k} \Gamma(m+k+\tfrac{1}{2})X_{k,m}(u_n)C_n^{2k-m}$$

$$= \frac{1}{(n+1)^k} \cdot \frac{P_k(u_n)}{C_n^{3k}}$$

where $P_k(u)$ is a polynomial of exact degree $7k$. On taking $\alpha_n = U_n/n!$ in Theorem 1, we now obtain the following result which was given in essentially the same form in [2].

THEOREM 2. *Let u_n be the positive solution of $u(u+1)e^u = n+1$ and let $C_n = u_n^2 + 3u_n + 1$. Then there exist polynomials $P_k(u)$ of exact degree $7k$ such that for each fixed $N \geqq 0$ we have, as $n \to \infty$,*

$$U_n = \sqrt{\frac{u_n+1}{2\pi(n+1)C_n}} \cdot \frac{n!}{u_n^n} e^{(n+1)/(u_n+1)} \left\{1 + \sum_{k=1}^{N} \frac{1}{(n+1)^k} \cdot \frac{P_k(u_n)}{C_n^{3k}} + O\left(\frac{\log n}{n}\right)^{N+1}\right\}.$$

In [3], we tabulated U_n and the (leading term) I_n of (27) and showed that the relative error did not exceed .73% for the nine values of $n = 16$; 25, 50, 75, $\cdots$, 200. Hence, the asymptotic expansion is very accurate even for $N = 0$. Curiously enough, for $n = 200$ the leading term I_n provides a better approximation than that obtained by taking $N = 1$.

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