A PATHOLOGICAL FIBER SPACE[1]

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1. Introduction

The following result is a consequence of the work of Dyer and Hamstrom [1]. Suppose K, X and Y are metric spaces, K compact X complete, the dimension of Y is equal to n and the space of homeomorphisms of K onto itself (*c-o* topology) is LC^{n+1}. Then if f is a completely regular mapping from X to Y such that the inverse of each point is homeomorphic to K, then f is locally trivial.

Since it is conjectured that the space of homeomorphisms of a manifold is locally connected in all dimensions, the above theorem gives rise to the question as to whether the local connectivity of the space of homeomorphisms could be replaced by assuming that K is a manifold or an absolute retract.

In [3] McAuley conjectured: Suppose that (E, p, B) is a Serre fibration and that E and B are finite-dimensional Peano continua. Then if each fiber is homeomorphic to a fixed Peano continuum, p is locally trivial.

In this paper an example is given which would answer the first question negatively for K an absolute retract and the example also shows that McAuley's conjecture is false even for Hurewicz fibrations.

2. Definitions

(2.1) A map p from a metric space E onto a metric space B is *completely regular* if given any $b \in B$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that if $b_1 \in B$ and $d(b, b_1) < \delta$ then there exists a homeomorphism from $p^{-1}(b)$ onto $p^{-1}(b_1)$ which moves no point as much as ε.

(2.2) A map p from a space E onto a space B is a *Hurewicz fibration* if the mapping

$$p^* : E^I \to Z = \{(e, f) \in E \times B^I \mid p(e) = f(0)\}$$

defined by $p^*(g) = (g(0), pg)$ admits a section.

It should be noted that if p is a Hurewicz fibration then p has the absolute covering homotopy property.

(2.3) A mapping p from a space E onto a space B is *locally trivial* if there exists a space F such that for each $b \in B$, there is an open neighborhood U of

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[1] This paper contains part of a doctoral dissertation written under the direction of Professor Louis McAuley while the author held an Academic Year Extension of the Research Participation Program for College Teachers.

[2] The author would like to express his appreciation to the referee whose suggestions considerably shortened the proofs of (3.3) and (3.5).

b in B together with a homeomorphism $\varphi_U : U \times F \to p^{-1}(U)$ of $U \times F$ onto $p^{-1}(U)$ satisfying $p\varphi_U(u, x) = u$ for $u \in U, x \in F$.

3. The example

The example in this paper is that of a Hurewicz fibration which is also a completely regular map. The total space is a two-dimensional absolute retract, the base space is the unit interval, and the fibers are one-dimensional absolute retracts. However this mapping is not locally trivial.

(3.2) *Description of example.* Let C be the usual Cantor set on [0, 1]. Let E be the following subset of Euclidean three space R^3:

$$E = \{(x, y, z) \in R^3 \mid z = 0 \text{ and } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\} \cup \{(x, y, z) \in R^3 \mid y = -cx + c \text{ and } 0 \leq z \leq (1/3^n)(1 - x) \text{ where } c \in C - \{0 \cup 1\} \text{ and } c = a/3^n \text{ is in reduced form}\} \cup \{(x, y, z) \in R^3 \mid y = (c - 1)x + 1 \text{ and } 0 \leq z \leq (1/3^n)x \text{ where } c \in C - \{0 \cup 1\} \text{ and } c = a/3^n \text{ is in reduced form}\}$$

Define $p : E \to I$ (the unit interval) by $p(x, y, t) = x$ (see Figure 1).

The function p is continuous. The total space E is easily seen to be a 2-dimensional absolute retract. If $t \in I$, then $p^{-1}(t)$ is homeomorphic to the following subset F of R^2:

$$F = \{(x, y) \in R^2 \mid x \in C - \{0 \cup 1\} \text{ and } 0 \leq y \leq (1/3^i) \text{ where } x = (a/3^i) \text{ in reduced form}\} \cup \{(x, y) \in R^2 \mid y = 0 \text{ and } 0 \leq x \leq 1\}$$

(See Figure 2).

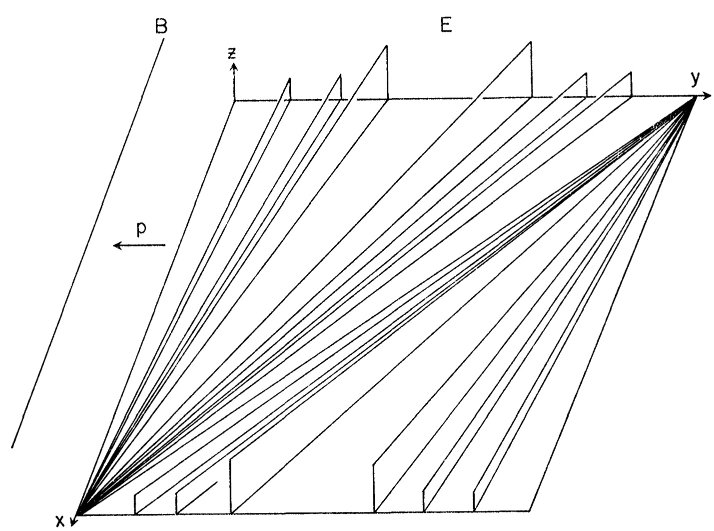

FIGURE 1

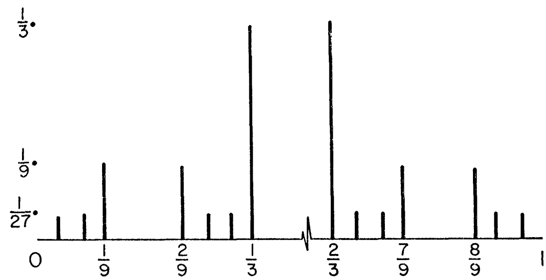

FIGURE 2

It is easily seen then that F is a one-dimensional absolute retract and that p is completely regular.

(3.3) The map p is not locally trivial.

Proof. This is an immediate consequence of the fact that if p is locally trivial it is trivial (i.e. a product) since the base is contractible.

The following lemma suggested by the referee will be used to show that p is a Hurewicz fibration.

(3.4) LEMMA. *Suppose p' is a Hurewicz fibration from E' to B, p is a map from E to B and r' is a fiber-preserving retract from E' to E. Then p is a Hurewicz fibration.*

Proof. Trivial.

(3.5) The map p is a Hurewicz fibration.

Proof. Let

$$E' = \{(x, y, z) \in R^3 \mid 0 \leq x \leq 1, 0 \leq y \leq 1, \text{ and } 0 \leq z \leq 1\}.$$

Define $p' : E' \rightarrow B$ by $p'(x, y, z) = x$ and note there is a fiber-preserving retraction of E' onto E. Hence by (3.4) the proof is complete.

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