A MONOTONIC MAPPING THEOREM FOR SIMPLY CONNECTED 3-MANIFOLDS[1]

BY

EDWIN E. MOISE

1. Statement of results

THEOREM. *Let M be a triangulated 3-manifold, and suppose that M is compact, connected and simply connected. Then there is a subcomplex K of a triangulation of the 3-sphere S^3, and a mapping*

$$f : S^3 \to M$$

of S^3 onto M, such that

(1) $\dim K \leqq 2$,

(2) $f \mid K$ *is simplicial (relative to K and a subdivision of M),*

(3) $f \mid (S^3 - K)$ *is one-to-one,*

(4) $f(K) \cap f(S^3 - K) = 0$,

(5) *f is monotonic, and*

(6) *Each set $f^{-1}(x)$ is either a point or a linear graph.*

Here (5) means that each set $f^{-1}(x)$ is connected. By a linear graph we mean a 1-dimensional polyhedron.[2]

2. Bing's example

R. H. Bing [B] has given a curious example of a mapping of the sort described in the above theorem. In Bing's example, M is S^3, but the inverse-image sets $f^{-1}(x)$ are of an unexpected sort. Consider (as shown on the left in Figure 1) two circular disks D_1, D_2 which intersect each other in a common radius. Let their boundaries be the circles C_1 and C_2. Each of these is decomposed into concentric circles. (In the figure, we show one such circle J_1 in D_1, and one such circle J_2 in D_2.) Thus we have a collection G of sets, consisting of (1) the points of $S^3 - (D_1 \cup D_2)$, (2) the circles C_1 and C_2 and (3) infinitely many "figure 8's" of the type $J_1 \cup J_2$.

The collection G is upper-semicontinuous in the usual sense: if X is any closed set in S^3, then the union of all elements of G that intersect X is also a closed set [K]. Thus we can define a Hausdorff topology in G, by saying

Received March 16, 1967.

[1] A portion of the work reported here (in Sections 3 through 10 below) was done while the author held a Guggenheim Memorial Foundation Fellowship. This portion of the paper was also sponsored by the National Science Foundation and the Institute for Advanced Study.

[2] Theorem 3.1 below was announced in [M] (see the bibliography at the end), and earlier, in colloquia at Warsaw and Madison. Since then, a weaker version of the theorem has been proved by Wolfgang Haken [H_1].

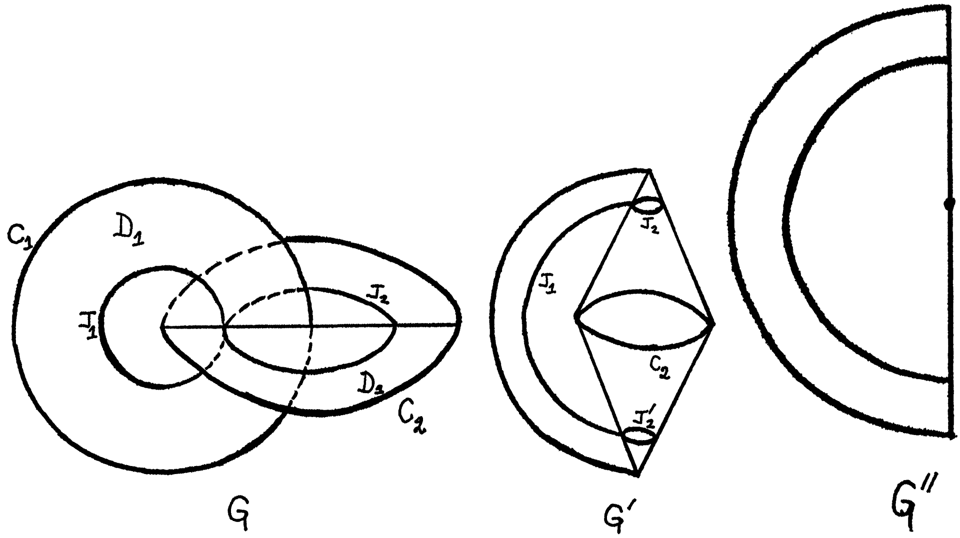

FIGURE 1

that a set $H \subset G$ is open in the space G if the union of its elements is open in the space S^3.

It was shown by Bing that the space G is homeomorphic to S^3. Following is a proof of this result, different from his.

Let us split D_2 into two conical surfaces, as shown in the middle of Figure 1. Under this operation, C_2 is fixed. To each other circle J_2 in D_2 there correspond two circles J_2, J_2', on the respective cones; and to the center of D_2 there correspond two points N and S. Thus we get a new space G' whose points are (1) the arc from N to S (corresponding to C_1) (2) sets of the type $J_2 \cup J_1 \cup J_2'$ (3) C_2 and (4) the points of the exterior of the figure. The region in the interior of the two conical surfaces is regarded as empty. While the splitting operation $G \to G'$ is not continuous, or even one-to-one, if regarded as an operation in the 3-sphere, it is rather easy to see that it induces a homeomorphism between G and G'; the obvious correspondence $G \leftrightarrow G'$ is one-to-one, and is continuous both ways. The point is that when a circle in D_2 is split into two parallels of latitude J_2, J_2', these sets are still joined by an arc J_1.

Each circle J_2 or J_2' is the boundary of a plane disk. To get the space G'', we map each such disk onto a point, by a mapping $\phi : S^3 \to S^3$ which is a homeomorphism except on the union of the disks (that is, except on the closed interior of the union of the two cones.) Obviously G' and G'' are homeomorphic, because ϕ induces a one-to-one continuous mapping $G' \leftrightarrow G''$.

It is now easy to see that the arcs in G'' can be mapped onto points by a mapping which is one-to-one elsewhere in S^3. Therefore G is homeomorphic to S^3.

3. A weaker form of the monotonic mapping theorem

For the sake of convenience, we state a weaker form of the Monotonic Mapping Theorem, incorporating into it some of the apparatus to be used in the proof. Sections 3 through 10 will be devoted to the proof of Theorem 3.1. In the rest of the paper, we shall show f can be chosen in such a way that each set $f^{-1}(P)$ is a point or a linear graph.

THEOREM 3.1. *Let M be a triangulated 3-manifold, and suppose that M is compact, connected and simply connected. Then there are subcomplexes K and D of a subdivision of the 3-sphere S^3, a subcomplex L of a subdivision of M, and a mapping*

$$f : S^3 \to M$$

of S^3 onto M, such that

(1) *$M - L$ is an open 3-cell,*
(2) $\dim L = 2$,
(3) $\dim K \leqq 2$,
(4) *$f \mid K$ is simplicial,*
(5) *$f(K)$ is the 1-skeleton L^1 of L,*
(6) *f is monotonic,*
(7) *$f \mid (S^3 - K)$ is one-to-one,*
(8) $f(K) \cap f(S^3 - K) = 0$,
(9) $f(D) = L$,
(10) *for each 2-simplex τ^2 of L there is exactly one 2-simplex σ^2 of D such that $f \mid \sigma^2$ is a simplicial homeomorphism of σ^2 onto τ^2.*

The complex L is of a familiar type. If we represent M in the usual way as a singular 3-cell with singularities only on its boundary, then L is the image of the boundary. K is like the set $D_1 \cup D_2$ in Bing's example. Note, however, that under the conditions of the theorem, 2-simplices of K may be mapped onto points. Note also that while Bing's $D_1 \cup D_2$ is contractible, Theorem 3.1 tells us nothing at all about the topology of K, except that its dimension is $\leqq 2$. (Obviously $K \cup D$ must be contractible: $M - L$ is an open 3-cell,

$$f(S^3 - [K \cup D]) = M - L,$$

and f is a homeomorphism except on K. Therefore $S^3 - [K \cup D]$ is an open 3-cell, and its complement $K \cup D$ is contractible.)

4. The topological contraction cell

If A is an n-manifold with boundary, then Int A denotes the interior of A, that is, the set of all points of A that have open neighborhoods U in A, homeomorphic to Euclidean n-space E^n. The "intrinsic boundary" A − Int A of A is denoted by Bd A. If A is a subset of a space S, then FrA is the boundary (or frontier) of A relative to S, that is, Cl $(A) \cap$ Cl $(S - A)$.

Given a 3-manifold M as in Theorem 3.1, we first represent M as a singular

3-cell with singularities only on its boundary. That is, we define a mapping

$$\phi : \sigma^3 \to M$$

of a 3-simplex onto M, such that (1) ϕ is simplicial, relative to M and a subdivision of σ^3 and (2) $\phi \mid \text{Int } \sigma^3$ is a homeomorphism. It follows, of course, that ϕ maps no edge or 2-face of Bd σ^3 onto a point, and that the 2-simplices of the subdivision of Bd σ^3 are identified in pairs by the mapping ϕ. Let

$$L = \phi(\text{Bd } \sigma^3).$$

After a suitable subdivision, this L will be the L of Theorem 3.1.

(Such a ϕ and L can be constructed by the following well known process. Let σ^3 be any 3-simplex of M, let $N = M - \text{Int } \sigma^3$, and let $\phi_1 : \sigma^3 \to \sigma^3$ be the identity. Inductively, suppose that we have given a piecewise linear mapping $\phi_i : \sigma^3 \to M_i$ of σ^3 onto a set M_i which is the union of some or all of the 3-simplices of M, such that $\phi_i \mid \text{Int } \sigma^3$ is a homeomorphism. If M_i is not all of M, then there is a 3-simplex τ^3 of which does not lie in M_i but has a 2-face τ^2 in common with Fr M_i. There is therefore a piecewise linear mapping $\psi : M_i \to M_i \cup \tau^3$, such that if $\phi_{i+1} = \psi\phi_i$, then $\phi_{i+1} \mid \text{Int } \sigma^3$ is a homeomorphism. Let k be the number of 3-simplices in M. Then ϕ_k is the ϕ that we were looking for.)

For each i, let

$$N_i = M - \phi_i (\text{Int } \sigma^3).$$

Then

$$N_1 = N = M - \text{Int } \sigma^3.$$

And if we carry out the above process in the usual way, then at each stage we have

$$N_{i+1} = N_i - \text{Int } \tau^3 \cup \text{Int } \tau^2.$$

Therefore N_{i+1} is a retract of N_i. By induction on i it follows that

PROPOSITION 4.1 *L is a retract of N.*

PROPOSITION 4.2. *N is contractible on itself to a point.*

Proof. This is obtainable by standard methods, as follows. By hypothesis, we know that the fundamental group $\pi(M)$ is $= 0$. It follows that the 1-dimensional homology group $H^1(M)$ (with integers as coefficients) is also $= 0$, because $H^1(M)$ is isomorphic to the factor group of $\pi(M)$ by its commutator subgroup. (See [ST, p. 173].) By the Poincaré Duality Theorem [ST, p. 245] it follows that $H^2(M) = 0$. Since $\pi(M) = 0$, it follows that M is orientable [ST, p. 206], so that $H^3(M)$ is isomorphic to the group $\mathbf{Z}$ of integers. Since M is connected, $H^0(M)$ is obviously isomorphic to $\mathbf{Z}$.

Similarly, $H^0(N) \approx \mathbf{Z}$. It is readily verifiable that $\pi(N) = 0$, because $M = N \cup \sigma^3$, and $N \cap \sigma^3$ is the 2-sphere Bd σ^3. Therefore $H^1(N) = 0$. We assert, finally, that $H^2(N) = 0$.

Proof. Let Z^2 be a 2-cycle on N. Then $Z^2 \sim 0$ on M, so that Z^2 is homologous on N to a 2-cycle Y^2 on Bd N. Since $\mathrm{H}^3(N) = 0$, and $H^3(M) \approx \mathbf{Z}$, it follows by the Mayer-Vietoris Theorem that every 2-cycle which generates $H^2(\mathrm{Bd}\ N)$ is homologous to zero not only on σ^3 but also on N. Therefore

$$Z^2 \sim Y^2 \sim 0 \quad \text{on } N,$$

which was to be proved.

This means that N satisfies the hypothesis of the classical contractibility theorem of W. Hurewicz [H_2]; and the proposition follows.

By the preceding two propositions we have immediately:

PROPOSITION 4.3. *L is contractible on itself to a point.*

We recall that L was defined as

$$L = \phi(\mathrm{Bd}\ \sigma^3),$$

where

$$\phi : \sigma^3 \to M$$

was a singular 3-cell with singularities only on its boundary. Let us now think of the domain of definition of ϕ as the closure $\mathrm{Cl}\ (S^3 - B)$ of the complement of a 3-simplex B in the 3-sphere. Thus we have a piecewise linear mapping

$$\phi : \mathrm{Cl}\ (S^3 - B) \to M,$$
$$: \mathrm{Bd}\ B \to L,$$

such that $\phi \mid (S^3 - B)$ is one to one. Since L is contractible, the mapping $\phi : \mathrm{Bd}\ B \to L$ can be extended to give a mapping $B \to L$. Thus we have the following:

PROPOSITION 4.4. *There is a 3-simplex B in the 3-sphere, and a mapping*

$$\phi : S^3 \to M$$

such that

(1) *$\phi \mid (S^3 - B)$ is one-to-one,*
(2) *$\phi \mid \mathrm{Bd}\ B$ is simplicial, relative to a suitable triangulation of B,*
(3) *$\phi(B) \cap \phi(S^3 - B) = 0$, and*
(4) *$\phi(B) = L$.*

We might have added that (5) $\phi \mid (S^3 - B)$ is piecewise linear. But this fact will not be needed, and will not be preserved under geometric operations soon to be performed.

5. The relative simplicial approximation theorem

Given a mapping

$$\phi \mid S^3 \to M,$$

as in Proposition 4.4, it follows from Zeeman's relative simplicial approximation theorem [Z] that there is a mapping

$$\Phi : S^3 \to M,$$

such that (1) $\Phi \mid (S^3 - B) = \phi \mid (S^3 - B)$, (2) $\Phi(B) = L$, and (3) Φ is simplicial (relative to M and a suitable subdivision of S^3. To sum up:

THEOREM 5.1. *There is a simplex B in the* 3-*sphere, and a mapping*

$$\Phi : S^3 \to M$$

such that

(1) $\Phi \mid (S^3 - B)$ *is one-to-one,*
(2) $\Phi \mid B$ *is simplicial* (*relative to subdivisions of B and M*),
(3) $\Phi(B) \cap \Phi(S^3 - B) = 0$, *and*
(4) $\Phi(B) = L$.

Hereafter, when we speak of a simplex of B, M or L, we shall mean a simplex of one of the subdivisions referred to in condition (2).

6. The operation α and the definitions of f, K and D

Consider the union W of two 3-simplices σ^3, τ^3 whose intersection is a face σ^2 of each of them. Suppose that we have a mapping

$$\psi : W \to X,$$

of W onto a subcomplex X of M, such that

$$\psi \mid \tau^3 \text{ is one-to-one,}$$

$$\psi \mid \sigma^3 \text{ is simplicial,}$$

$$\psi(v_3) = \psi(v_4), \text{ and}$$

$$\psi \mid \sigma^2 \text{ is one-to-one.}$$

(Here the condition that $\psi \mid \tau^3$ be one-to-one is not as restrictive as it looks; in practice, under the scheme now to be described, σ^3 will be a simplex of the complex K on which a given mapping fails to be one-to-one, and σ^2 will lie in Fr K. We then take v_0 as we please, close to the barycenter of σ^2, in the complement of K.)

Under these conditions, the sets $\psi^{-1}(x)$ $(x \in X)$ are (1) the points of $\tau^3 - \sigma^2$, (2) the points of v_1v_2 and (3) infinitely many linear segments in σ^3, one of these being v_3v_4 and the others being parallel to v_3v_4.

Obviously X is a 3-cell, and

$$\text{Bd } X = \psi \text{ Bd } W.$$

Now the sets

$$\psi^{-1}(x), \qquad x \in \text{Bd } X$$

form a hyperspace in Bd W; and this hyperspace (under the natural topology)

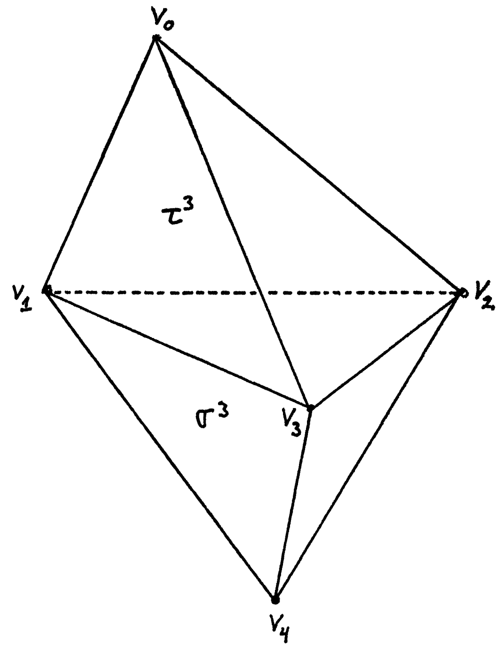

FIGURE 2
$W = \sigma^3 \cup \tau^3$

is a 2-sphere. In fact, it is easy to see that there is a mapping

$$\rho : W \to \tau^3$$

of W onto τ^3, such that

(1) $\rho(v_1 v_3 v_4) = v_1 v_3$
(2) $\rho(v_2 v_3 v_4) = v_2 v_3$
(3) $\rho(v_1 v_2 v_4) = v_1 v_2 v_3$, linearly, and
(4) $\rho \mid \mathrm{Cl}(\mathrm{Bd}\ \tau^3 - \sigma^2)$ is the identity.

To get such a mapping, we mash σ^3 against σ^2 and slightly past σ^2, allowing the image to protrude slightly into τ^3.

Now let

$$\psi' : W \to X$$

be defined by the condition

$$\psi' = \psi\rho.$$

When we replace ψ by ψ', the effect is to delete Int σ^3 from the set on which ψ fails to be one-to-one. The operation α is the operation which replaces ψ by ψ'. Thus

$$\alpha\psi = \psi' : W \to X = \psi(W).$$

Starting with the mapping Φ given by Theorem 5.1, we shall construct a new mapping by repeated applications of the operation α.

Let σ_1^2 be a 2-simplex of Bd B. Then σ_1^2 is a face of exactly one 3-simplex σ^3 of B; $\Phi \mid \sigma_1^2$ is simplicial and one-to-one; $\Phi(\sigma_1^2) = \Phi(\sigma^3)$; and obviously there is a 3-simplex τ^3, with σ_1^2 as a face, such that

$$\tau^3 \cap B = \sigma_1^2 .$$

We now apply the operation α. This gives a mapping

$$\Phi' = \alpha\Phi : S^3 \rightarrow M.$$

And we have added Int σ^3 to the set on which Φ is one-to-one.

Let

$$B_1 = B - (\text{Int } \sigma^3 \cup \text{Int } \sigma_1^2).$$

Then B_1 is not necessarily a manifold with boundary. But there is a 2-simplex σ_2^2 of Fr B_1 such that

$$\Phi'(\sigma_2^2) = \tau^2 = \Phi(\sigma_1^2).$$

If σ_2^2 lies in a 3-simplex σ_2^3 of B_1, we repeat the operation α, so as to delete Int $\sigma_2^3 \cup$ Int σ_2^2 from B_1. In a finite number of such steps, we get a complex B_n, a mapping

$$\Phi_n : S^3 \rightarrow M,$$

and a 2-simplex σ_n^2 of B_n, such that Φ_n is a simplicial homeomorphism of σ_n^2 onto τ^2, and σ_n^2 lies in no 3-simplex of B_n. Here σ_n^2 is one of the two 2-simplices of Bd B which are mapped onto τ^2 by Φ. Of course, $\Phi_n \mid (S^3 - B_n)$ is a homeomorphism; this follows by an easy induction. Note also that B_n contains σ_n^2.

We do this for every 2-simplex τ^2 of L. Given τ^2, there are always exactly two 2-simplices of Bd B which are mapped onto τ^2; we choose one of them, repeat the above process, and get a σ^2 which is mapped onto τ^2 and which lies in the interior of the set on which the new mapping is one to one. Let the final mapping thus obtained be f, and let D be the complex whose simplices are the 2-simplices σ^2 and their faces. Let B_p be the "ultimate B_n", consisting of all simplices remaining in B after the operations just performed. Thus $B_p = D \cup K$, where K is the set of all simplices of B_p other than the σ^2's. Note that it is not necessarily true that $f(K) \cap f(S^3 - K) = 0$, because K may contain 3-simplices σ^3 such that $f(\sigma^3) = \tau^2 \in L$. The properties of f, K, and D are described in the following propositions.

PROPOSITION 6.1. *$K \cup D$ is a subcomplex of a subdivisoin of S^3 and $f \mid (K \cup D)$ is simplicial.*

(Because $K \cup D$ is a subcomplex of B, and $f \mid (K \cup D) = \Phi \mid (K \cup D)$.)

PROPOSITION 6.2. *$f(D) = L$. And for each $\tau^2 \in L$ there is exactly one $\sigma^2 \in D$ such that f maps σ^2 simplicially onto τ^2.*

By construction.

PROPOSITION 6.3. *$f \mid (S^3 - K)$ is one-to-one.*

By induction.

PROPOSITION 6.4. $f(\text{Bd } K) \cap f(S^3 - K) = 0$.

By induction.

PROPOSITION 6.5. $f(K) \subset L$.

Because $f \mid K = \Phi \mid K$, and $K \subset B$.

PROPOSITION 6.6. *$f \mid \text{Fr } K$ is monotonic.*

This calls for a proof. We recall that

$$\Phi : \text{Cl } (S^3 - B) \to M$$

can be regarded as an identification mapping, representing M as a singular 3-cell with singularities only on its boundary. We got f from Φ by a sequence of operations α. Thus we have a sequence

$$\Phi, \Phi_1, \Phi_2, \cdots, \Phi_p = f;$$

and we have a corresponding sequence of complexes

$$B, B_1, B_2, \cdots, B_p = K \cup D.$$

Let $C = \text{Cl } (S^3 - B)$; and for each i let

$$C_i = \text{Cl } (S^3 - B_i).$$

Let ξ_i be the identification mapping on C_i which identifies two points x and y of C_i if (1) $\Phi_i(x) = \Phi_i(y)$, and this point lies in the interior of a 2-simplex of L or (2) x and y lie in the same *component* of the same set

$$\Phi_i^{-1}(z) \cap \text{Fr } C_i \qquad (z \in L).$$

We define ξ similarly for C. This gives a sequence of spaces

$$\xi C, \xi_1 C_1, \xi_2 C_2, \cdots, \xi_p C_p.$$

We assert that ξC is a 3-manifold, homeomorphic to M. The proof is as follows. We know by rule (1) that in the interiors of the 2-simplices of Bd C, ξ performs all the identifications performed by Φ. Since $\{\Phi_i^{-1}(z)\}$ forms an upper-semicontinuous collection, so also does $\{\Phi_i^{-1}(z) \cap \text{Fr } C_i\}$; and since the union of the latter sets is compact, it follows that the set of all their components forms an upper-semicontinuous collection. We see by continuity that for each σ_1^2, σ_2^2 in Bd C, $\xi(\sigma_1^2) = \xi(\sigma_2^2)$ if and only if $\Phi(\sigma_1^2) = \Phi(\sigma_2^2)$. But when a 3-manifold is represented by making identifications on the boundary of a 3-cell, edge—and vertex identifications are made if and only if they are consequences (by continuity) of the 2-face-identifications. It follows that for *points* x, y, $\xi(x) = \xi(y)$ if and only if $\Phi(x) = \Phi(y)$.

But it is also easy to see, by a re-examination of the operation α, that $\xi_{i+1} C_{i+1}$ is homeomorphic to $\xi_i C_i$ for each i. Therefore $\xi_p C_p$ is a 3-manifold.

Now

$$\begin{aligned} C_p &= \text{Cl } (S^3 - B_p) \\ &= \text{Cl } [S^3 - (K \cup D)] \\ &= \text{Cl } (S^3 - K). \end{aligned}$$

Consider the identification mapping ξ' on C_p, defined by the condition that $\xi'(x) = \xi'(y)$ if $f(x) = f(y)$. Then $\xi' C_p$ is a 3-manifold, because $\xi' C_p$ is homeomorphic to M. If ξ' performed any additional identifications, not performed by ξ_p, then these additional identifications would apply to the 1-dimensional set ξ_p Fr (K), and so they would destroy the property of being a 3-manifold. Therefore $\xi_p = \xi'$, and so each set $f^{-1}(z) \cap \text{Fr } K$ has only one component, which was to be proved.

PROPOSITION 6.7. *f, K and D can be chosen in such a way that if v is a vertex of L, then $S^3 - f^{-1}(v)$ is connected.*

(From this it can be shown that *every* set $S^3 - f^{-1}(z) (z \in M)$ is connected. But we shall not need this fact.)

Proof. Suppose that for the given f, some set $S^3 - f^{-1}(v)$ is not connected. Some one component U of $S^3 - f^{-1}(v)$ contains $S^3 - K$. Let V be the union of all the others. Then Cl (V) forms a subcomplex of K, because Fr V does. We now define a new mapping

$$f' : S^3 \to M$$

by providing that

$$f' \mid (S^3 - V) = f \mid (S^3 - V)$$

and

$$f'(V) = f(v).$$

In a finite number of such steps we obtain the desired f.

Thus we have an f, K, D satisfying the conditions of Propositions 6.1—6.7. Let n be the number of 3-simplices of K. The next few sections will be devoted to the proof of the fact that if f, K and D satisfy these conditions, and *are chosen so as to minimize n*, then $n = 0$ and $\dim K \leqq 2$. This will complete the proof of Theorem 3.1, because in this case Fr $K = K$.

Essentially, the proof is constructive; the geometric operations described below can be used to eliminate the 3-simplices of a given K, one at a time. The notation is simpler, however, if we avoid the problem of giving names to the objects which appear in the intermediate stages.

7. The operations β, γ and δ

Consider the union W of two 3-simplices σ^3, τ^3 whose intersection σ^2 is a face of each of them. (See Figure 2.) Suppose that we have a mapping

$$\psi : W \to X,$$

such that $\psi(\sigma^3)$ is a point and $\psi \mid (\tau^3 - \sigma^2)$ is a homeomorphism. Evidently the hyperspace formed by the sets $\psi^{-1}(x)$ is a 3-cell.

It follows that there is a mapping $\psi' : W \to X$, such that $\psi' \mid \text{Bd } W = \psi \mid \text{Bd } W$ and $\psi' \mid \text{Int } W$ is a homeomorphism. When we replace ψ by ψ', the effect is to delete Int σ^3 from the set on which ψ fails to be one-to-one. The operation β is the operation which replaces ψ by ψ'. Thus

$$\beta\psi = \psi' : W \to X = \psi(W).$$

PROPOSITION 7.1. *If f, K and D satisfy the conditions of Propositions* 6.1–6.7, *and n is minimal, then K does not contain a* 3-*simplex σ^3, with a* 2-*face σ^2 in* Fr K, *such that $f(\sigma^3)$ is a point.*

Proof. If there were such a σ^3, we could reduce n by the operation β. We need to verify, of course, that β preserves the conditions of Propositions 6.1–6.7; but all these verifications are trivial.

Consider now

$$W = \sigma^3 \cup \tau^3,$$

as before, with

$$\sigma^3 \cap \tau^3 = \sigma^2.$$

Suppose that we have a mapping

$$\psi : W \to X.$$

$\psi(v_2 v_3 v_4)$ is a point, $\psi \mid \sigma^3$ is simplicial, $\psi(v_1) \neq \psi(v_2)$, and $\psi \mid (\tau^3 - \sigma^2)$ is a homeomorphism. Thus the sets $\psi^{-1}(x)$ are (1) the points of $\tau^3 - \sigma^2$ (2) v_1 and (3) an infinite collection of 2-simplices in planes parallel to the plane of $v_2 v_3 v_4$. As before, X is a 3-cell. Now let $H^{\cdot}$ be the space whose points are (1) the points of Int W and (2) the sets $\psi^{-1}(x) \cap \text{Bd } W$. Then H is a 3-cell. It follows (as in the definition of β above) that $\psi \mid \text{Bd } W$ has an extension

$$\psi' : W \to X$$

such that $\psi \mid \text{Int } W$ is a homeomorphism. Let

$$\gamma\psi = \psi'.$$

PROPOSITION 7.2. *If f, K and D satisfy the conditions of Propositions* 6.1–6.7, *and n is minimal, then K does not contain a* 3-*simplex $\sigma^3 = v_1 v_2 v_3 v_4$ such that $v_1 v_2 v_3 \in$* Fr K *and f maps v_1 and $v_2 v_3 v_4$ onto two different points.*

Proof. If there were such a σ^3, then n could be reduced by the operation γ. (As before, we verify trivially that γ preserves the conditions of Theorems 6.1–6.6.)

Consider next $W = \sigma^3 \cup \tau^3$ and $\psi : W \to X$; and suppose that (1) $\psi \mid (\tau^3 - \sigma^2)$ is one-to-one, (2) $\psi \mid \sigma^3$ is simplicial and (3) $\psi(v_1 v_3)$ and $\psi(v_2 v_4)$ are two different points. The sets $\psi^{-1}(x)$ are then (1) the points of $\tau^3 - \sigma^2$, (2) $v_1 v_3$, (3) $v_2 v_4$ and (4) an infinite collection of quadrilateral regions lying in parallel planes. In the figures, we show two quadrilateral regions $\psi^{-1}(x)$, one lying close to $v_1 v_3$ and the other lying close to $v_2 v_4$.

As in the preceding cases, the mapping $\psi \mid \text{Bd } W$ has an extension

$$\psi' : W \to X,$$

such that $\psi' \mid \text{Int } W$ is one to one. The verification is entirely analogous to the preceding ones. Let

$$\delta\psi = \psi'.$$

PROPOSITION 7.3. *If f, K and D satisfy the conditions of Propositions* 6.1–6.6, *and n is minimal, then K does not contain a* 3*-simplex $\sigma^3 = v_1 v_2 v_3 v_4$ such that $v_1 v_2 v_3 \in \text{Fr } K$ and f maps $v_1 v_3$ and $v_2 v_4$ onto two different points.*

The proof is like the preceding ones.

8. The operations ϵ and α'

If we think of the proof of the Monotonic Mapping Theorem as a sequence of operations which replace a given mapping by a monotonic one, it is plain that not much of consequence has happened so far: α, β, γ and δ give monotonic mappings only when monotonic mappings were given to them. Under the conditions of Theorem 5.1, it is quite possible that some components of some

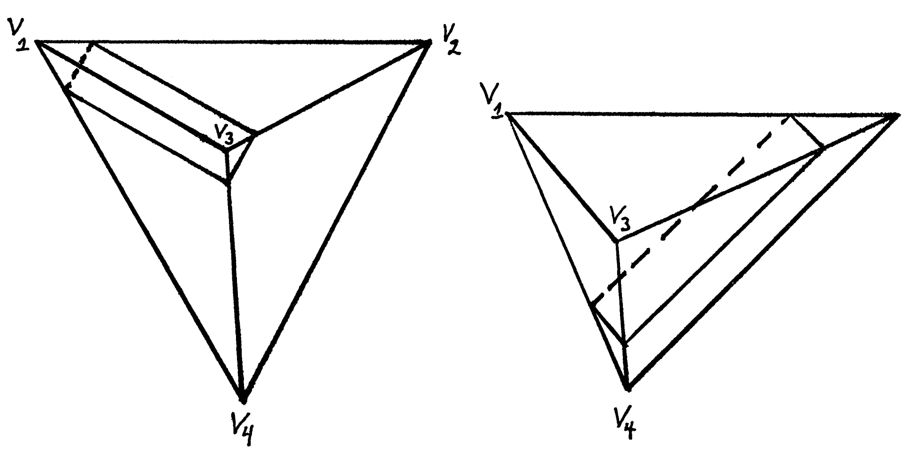

FIGURE 3

sets $\Phi^{-1}(x)$ lie entirely in Int B; and if this is true, it remains true after any number of applications of α, β, γ, and δ. In this section we describe a method of eliminating such components.

Consider $W = \sigma^3 \cup \tau^3$, as before. (See Figure 2.)

Suppose that (1) $\sigma^3 \in K$, (2) $\sigma^2 = v_1 v_2 v_3 \in \text{Fr } K$, (3) $f(v_1) = f(v_2)$ and (4) $f \mid v_1 v_3 v_4$ and $f \mid v_2 v_3 v_4$ are one-to-one. This means, of course, that f maps σ^3 simplicially onto a 2-simplex ρ^2 of L.

We assume further that (5) f maps $v_0 v_2 v_3$ simplicially onto ρ^2, (6) $f \mid (\tau^3 - \sigma^2)$ is one-to-one, and (7) $v_0 v_1 \notin K \cup D$.

Here condition (5) implies that $v_0 v_2 v_3 \in D$. We know that there is a simplex of D which is mapped simplicially onto ρ^2; and since $f \mid (S^3 - K)$ is one-to-one, this simplex must be $v_0 v_2 v_3$.

These are the hypotheses for the operation ε. Note that (5) is a very strong and special hypothesis. In the following section we shall show how one can get along without it.

The first stage in the operation ε is a sort of simplified inverse of the operation α. By (7), there is a polyhedral 3-cell E, containing $v_0 v_1 v_2$, such that

$$(\text{Bd } E) \cap v_0 v_1 v_2 = v_1 v_2 \cup v_0 v_2 = E \cap (K \cup D).$$

Let $\psi = f \mid E$. Then there is a mapping $\psi' : E \to f(E)$, such that (i) $\psi' \mid \text{Bd } E = \psi \mid \text{Bd } E$, (ii) ψ' maps $v_0 v_1 v_2$ simplicially onto a 1-simplex, and (iii) $\psi \mid (E - v_0 v_1 v_2)$ is a homeomorphism. The operation α' replaces ψ by ψ', leaving f unchanged on $S^3 - E$.

The next stage is to replace the resulting mapping by a mapping f' which maps τ^3 simplicially onto ρ^2. We get such an f' by applying the inverse α^{-1} of the α defined in Sec. 6.

Now let v be any point of the interior of σ^2; and let W' be the subdivision of W in which v is the only new vertex. We define a new mapping f'' by the following conditions:

$$f'' \mid \text{Cl } (S^3 - W) = f' \mid \text{Cl } (S^3 - W),$$

$$f''(v) = f(v_0)(=f(v_4)), \text{ and}$$

$$f'' \mid W' \text{ is simplicial.}$$

It may be easier to see what is happening here if we draw 2-dimensional figures. We started with a situation whose 2-dimensional analogue looks like Figure 4. Here the concentric circles in the annulus are mapped onto points; and the annulus and the vertical segment are mapped by f onto the same 1-simplex. The first step is to introduce a new 2-simplex (see Figure 5). This shows inverse-images under f'. Next we get f'', for which the inverse image sets look like this (see Figure 6). Intuitively speaking, what we have done is to dig a hole in K so that components of sets $f^{-1}(x)$ which were buried in Int K can get access to Fr K.

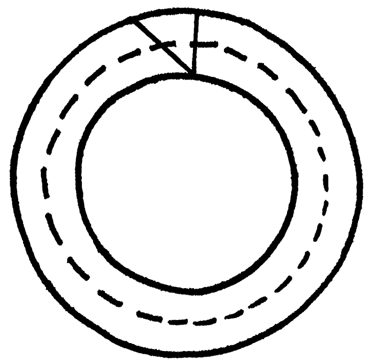

FIGURE 4

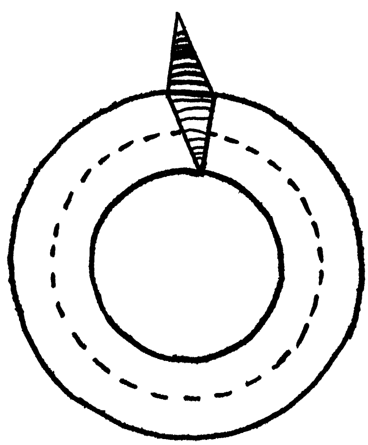

FIGURE 5

Now let

$$K' = (K - \sigma^3) \cup W';$$

let

$$\omega_1 = vv_0 v_2 v_3 ;$$

let

$$\omega_2 = vv_2 v_3 v_4 ;$$

and given ω_i, let ω_{i+1} be the 3-simplex of K' such that (1) $f''(\omega_{i+1}) = \rho^2$ and (2) $\omega_{i+1} \cap \omega_i$ is a 2-simplex whose image is also ρ^2, and (3) $\omega_{i+1} \neq \omega_{i+1}$, if such an ω_{i+1} exists. Obviously this process terminates, with a certain ω_p ; and ω_p must be $vv_0 v_1 v_3$. The reason is that ω_p has a 2-face, lying in Fr K', which is mapped simplicially by f'' onto ρ^2; only two 3-simplices of K' have this property, one of them being ω_1 and the other being $vv_0 v_1 v_3$.

We now eliminate $\omega_1, \omega_2, \cdots, \omega_p$ from K', in the *reverse* of the stated order,

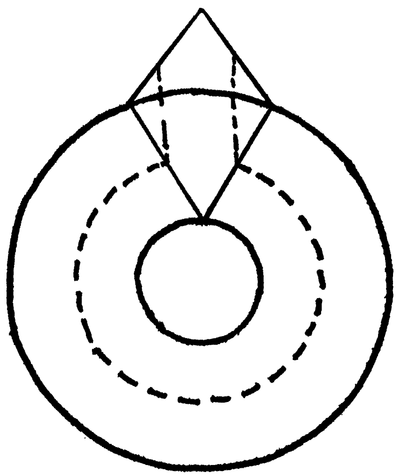

FIGURE 6

by repeated applications of the operation α. This gives a new mapping $f^{(3)}$ and a new "singularity complex"

$$K'' = K' - \{\omega_1, \omega_2, \cdots, \omega_p\}.$$

Note that since we eliminated the ω_i's in reverse order, the new D is the same as the old one.

Thus we have eliminated σ^3 and other 3-simplices from K. But we have added to K the 3-simplices $vv_0 v_1 v_2$ and $vv_4 v_1 v_2$. We get rid of these, in the order named, by two applications of the operation δ. The final result is a mapping satisfying all the conditions of Propositions 6.1–6.7, for which the associated complex K has fewer 3-simplices than the given one.

The only non-trivial verification required is that $f^{(3)} \mid \text{Fr } K''$ is monotonic. The only points where this condition might fail are the points y of Int $v_2 v_3$. But it is easy to see, inductively, that each such y is joined to the corresponding $y' \in \text{Int } v_1 v_3$ by a broken line in Fr $K'' \cap \bigcup \text{Bd } \omega_i$.

The total operation just described is ε. If n is minimal, then the hypotheses for this operation must not be satisfied. Thus we have the following:

PROPOSITION 8.1. *If f, K and D satisfy the conditions of Propositions* 6.1–6.6, *and n is minimal, then there do not exist* 3*-simplices* $\sigma^3 = v_1 v_2 v_3 v_4$, $\tau^3 = v_0 v_1 v_2 v_3$ *such that*

(1) $\sigma^3 \in K$,
(2) $\sigma^2 = v_1 v_2 v_3 \in \text{Fr } K$,
(3) $f(v_1) = f(v_2)$,
(4) $f \mid v_1 v_3 v_4$ *and* $f \mid v_2 v_3 v_4$ *are one-to-one,*
(5) f *maps* $v_0 v_2 v_3$ *simplicially onto* $f(v_1 v_3 v_4)$ *and*
(6) $f \mid (\tau^3 - \sigma^2)$ *is one-to-one, and*
(7) $v_0 v_1 \notin K \cup D$.

In the following section, we shall refer to conditions (1)–(7) as *the hypothesis for* ε.

9. A reduction of the hypothesis for ε

To apply the operation ε to a 3-simplex $\sigma^3 = v_1 v_2 v_3 v_4$, we needed to know that there was a 2-simplex $v_0 v_2 v_3$ of D, in exactly the right position, such that $f(v_0 v_2 v_3) = \rho^2 = f(v_2 v_3 v_4)$. Under the conditions for f, K and D in Sec. 6, all that we know is that there is *some* 2-simplex τ^2 of D which is mapped simplicially onto ρ^2. Thus we need to show that τ^2 can be moved into the position required for the operation ε. What we need is the following:

PROPOSITION 9.1. *Given f, K and D, satisfying the conditions of Propositions 6.1–6.6, and a 3-simplex $\sigma^3 = v_1 v_2 v_3 v_4$ with a 2-face $\sigma^2 = v_1 v_2 v_3$, satisfying conditions (1)–(4) of the hypothesis for ε. Then there exist f', K', D', satisfying the same conditions, such that the 3-simplices of K' are those of K, and such that f', K', D', and σ^3 satisfy the entire hypothesis for ε.*

Proof. We recall that S^3 has a triangulation T in which $K \cup D$ forms a subcomplex. We subdivide this T by introducing, as new vertices, the barycenters of the 2-faces and 3-simplices of T that do not lie in K. Let T' be the resulting subdivision of T. Then K is a subcomplex of T', but D is not; the latter creates a slight technical problem, to be taken care of presently. Note that every simplex of T' intersects K in a simplex (or in the empty set.)

Let $\tau_1^3 = v_0 v_1 v_2 v_3$ be the 3-simplex of T' which intersects σ^3 in $\sigma^2 = v_1 v_2 v_3$. Let G be the complex formed by all 3-simplices τ of T, not lying in K, such that $\tau \cap K$ is a 1- or 2-simplex $w_0 w_1$ or $w_0 w_1 w_2$ such that $f(w_0 w_1) = f(v_2 v_3)$ (or $f(w_0 w_1 w_2) = f(v_2 v_3)$). Then the 3-simplices of G are arranged in a natural cyclic order

$$\tau^3 = \tau_1^3, \tau_2^3, \cdots, \tau_p^3,$$

such that for each i, $\tau_{i-1}^3 \cap \tau_i^3$ is a 2-simplex τ_i^2, not lying in K, but having an edge τ_i^1 such that $f(\tau_i^1) = f(v_1 v_3)$. To see this, let $\tau_1^2 = v_0 v_1 v_3$, $\tau_1^1 = v_1 v_3$, $\tau_2^2 = v_0 v_2 v_3$, $\tau_2^1 = v_2 v_3$. Let τ_2^3 be the other 3-simplex of T' (that is, the one not mentioned so far) that contains τ_2^2. If $\tau_2^3 \cap K = \tau_2^1$, let $\tau_3^1 = \tau_2^1$; if $\tau_2^3 \cap K$ is a 2-simplex τ^2, let τ_3^1 be the other edge of τ^2 for which $f(\tau_3^1) = f(\tau_2^1)$ $(= f(\tau_1^1))$; in either case, let τ_3^2 be the 2-face of τ_2^3 which contains τ_3^1 but does not lie in K or in τ_1^3, and let τ_3^3 be the other 3-simplex of T' that contains τ_3^2. Inductively, this defines a sequence τ_1^3, τ_2^3, $\cdots$. The sequence ultimately repeats, with $\tau_{p+1}^3 = \tau_1^3$ for some (minimal) p. Evidently each set $f(\tau_i^3)$ is a 3-cell, because each set $\tau_i^3 \cap K$ is an edge or 2-simplex τ in $\text{Bd}\, \tau_i^3$, and $f(\tau) = f(v_1 v_3)$. And each set $f(\tau_i^3)$ $(i > 1)$ intersects the union of its predecessors in a disk, namely, the disk $f(\tau_i^2)$. It follows that $\bigcup_{i=1}^p f(\tau_i^3)$ is a 3-cell, whose interior contains $\text{Int}\, f(v_1 v_3)$. Since $M = f(S^3)$ is locally Euclidean, $\text{Int} \bigcup_{i=1}^p f(\tau_i^3)$ is open in M; and this means that $\bigcup_{i=1}^p \tau_i^3$ is all of G.

Now let d be a 2-simplex of D such that $f(d)$ contains the edge $f(\sigma^2)$ of L. Since $\bigcup \tau_i^3$ is all of G, it follows that some τ_{k+1}^2 lies in d.

LEMMA 9.1.1. *If none of the simplices* $\tau_1^2, \tau_2^2, \cdots, \tau_k^2$ *lie in* D, *then there are objects* f', K', D', *satisfying the conclusion of Proposition* 9.1, *such that* (1) $f'(\tau_1^2) = f(d)$, (2) $K' \cup D'$ *is a subcomplex of* T', *and* (3) $D' \cap \bigcup_{i=1}^{k+1} \tau_i^2 = \tau_1^2$.

Proof of lemma. Let $d = w_0 w_1 w_2$, where $w_1 w_2 \in K$ and $w_0 \notin K$; and let w be the barycenter of d, so that $\tau_{k+1}^2 = w w_1 w_2$. By two applications of the operation α', defined in the preceding section, we can get a mapping f_1, such that (1) f_1 agrees with f except in a small neighborhood of Int d, (2) $f_1(w) = f_1(w_0) = f(w_0)$, and (3) $f \mid w w_0 w_1$ and $f \mid w w_0 w_2$ are linear. Thus we have added $w w_0 w_1$ and $w w_0 w_2$ to K, and replaced d by τ_{k+1}^2 in D.

We repeat this operation, in exactly the same form, for each 2-simplex d' of D which contains a 2-simplex τ_i^2. Finally, we repeat it for the other 2-simplices of D. This gives a new mapping f_2, and a new complex D_2, having the stated properties of D, such that D_2 is a subcomplex of T'.

There are now two cases to consider.

Case 1. $\tau_k^3 \cap K$ is a 2-simplex. Let $\tau_k^3 = w x_1 x_2 x_3$, with $x_1 x_2 x_3 \in K$, $f_2(x_1) = f_2(x_2)$, $f_2(x_1 x_2 x_3) = f(v_2 v_3)$. By one application of α', we can get a mapping f_3 such that (1) f_3 agrees with f_2 except in a small neighborhood of Int $w x_1 x_2 \cup$ Int $w x_2$ and (2) $f_3 \mid w x_1 x_2$ is linear. Thus we have added $w x_1 x_2$ to K. By one application of the operation α^{-1}, we can get a mapping f_4 such that (1) f_4 agrees with f_3 except in a small neighborhood of Int $\tau_k^3 \cup$ Int $w x_1 x_3$ and (2) $f_4 \mid \tau_k^3$ is linear. By one application of α, we can get a mapping f_5 such that (1) f_5 agrees with f_4 except in a small neighborhood of Int $\tau_k^3 \cup$ Int $w x_2 x_3$, (2) $f_5 \mid$ Int τ_k^3 is one-to-one, and (3) $f_5(w x_1 x_3) = f_4(w x_2 x_3)$.

But $w x_2 x_3 = \tau_{k+1}^2 \subset d$, and $w_1 x_3 = \tau_k^2$. Thus the effect of our operations so far has been to replace d by τ_k^2 in D.

Case 2. $\tau_k^3 \cap K$ is a 1-simplex. Let $\tau_k^3 = w w_1 x_2 x_3$, with $\tau_{k+1}^2 = w x_2 x_3$, $\tau_k^2 = w_1 w_2 w_3$, $f(x_2 x_3) = f(v_2 v_3)$. The method here is precisely analogous to that used in Case 1: first we incorporate $w w_1 x_2$ and $w w_1 x_3$ into K (by two applications of α') and then we replace τ_{k+1}^2 by τ_k^2 in D (by α^{-1}, followed by α).

In k steps of this kind, we can replace d by τ_1^2 in D, which is what we wanted in the conclusion of the lemma.

We now conclude the proof of Proposition 9.1. If the d of the lemma is such that $f(d) = f(\sigma^3)$, then Proposition 9.1 follows immediately from the lemma. If not, we apply the lemma to d, thus "moving d to the position τ_1^2"; we then subdivide T', just as we subdivided T, getting a complex T''; we form a *new* sequence $\tau_1^3, \tau_2^3, \cdots, \tau_q^3$ of 3-simplices of T'', and apply the lemma to the first τ_{i+1}^2 that lies in a simplex of D. Since D is a finite complex, this process terminates, giving a mapping of the sort desired in the conclusion of Proposition 9.1.

10. Proof of Theorem 3.1: conclusion

Consider now f, K, and D, satisfying the conditions of Propositions 6.1–6.7, such that the number n of 3-simplices of K is minimal.

Suppose that K contains a 3-simplex; and let K^3 be the complex consisting of the 3-simplices of K and their faces.

(1) If $\sigma^2 \in \text{Fr } K^3$, then $f(\sigma^2)$ is not a 2-simplex. (If it were, $f \mid \text{Fr } K$ could not be monotonic.)

(2) If $\sigma^2 \in \sigma^3 \in K^3$, and $\sigma^2 \in \text{Fr } K^3$, then $f(\sigma^2)$ is not a 1-simplex.

Proof. If σ^3 is mapped onto the same 1-simplex, then n can be reduced by one of the operations γ, δ. If $f(\sigma^3)$ is a 2-simplex, then n can be reduced by Proposition 9.1 and the operation ε.

(3) It follows from (1) and (2) that every 2-simplex of Fr K^3 is mapped into a point. Let

$$V = \text{Fr } (S^3 - K^3),$$

and let W be a component of V. Then W is the union of a finite number of 2-simplices of Fr K^3; and since W is connected, $f(W)$ is a point. If $\sigma^2 \in V$, and $\sigma^2 \in \sigma^3 \in K^3$, then $f(\sigma^3)$ cannot be the point $f(\sigma^2)$, because n could then be reduced by operation β. On the other hand, $f(\sigma^3)$ cannot be a 1-simplex, because then $f^{-1}f(\sigma^2)$ would separate S^3, which contradicts Proposition 6.7.

Therefore the assumption $K^3 \neq 0$ is false, and $\dim K \leqq 2$. As indicated at the end of Sec. 6, this is sufficient to complete the proof of Theorem 3.1.

11. First modification of the f of Theorem 3.1

The f and K given by Theorem 3.1 satisfy all the conditions of the Monotonic Mapping Theorem, except that some of the inverse-image sets $f^{-1}(x)$ may be 2-dimensional. It remains, therefore, to get a mapping for which all inverse-image sets are linear graphs.

PROPOSITION 11.1. *There is a subcomplex K' of a subdivision of S^3, and a mapping*

$$f' : S^3 \to M,$$

such that

(1) $f' \mid (S^3 - K')$ *is one-to-one,*

(2) $f' \mid K'$ *is piecewise linear,*

(3) $f'(K') \cap f'(S^3 - K') = 0,$

(4) f' *is monotonic and*

(5) *every set $f'^{-1}(x)$ is either a point or the union of a linear graph and a 3-manifold with boundary.*

Proof. Step 1. Let σ^2 be a 2-simplex of the K of Theorem 3.1, such that $f(\sigma^2)$ is a point. (It follows, of course, that $f(\sigma^2)$ is a vertex of L.) Let σ^3 be a 3-simplex such that $\sigma^3 \cap K = \sigma^2$ and σ^2 is a face of σ^3; let

$$\beta = \text{Cl } (\text{Bd } \sigma^3 - \sigma^2);$$

and let

$$\phi : \beta \rightarrow \sigma^2$$

be a piecewise linear homeomorphism of β onto σ^2, such that $\phi \mid \text{Bd } \beta$ is the identity. We define $\phi \mid \sigma^2$ to be the identity. Then ϕ can be extended to give a piecewise linear mapping

$$\phi : \text{Cl }(S^3 - \sigma^3) \rightarrow S^3 \quad \text{(onto)},$$

such that $\phi \mid (S^3 - \sigma^3)$ is one-to-one. For each $p \in S^3 - \sigma^3$, let

$$g(p) = f\phi(p);$$

and let

$$g(\sigma^3) = f(\sigma^2).$$

Then $g \mid (K \cup \sigma^3)$ is piecewise linear.

We perform this process for each $\sigma^2 \in K$ for which $f(\sigma^2)$ is a point; for each σ^2, we let $\sigma^3 = v\sigma^2$, where v is very close to the barycenter of σ^2; and so different 3-simplices σ_i^3, σ_j^3 intersect one another only where they must, in the corresponding sets $\sigma_i^2 \cap \sigma_j^2$. But K is a finite complex. Therefore, in a finite number of such steps (one for each such σ^2), we get an f_1, K_1 which satisfy (1)–(4) of Proposition 11.1 and also

(5′) Every set $f_1^{-1}(x)$ is a point, a linear graph, or a finite union of linear graphs and 3-simplices which intersect one another only in edges and vertices.

Step 2. Let e be an edge of a 3-simplex of K_1 which is mapped onto a point by f_1, and let V be the union of all 3-simplices of K_1 that have e as an edge. Thus

$$V = \sigma_1^3 \cup \sigma_2^3 \cup \cdots \cup \sigma_n^3,$$

where the σ_i^3's are listed in the cyclic order in which they appear around e in S^3. Then V is not a neighborhood of Int e in S^3, because no two 3-simplices of K_1 have a 2-face in common. But for each pair σ_i^3, σ_{i+1}^3 there is a polyhedral 3-cell Σ such that $\Sigma \cap V$ is a polyhedral disk d_1, lying in Bd $\sigma_i^3 \cup$ Bd σ_{i+1}^3, containing Int e in its interior, and such that Σ intersects K_1 only in d_1. Let

$$d_2 = \text{Cl }(\text{Bd } \Sigma - d_1),$$

and let ϕ be a piecewise linear homeomorphism d_2 onto d_1, such that $\phi \mid \text{Bd } d_2$ is the identity. We define $\phi \mid d_1$ as the identity. Then ϕ can be extended to give a piecewise linear mapping

$$\phi : \text{Cl }(S^3 - \Sigma) \rightarrow S^3 \quad \text{(onto)},$$

such that $\phi \mid (S^3 - \Sigma)$ is one-to-one. For each $p \in S^3 - \Sigma$, let

$$g(p) = f_1\phi(p);$$

and let

$$g(\Sigma) = f_1(d_1).$$

Then $g \mid (K_1 \cup \Sigma)$ is piecewise linear. In a finite number of such steps we get an f_2, K_2 which satisfy (1)–(4) of Theorem 1 and also

(5″) Every set $f_2^{-1}(x)$ is a finite polyhedron. This polyhedron is a point, or a linear graph, or the union of a linear graph and a set in which all but a finite number of points have 3-cell neighborhoods.

Under condition (5″), if $v \in f_2^{-1}(x)$, and U is a small convex polyhedral neighborhood of v in S^3, then $f_2^{-1}(x) \cap \text{Bd } U$ is the union of a finite set and a 2-manifold with boundary (the latter being not necessarily connected.) Let F_x be the union of the 3-simplices in $f_2^{-1}(x)$. Then (a) $F_x \cap U$ is empty, or (b) $F_x \cap U$ is a 3-cell, or (c) $F_x \cap \text{Bd } U$ is not connected, or (d) $\text{Bd } U - F_x$ is not connected. If (a) or (b) hold, we have no problem. And (c) and (d) hold, at most, at a finite number of points v, because such a v must be a vertex of $f_2^{-1}(x)$. Steps 3 and 4 below apply in cases (c) and (d) respectively.

Step 3. If (c) holds at v, then there is a polyhedral disk d, containing v in its interior, intersecting $f_2^{-1}(x)$ only at v, and separating S^3 locally into two connected sets each of which intersects $f_2^{-1}(x)$. If d is taken in general position, then d will intersect each set $f_2^{-1}(y)$ only in isolated points. We shall think of S^3 as Euclidean 3-space E^3, compactified at infinity. We may then assume that d is a 2-simplex in a horizontal plane, since the given d can be mapped onto such a simplex by a piecewise linear homeomorphism of S^3 onto itself. (We recall that f_2 is supposed to be merely piecewise linear, and not necessarily simplicial.) Let σ_1^3 and σ_2^3 be 3-simplices such that $\sigma_1^3 \cap \sigma_2^3 = d$, and such that v lies on the linear segment joining the fourth vertices of σ_1^3 and σ_2^3. Let

$$d_1 = \text{Cl } (\text{Bd } \sigma_1^3 - d),$$

and let

$$d_2 = \text{Cl } (\text{Bd } \sigma_2^3 - d).$$

Let

$$\phi : \text{Cl } (S^3 - W) \to S^3 \quad \text{(onto)}$$

be a piecewise linear mapping such that (1) $\phi \mid (S^3 - W)$ is one-to-one, (2) $\phi \mid \text{Bd } d$ is the identity, (3) $\phi \mid d_1$ is the vertical projection of d_1 onto d and (4) $\phi \mid d_2$ is the vertical projection of d_2 onto d.

We now define a new mapping $g : S^3 \to M$, as follows:

(1) If $p \in \text{Cl } (S^3 - W)$, then

$$g(p) = f_2 \phi(p).$$

(2) If p lies on a vertical segment xx' ($x \in d_1$, $x' \in d_2$), then $g(p) = g(x)$.

Consider now the points x of d_1 for which $\phi(x)$ is in K. The set of all such points forms a polyhedral linear graph A, and thus forms a subcomplex of a triangulation of d_1. If τ^2 is a 2-simplex of such a triangulation of d_1, and

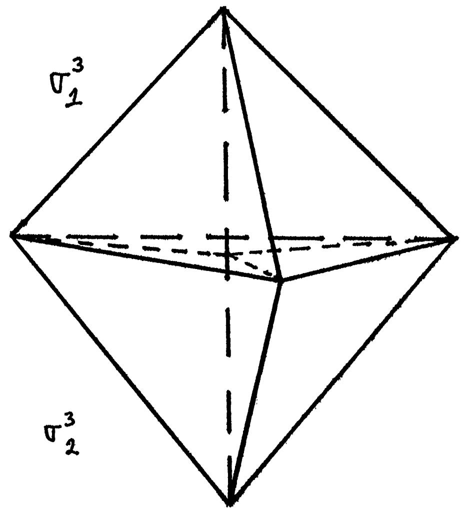

FIGURE 7
$W = \sigma_1^3 \cup \tau_2^3$

$y \in \operatorname{Int} \tau^2$, then $g^{-1}g(y) = yy'$ can be eliminated by repeated applications of the operation α.

When we replace f_3 by g, we get a new "singularity complex" K_g, on which g is piecewise linear, and we have reduced by 1 the number of points at which (c) holds. In a finite number of such steps we obtain an f_3, K_3 which satisfy (1)–(4) and also

(5‴) If $v \in f_3^{-1}(x)$, then v satisfies (a), (b), or (d).

Step 4. If $v \in f_3^{-1}(x)$, and v satisfies (d), then there is a polyhedral disk d, with v in its interior, such that

$$d - v \subset f_3^{-1}(x) - \operatorname{Fr} f_3^{-1}(x)$$

and such that d separates S^3 locally into two connected sets each of which intersects $\operatorname{Fr} f_3^{-1}(x)$.

As before, we suppose that d is a simplex lying in a horizontal plane; we take

$$W = \sigma_1^3 \cup \sigma_2^3,$$

d_1, d_2 and ϕ as in Step 3; and we define a new mapping

$$g : S^3 \to M$$

by the following conditions

(1) If $p \in \operatorname{Cl}(S^3 - W)$, then

$$g(p) = f_3\phi(p).$$

(2) $g(W) = f_3(d)$.

In a finite number of such steps, we get an f', K' of the sort described in Proposition 11.1.

12. Fox's Theorem. An unknotting process

The following theorem has been proved by Ralph H. Fox [F_2]:

THEOREM (Fox). *Let W be a polyhedral 3-manifold with boundary, in S^3. Then there is a piecewise linear homeomorphism ϕ, of W into S^3, such that* $\mathrm{Cl}\,[S^3 - \phi(W)]$ *is a tube.*

Here by a tube we mean a set T which is homeomorphic to a regular neighborhood of a polyhedral linear graph. This is equivalent to the statement that T contains a finite collection $d_1, d_2, \cdots, d_k$ of disjoint polyhderal disks, such that $\mathrm{Bd}\, d_i \subset \mathrm{Bd}\, T$ for each i, such that the closure of every component of $T - \bigcup d_i$ is a c-cell, and such that no set $\mathrm{Bd}\, d_i$ separates $\mathrm{Bd}\, T$.

A trivial illustration of the process involved in Fox's theorem is the case in which W is a knotted tube and ϕ maps W onto an unknotted tube. Obviously very non-trivial cases can occur.

Given f' and K' as in Proposition 11.1, let V be the union of all 3-simplices lying in sets $f'^{-1}(x)$, and let $W = \mathrm{Cl}\,(S^3 - V)$. We apply Fox's Theorem to this W, getting a mapping

$$\phi : W \to S^3$$

such that the set

$$T = \mathrm{Cl}\,[S^3 - \phi(W)]$$

is a tube. We now define the mapping

$$f'' : S^3 \to M$$

by the conditions

(1) $f'' \mid \phi(W) = f\phi^{-1}$,
(2) if A is a component of T, then

$$f''(A) = f''(\mathrm{Bd}\, A).$$

Thus we can rewrite Proposition 11.1, with condition (5) in a stronger form, as follows:

PROPOSITION 12.1. *There is a subcomplex K of a subdivision of the 3-sphere, and a mapping*

$$f : S^3 \to M$$

such that

(1) $f \mid (S^3 - K)$ *is one-to-one,*
(2) $f \mid K$ *is piecewise linear,*
(3) $f(K) \cap f(S^3 - K) = 0$,
(4) f *is monotonic and*
(5) *every set $f^{-1}(x)$ is a point, a linear graph or the union of a linear graph and a tube.*

Thus, to complete the proof of the Monotonic Mapping Theorem, we need to reduce to linear graphs the tubes mentioned in (5), and we need to make $f \mid K$ simplicial, rather than merely piecewise linear.

13. Conclusion

Let T be a polyhedral tube, such that Bd T lies in a set Fr $f^{-1}(x)$, as in Proposition 12.1. Let d be a (polyhedral) disk in T, with Bd $D \subset$ Bd T, as in the definition of a tube, at the beginning of Sec. 12, so that d does not separate T. We may assume that d is a convex polyhedral disk lying in a plane E, since this situation can be obtained by a piecewise linear homeomorphism of S^3 onto itself. And if d is in general position, then E will intersect K, in the neighborhood of d, in the union of d and a 1-dimensional set.

It is now an elementary matter to show that there is a mapping

$$\phi : S^3 \to S^3,$$

such that $\phi \mid (S^3 - d)$ is one-to-one, $\phi(d)$ is a point, and $\phi \mid K$ is piecewise linear. This gives us a new $K' = \phi(K)$, and a new mapping

$$f' = f\phi^{-1}.$$

We can now "pull $f'^{-1}f'(d)$ apart at $\phi(d)$," by the process used in Step 3 of the proof of Proposition 11.1. This reduces the 1-dimensional Betti number of T. Thus, in a finite number of such steps, we get a mapping f_1 and a complex K_1, satisfying (1)–(4) of Proposition 12.1 and also

(5′) Every set $f_1^{-1}(x)$ is a point, a linear graph, or a finite union of linear graphs and disjoint polyhedral 3-cells.

We can now define a mapping

$$\psi : S^3 \to S^3$$

such that $\psi \mid K_1$ is piecewise linear, ψ maps every 3-cell in $f^{-1}(x)$ onto a point, and ψ is one-to-one except on the union of these 3-cells. Let $K_2 = \psi(K_1)$, and let

$$f_2 = f_1 \psi^{-1}.$$

Then all of the sets $f_2^{-1}(x)$ are points or linear graphs. It remains only to show that f_2 is simplicial relative to a suitable subdivision of K_2.

We know that for every simplex σ of K_2, $f_2 \mid \sigma$ is linear, though not necessarily simplicial. For each vertex v of K_2, the set $f_2^{-1}f_2(v)$ is a linear graph. Let V be the union of these graphs. Then V decomposes each $\sigma^2 \in K_2$ into 2-simplices and quadrilateral regions. Decomposing each of the latter into two 2-simplices, using either diagonal, we get a subdivision relative to which $f_2 \mid K_2$ is simplicial.

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HARVARD UNIVERSITY
CAMBRIDGE, MASSACHUSETTS