CARTER SUBGROUPS AND FITTING HEIGHTS OF FINITE
SOVLABLE GROUPS

BY

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Let $G$ be a finite solvable group having Fitting height $h$ (as defined in [7]
or in §1 below). Let $H$ be a Carter subgroup of $G$ and $l$ be the length of a
composition series of $H$. We shall establish the correctness of a conjecture
of John Thompson (at the end of [7]) by proving that

$$h < 10(2^l - 1) - 4l.$$  

This is the result of Theorem 8.5 below, and the rest of this paper is a proof
of that theorem.

The upper bound for $h$ given by (0.1) is almost certainly too large. The
work of Shamash and Shult [6] leads one to conjecture that there is some
constant $K$ such that

$$h \leq Kl,$$

for all finite solvable groups $G$. The methods of this paper unfortunately
cannot give an upper bound whose order of magnitude is less than $2^l$. This
is caused by our very naive approach. Essentially we choose a normal
subgroup $P$ of prime order in $H$ and a suitable chain $A_1, \cdots, A_k$ of $H$-in-
variant sections of $G$. Obviously either $P$ centralizes $A_1, \cdots, A_{[h/2]}$ or
there exists a subchain $A_k, A_{k+1}, \cdots, A_{k+[h/2]}$ such that $P$ does not cen-
tralize $A_k$. In the latter case we construct (and this is the hard part of the
proof) an $H$-invariant chain $D_{k+j}, D_{k+j+1}, \cdots, D_{k+[h/2]}$ of sections of
sections $A_{k+j}, A_{k+j+1}, \cdots, A_{k+[h/2]}$ (respectively) such that $j$ is bounded and $P$
centralizes each $D_i$. In either case we obtain a chain of length "almost"
h/2 of sections of $G$ on which $H/P$ acts, and which satisfies suitable axioms so
that the process can be repeated (using a normal subgroup of prime order in
$H/P$, etc.) Obviously no method based on this process can give an upper
bound smaller than $2^l$.

There are many technical complications in the proof due to the difficulty
of handling the case $|P| = 3$ (among other things). But basically it is a
straightforward application of the methods of Hall and Higman [3]. The
few new concepts which are used are grouped together in Sections 1, 2 and 3.
They are the notions of Fitting chains (which are the "correct" chains of
sections $A_1, \cdots, A_n$ of $G$), of weak equivalence (which is used in place of
equivalence in Fitting chains because it is impossible to verify the latter after

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a complicated construction), of ample representations (which are just the
ones which are "good" in the Hall-Higman theory) and of the class $\mathcal{C}$ of groups
(which contains all the useful special groups and is closed under formation
of non-trivial sections). There is also something called an augmented Fitting
chain which was just introduced to handle the case $|P| = 3$.

The titles of the sections indicate pretty well the outline of the argument.
§1 is the obligatory list of notations. In §2 we introduce Fitting chains and
state the basic theorems we shall prove about them. In §3 we prove some
elementary facts about ample representations. In §4 we study closely a
certain situation of two steps in a chain in which a non-ample representation
appears. From this we conclude (in Theorem 4.20) that ample representa-
tions always appear in our chains after a bounded number of steps. Then we
show in §5 that, knowing we have ample representations in one step of our
chain, we can find "enough" ample representations at the next step. The
arguments here break down if $|P| = 3$. But in that case we have an aug-
mented Fitting chain. In §6 we use the additional structure to find "enough"
ample representations when $|P| = 3$. In §7, we put the results of the pre-
ceding sections together, add a few new ones, and prove the basic theorems of
§2. Finally in §8 we prove (0.1) from the established results of §2.

1. Notation

Let $G$ be any finite group. We denote by

- $Z(G)$ the center of $G$,
- $G'$ the derived group of $G$,
- $\Phi(G)$ the Frattini subgroup of $G$ (i.e., the intersection of all maximal subgroups
  of $G$),
- $F(G)$ the Fitting subgroup of $G$ (i.e., the largest normal nilpotent subgroup
  of $G$),
- $\text{Aut}(G)$ the automorphism group of $G$.

The Fitting series $F_n(G)$, $n = 0, 1, 2, \ldots$, is defined inductively by

$$F_0(G) = \{1\}$$

$$F_n(G) = \text{the inverse image in } G \text{ of } F(G/F_{n-1}(G)), \quad \text{for } n \geq 1.$$  

Evidently each $F_n(G)$ is a characteristic subgroup of $G$. If $G$ is solvable,
then there is some integer $h \geq 0$ such that $F_h(G) = G$. We call the least
such integer $h$ the Fitting height of $G$ and denote it by $h(G)$.

If each $S_i$, $i = 1, \ldots, k$, is an element or a subset of $G$, then $\langle S_1, \ldots, S_k \rangle$
will denote the subgroup of $G$ generated by $S_1, \ldots, S_k$.

If $\sigma, \tau \in G$, then we define

$$\sigma^\tau = \tau^{-1} \sigma \tau$$

$$[\sigma, \tau] = \sigma^{-1} \tau^{-1} \sigma \tau = \sigma^{-1} \sigma^\tau.$$
For any \( \tau_1, \ldots, \tau_n \in G \) and any integers \( a_1, \ldots, a_n \), we define

\[
g^{a_1 \tau_1 + \cdots + a_n \tau_n} = (g^{a_1})^{\tau_1} (g^{a_2})^{\tau_2} \cdots (g^{a_n})^{\tau_n}, \quad \text{for all } \sigma \in G.
\]

Thus \( [\sigma, \tau] = g^{-1+\tau} \), for all \( \sigma, \tau \in G \). If \( \rho_1, \ldots, \rho_m \) are also elements of \( G \), then we define

\[
g^{(a_1 \rho_1 + \cdots + a_n \rho_n)(b_1 \rho_1 + \cdots + b_m \rho_m)} = [g^{a_1 \rho_1 + \cdots + a_n \rho_n}]^{b_1 \rho_1 + \cdots + b_m \rho_m}, \quad \text{for all } \sigma \in G.
\]

Thus

\[
[\sigma, \tau, \rho] = [[\sigma, \tau], \rho] = g^{(\rho_1^{\sigma})^{-1}} \rho, \quad \text{for all } \sigma, \tau, \rho \in G.
\]

Obviously this definition can be repeated to define \( g^{f_1 \cdots f_t} \), where each \( f_i \) has the form \( a_1 \tau_1 + \cdots + a_n \tau_n \), for some integers \( a_1, \ldots, a_n \) and some elements \( \tau_1, \ldots, \tau_n \) of \( G \).

If \( A, B \) are two subgroups of \( G \), then \( [A, B] \) will denote the subgroup generated by all \( [\sigma, \tau] \), where \( \sigma \in A \), \( \tau \in B \). We define \( [A, B]^n \), for all integers \( n \geq 0 \), by

\[
[A, B]^0 = A, \quad [A, B]^n = [(A, B)^{n-1}, B], \quad \text{for } n > 0.
\]

Thus \( [A, B]^2 = [A, B, B] = [[A, B], B] \).

By \( A \leq G \) we mean "\( A \) is a subgroup of \( G \)" as opposed to \( A \subseteq G \), which means "\( A \) is a subset of \( G \)". By \( A \trianglelefteq G \) we mean "\( A \) is a normal subgroup of \( G \)".

A section of \( G \) is a factor group \( A/B \) where \( B \trianglelefteq A \leq G \). The section \( A/B \) equals another section \( C/D \) if and only if \( A = C \) and \( B = D \). A subgroup \( E \) of \( G \) covers the section \( A/B \) if \( (E \cap A)B = A \) and avoids \( A/B \) if \( E \cap A = E \cap B \).

If \( G \) is solvable, then \( l(G) \) is defined to be the length of a composition series of \( G \). If we write the order \( |G| \) as a product of (not necessarily distinct) primes: \( |G| = p_1 \cdots p_l \), then \( l(G) = l \).

If \( G \) is a non-trivial \( p \)-group, for some prime \( p \), we write \( p = p(G) \).

Let \( F \) be any field. We denote by \( F[G] \) the group algebra of \( G \) over \( F \). By an "\( F[G] \)-module", we understand a right \( F[G] \)-module on which the identity of \( F[G] \) acts as the identity transformation and which is finite-dimensional as a vector space over \( F \).

If \( V \) is an \( F[G] \)-module and \( H \) is any subgroup of \( G \), then \( V_H \) will denote the restriction of \( V \) to an \( F[H] \)-module. If \( U \) is any \( F[H] \)-module, then \( U^0 \) will denote the \( F[G] \)-module induced from \( U \).

An \( F[G] \)-module \( V \) is trivial if \( G \) centralizes it. It is completely reducible if it is a direct sum of irreducible \( F[G] \)-submodules. If \( |G| \) is relatively prime to the characteristic of \( F \), then every \( F[G] \)-module is completely reducible.

For any \( F[G] \)-module \( V \), there exists some \( F[G] \)-composition series \( \{0\} = V_0 < V_1 < \cdots < V_n = V \). We call the composition factors \( V_i/V_{i-1} \), \( i = 1, \ldots, n \), the irreducible \( F[G] \)-components of \( V \). Of course, these irreducible components are unique up to order and \( F[G] \)-isomorphism.
An $F[G]$-module $V$ is called primary if all of its irreducible $F[G]$-components are isomorphic to each other.

If $V$ is an $F[G]$-module and $H$ is a subgroup of $G$, then

$[V, H]$ is the $F$-subspace spanned by all $v(\sigma - 1), v \in V, \sigma \in H$.

$[V, H]^n = V$.

$[V, H]^n = [[V, H]^{n-1}, H]$, for all $n > 0$.

$C_V(H)$ is the $F$-subspace of all $v \in V$ such that $v\sigma = v$ for all $\sigma \in H$.

Evidently $[V, H], [V, H]^n$ and $C_V(H)$ are all $F[H]$-submodules of $V_H$.

For any integer $n \geq 1$ and any $F[G]$-module $V$, we define the $F[G]$-module $n \times V$ by:

$$n \times V = V \oplus \cdots \oplus V.$$ 

If $V$ is any $F[G]$-module, then the dual $F$-vector space $\text{Hom}_F(V, F)$ is made into the dual $F[G]$-module by

$$(f\sigma)(v) = f(v\sigma^{-1}), \quad \text{for all } f \in \text{Hom}_F(V, F), \quad \sigma \in G, \quad v \in V.$$ 


(1.1) If $U, V$ are weakly $F[G]$-equivalent $F[G]$-modules and $H \leq G$, then $U_H, V_H$ are weakly $F[H]$-equivalent.

Indeed, any non-trivial irreducible $F[H]$-component of $U_H$ must be $F[H]$-isomorphic to an $F[H]$-component of some non-trivial $F[G]$-component of $U$, and hence to an $F[H]$-component of $V$. Statement (1.1) follows immediately from this.

Another remark about weak equivalence has to do with field extensions:


An action of a group $K$ on a group $G$ will be a homomorphism of $K$ into $\text{Aut}(G)$. Since we seldom need consider two different actions of $K$ on $G$, we usually write "$(K$ on $G)$" to denote that action of $K$ on $G$ which is being considered at a given point in the argument. If $\sigma \in K$, then we write $\tau^\sigma$ for the image of $\tau \in G$ under the automorphism of $G$ which is the image of $\sigma$ in
Aut \( (G) \). We may always form the semidirect product \( KG \) in which \( \tau \) becomes \( \sigma^{-1} \tau \sigma \), for all \( \sigma \in K \), \( \tau \in G \). This enables us to define \( [G, K] \) and \( [G, K]^\sigma \) as usual. We may also define the centralizers \( C_\sigma(K) \) of \( K \) in \( G \) and \( C_\sigma(G) \) of \( G \) in \( K \). For the latter we usually use the alternative notation \( \text{Ker}(K \text{ on } G) = C_\sigma(G) \), since it is the kernel of the representation of \( K \) on \( G \) given by \( (K \text{ on } G) \).

We denote the image of \( K \) in \( \text{Aut}(G) \) by \( K_\alpha \). Often we also consider \( K_\alpha \) to be the section \( K/\text{Ker}(K \text{ on } G) \) of \( K \). This identification seldom causes confusion.

If \( G \) is an abelian group, we denote by \( G^+ \) the group \( G \) written additively. When \( G \) is an elementary abelian \( p \)-group (i.e., when \( G \) is abelian with prime exponent \( p \)), we make \( G^+ \) into a vector space over the field \( \mathbb{F}_p \) of \( p \) elements in the natural way. If another group \( K \) acts on \( G \), then \( G^+ \) becomes a \( \mathbb{F}_p[K] \)-module.

Suppose a group \( K \) acts on a finite solvable group \( G \). Then each \( K \)-composition factor \( A/B \) of \( G \) is an elementary abelian \( p \)-group, for some prime \( p \). So \( [A/B]^+ \) is an irreducible \( \mathbb{F}_p[K] \)-module, which we call an irreducible component of \( (K \text{ on } G) \). If \( K \) also acts on another finite solvable group \( H \), then \( (K \text{ on } G) \) and \( (K \text{ on } H) \) are weakly equivalent if each nontrivial irreducible component of \( (K \text{ on } G) \) is \( K \)-isomorphic to an irreducible component of \( (K \text{ on } H) \) and vice versa. Obviously this is an equivalence relation among \( K \)-groups. As in (1.1) we have

\[
(1.3) \quad \text{If } (K \text{ on } G) \text{ is weakly equivalent to } (K \text{ on } H) \text{ and } L \leq K, \text{ then } (L \text{ on } G) \text{ is weakly equivalent to } (L \text{ on } H),
\]

where, of course, the actions of \( L \) are restricted from those of \( K \).

Suppose that a group \( K \) acts on a group \( G \). A section \( A/B \) of \( G \) is \( K \)-invariant if both \( A \) and \( B \) are \( K \)-invariant subgroups of \( G \). We also say that "\( K \) normalizes \( A/B \)". In this case \( K \) acts naturally on the factor group \( A/B \). To say that a section \( C/D \) of \( K \) normalizes \( A/B \) means that \( C \) normalizes \( A/B \) and \( D \leq \text{Ker}(C \text{ on } A/B) \). Then \( C/D \) acts naturally on \( A/B \).

Let a group \( K \) act on a group \( G \) and another group \( L \) act on both \( K \) and \( G \). We say that \( (K \text{ on } G) \) is \( L \)-invariant if \( (\sigma') = (\sigma^\rho)^\tau \), for all \( \sigma \in G, \tau \in K, \rho \in L \). In that case we may form the "triple semi-direct product" \( LKG \).

If \( K \) acts on \( G \) and \( L \) acts on \( K \), then \( (K \text{ on } G) \) is weakly \( L \)-invariant if the actions \( (K \text{ on } G) \) and \( (K \text{ on } G)^\tau \), the latter given by

\[
\tau \rightarrow (K \text{ on } G)(\tau^{-1}) \quad \text{for} \quad \tau \in K,
\]

are weakly equivalent for all \( \sigma \in L \). We define weak \( L \)-invariance similarly for \( \mathbb{F}[K] \)-modules \( V \) over any field \( F \), using weak \( \mathbb{F}[K] \)-equivalence.

We define \( \mathcal{C} \) to be the family of all finite groups \( A \) satisfying:

\[
(1.4a) \quad A \text{ is a non-trivial } p\text{-group, for some prime } p.
(1.4b) \quad \Phi(A) \leq Z(A).
\]
(1.4c) \( \Phi(\Phi(A)) = \{1\} \).
(1.4d) If \( p \) is odd, then \( A \) has exponent \( p \).

Evidently all special groups \( A \) (in the sense of [3]) lie in \( \mathfrak{g} \) provided they are non-trivial and satisfy (1.4d). However \( \mathfrak{g} \) obviously has the following important property which special groups lack:

(1.5) Any non-trivial section \( B/C \) of a group \( A \in \mathfrak{g} \) also lies in \( \mathfrak{g} \).

If \( A \in \mathfrak{g} \), we define \( \bar{A} \) to be the \( Z_{p(A)} \)-vector space \([A/\Phi(A)]^+\). It follows easily from (1.4b) that the map \( f_A \) defined by

(1.6) \( f_A(\sigma\Phi(A), \tau\Phi(A)) = [\sigma, \tau] \), for all \( \sigma, \tau \in A \),

is an alternating, bilinear map of \( \bar{A} \times \bar{A} \) into \( \Phi(A)^+ \) (note that \( \Phi(A)^+ \) is also a \( Z()- \)vector space by (1.4c)). It is clear from (1.6) that the radical of \( f_A \) (i.e., the set of all \( \bar{\sigma} \in \bar{A} \) such that \( f_A(\bar{\sigma}, \bar{A}) = \{0\} \)) is precisely \([Z(A)/\Phi(A)]^+\).

2. Fitting chains

The simplest way of thinking about the Fitting height of a finite solvable group \( G \) is to consider chains \( A_1, \ldots, A_t \) of sections of \( G \) satisfying the following conditions:

(2.1a) Each \( A_i \), \( i = 1, \ldots, t \), is a non-trivial \( p_i \)-group, for some prime \( p_i \).
(2.1b) \( A_i \) normalizes \( A_{i+1} \), for \( i = 1, \ldots, t - 1 \).
(2.1c) \( \ker(A_i \text{ on } A_{i+1}) = \{1\} \), for \( i = 1, \ldots, t - 1 \).
(2.1d) \( p_i \neq p_{i+1} \) for \( i = 1, \ldots, t - 1 \).

It is easy to verify that the Fitting height \( h(G) \) is merely the maximum of the lengths \( t \) of all such chains of sections of \( G \) (see Lemma 8.2 below for part of the argument).

The basic idea behind our proof of Thompson's conjecture is that one should forget about the group \( G \) and consider only chains \( A_1, \ldots, A_t \) of groups, each acting on the next, which satisfy axioms similar to (2.1). From this point of view the Carter subgroup \( H \) of \( G \) becomes a group outside the chain acting on each \( A_i \) and leaving invariant each action \( (A_i \text{ on } A_{i+1}) \). Under certain conditions, which Carter subgroups and appropriate chains of sections of \( G \) can be shown to satisfy, we prove that the length \( t \) of such a chain must be bounded as a function of \( l(H) \).

To make this program more explicit, we first consider the axioms which our chains \( A_1, \ldots, A_t \) must satisfy. Obviously we want the groups \( A_i \) to have as uncomplicated a structure as possible. The Hall-Higman theory suggests that we take them to be special. However, the class of special groups is not closed under subgroups and epimorphic images, which makes it awkward to use in complicated constructions. So we choose the \( A_i \) instead from the class \( \mathfrak{g} \), which does have the desired closure properties by (1.5) and contains enough special groups for our purposes.
A little experimentation soon demonstrates that we cannot allow the actions \((A_{i-1} \text{ on } A_i)\) and \((A_i \text{ on } A_{i+1})\) to be completely independent of each other. It is tempting to make the representation \((A_i \text{ on } A_{i+1})\) invariant under \(A_{i-1}\). However, in practice this condition is much too difficult to verify after a construction. So we only insist that \((A_i \text{ on } A_{i+1})\) be weakly \(A_{i-1}\)-invariant, which turns out to be sufficient in general to establish what we need.

Another axiom suggested by the Hall-Higman theory is that \(A_i\) centralizes \(\Phi(A_{i+1})\). This condition turns out to be vital in many of our proofs.

Combining the above ideas, we define a Fitting chain to consist of groups \(A_1, \ldots, A_t\) and actions \((A_i \text{ on } A_{i+1})\) for \(i = 1, \ldots, t - 1\), satisfying:

\begin{align*}
(2.2a) & \quad A_i \in \alpha_i, \text{ for } i = 1, \ldots, t. \\
(2.2b) & \quad p(A_i) \neq p(A_{i+1}), \text{ for } i = 1, \ldots, t - 1. \\
(2.2c) & \quad \Phi(A_{i+1}), A_i = \{1\}, \text{ for } i = 1, \ldots, t - 1. \\
(2.2d) & \quad \text{Ker}(A_i \text{ on } A_{i+1}) = \{1\}, \text{ for } i = 1, \ldots, t - 1. \\
(2.2e) & \quad (A_{i+1} \text{ on } A_{i+2}) \text{ is weakly } A_i\text{-invariant, for } i = 1, \ldots, t - 2.
\end{align*}

Usually we speak of “the Fitting chain \(A_1, \ldots, A_t\)” leaving the actions \((A_i \text{ on } A_{i+1})\) to be understood.

Suppose that \(A_1, \ldots, A_t\) is a Fitting chain and that \(D_i\) is a section of \(A_i\), for \(i = 1, \ldots, t\). If the action of \(A_i\) on \(A_{i+1}\) induces an action of \(D_i\) on \(D_{i+1}\), for \(i = 1, \ldots, t - 1\), and if these actions make \(D_1, \ldots, D_t\) a Fitting chain, we say that \(D_1, \ldots, D_t\) is a Fitting subchain of \(A_1, \ldots, A_t\). Notice that some of the axioms (2.2) for \(D_1, \ldots, D_t\) are free by

\begin{align*}
(2.4a) & \quad D_i \neq \{1\}. \\
(2.4b) & \quad D_i \text{ normalizes } D_{i+1}, \text{ for } i = 1, \ldots, t - 1. \\
(2.4c) & \quad \text{Ker}(D_i \text{ on } D_{i+1}) = \{1\} \text{ for } i = 1, \ldots, t - 1. \\
(2.4d) & \quad (D_{i+1} \text{ on } D_{i+2}) \text{ is weakly } D_i\text{-invariant, for } i = 1, \ldots, t - 2.
\end{align*}

**Proof.** If \(D_1, \ldots, D_t\) is a Fitting subchain it certainly satisfies (2.4) by (2.2) and (1.4a).

Conversely, suppose that \(D_1, \ldots, D_t\) satisfies (2.4). Then (2.2a) and (2.4a) imply \(D_i \in \alpha_i\) by (1.5). Suppose we know that \(D_i \in \alpha_i\) for some \(i = 1, \ldots, t - 1\). Then \(D_i = \{1\}\), by (1.4a). So (2.4b, c) imply that \(D_{i+1} = \{1\}\). Since \(D_{i+1}\) is a section of \(A_{i+1}\), this, (1.5), and (2.2a) give \(D_{i+1} \in \alpha\). By induction, (2.2a) holds for \(D_1, \ldots, D_i\).

Clearly \(p(D_i) = p(A_i)\) and \(p(D_{i+1}) = p(A_{i+1})\). Therefore (2.2b) for \(A_1, \ldots, A_t\) implies (2.2b) for \(D_1, \ldots, D_t\).

Because \(A_i\) centralizes \(\Phi(A_{i+1})\) (by (2.2c)), so does \(D_i\). If \(D_{i+1} = E/F\), then it follows that \(D_i\) centralizes \(\Phi(E) \leq E \cap \Phi(A_{i+1})\). Since \(\Phi(D_{i+1})\) is the image in \(D_{i+1}\) of \(\Phi(E)\), it is centralized by \(D_i\). So (2.2c) holds for \(D_1, \ldots, D_t\).
Finally (2.2d, e) for $D_1, \cdots, D_t$ are just (2.4c, d). Therefore (2.2) holds for $D_1, \cdots, D_t$, which proves the proposition.

We say that a group $H$ acts on a Fitting chain $A_1, \cdots, A_t$ if $H$ acts on each group $A_i$, $i = 1, \cdots, t$, leaving invariant each action ($A_i$ on $A_{i+1}$), for $i = 1, \cdots, t - 1$. A Fitting subchain $D_1, \cdots, D_t$ is then $H$-invariant if each $D_i$, $i = 1, \cdots, t$, is an $H$-invariant section of $A_i$. In that case $H$ clearly acts on the Fitting chain $D_1, \cdots, D_t$ in the natural manner.

The first two of our three basic theorems concern the situation in which

(2.5a) A group $H$ acts on a Fitting chain $A_1, \cdots, A_t$,

(2.5b) $H$ has a normal subgroup $P$ of prime order $p$,

(2.5c) $[A_1, P] \neq \{1\}$.

The theorems, whose proofs will follow later (see §7), are:

**Theorem 2.6.** If $t \geq 3$ and $p$ does not divide $\prod_{i=1}^{t} |A_i|$, then there exists an $H$-invariant Fitting subchain $D_3, D_4, \cdots, D_t$ of $A_3, \cdots, A_t$ such that $P$ centralizes each $D_i$, $i = 3, \cdots, t$.

**Theorem 2.7.** If $t \geq 4$ and $p \geq 5$, then there exists an $H$-invariant Fitting subchain $D_4, D_5, \cdots, D_t$ of $A_4, \cdots, A_t$ such that $P$ centralizes each $D_i$, $i = 4, \cdots, t$.

Assuming these two theorems, we now prove

**Theorem 2.8.** Let $H$ be a finite group acting on a Fitting chain $A_1, \cdots, A_t$ such that no non-trivial section of any $A_i$, $i = 1, \cdots, t$, is centralized by $H$. Assume further that $H$ is a supersolvable group whose order is not divisible by 6.

**Proof.** We use induction on $l(H)$. If $l(H) = 0$, then $H = \{1\}$. Since no non-trivial section of any $A_i$, $i = 1, \cdots, t$, is centralized by $H$, each $A_i$ must be $\{1\}$. By (2.2a) and (1.4a), this implies that $t = 0 = 3(2^6 - 1)$. So the theorem is true in this case.

Now suppose that $l = l(H) > 0$ and that the theorem is true for all smaller values of $l(H)$. Since $H$ is supersolvable it has a normal subgroup $P$ whose order is the largest prime $p$ dividing $|H|$ (see Theorem VI, 9.1 of [4]).

Suppose that $P$ centralizes $A_1, \cdots, A_s$, for some integer $s = 1, \cdots, t$. Then $H/P$ acts on the Fitting chain $A_1, \cdots, A_s$. Obviously $H/P$, and $A_1, \cdots, A_s$, satisfy all the hypotheses of the theorem with $l(H/P) = l - 1$. So induction tells us that $s \leq 3(2^{l-1} - 1)$.

If $t \leq 3(2^{l-1} - 1) + 3$, then

$$t \leq 3(2^{l-1} - 1) + 3 + 3(2^{l-1} - 1) = 3(2^{l-1} - 1)$$

and the theorem is true. So assume that $t > 3(2^{l-1} - 1) + 3$. The argument of the preceding paragraph gives us an integer

$$s = 1, 2, \cdots, 3(2^{l-1} - 1) + 1$$

such that $P$ does not centralize $A_s$. Furthermore, the length $t - s + 1$ of
the Fitting chain $A_s, A_{s+1}, \ldots, A_t$ is at least

$$[3(2^{t-1} - 1) + 4] - [3(2^{t-1} - 1) + 1] + 1 = 4.$$  

If $p \geq 5$, then Theorem 2.7 applied to $H$, $P$ and $A_s, \ldots, A_t$ gives us an $H$-invariant Fitting subchain $D_{s+2}, D_{s+3}, \ldots, D_t$ of $A_{s+3}, \ldots, A_t$, which is centralized by $P$. Evidently $H/P$, and $D_{s+3}, \ldots, D_t$, satisfy the hypotheses of the theorem with $l(H/P) = l - 1$. So induction tells us that

$$t - (s + 3) + 1 \leq 3(2^{t-1} - 1).$$  

Hence

$$t \leq s + 3(2^{t-1} - 1) + 2$$

$$\leq [3(2^{t-1} - 1) + 1] + 3(2^{t-1} - 1) + 2 = 3(2^t - 1),$$

and the theorem is true in this case.

If $p = 2$ or $3$, then $H$ is a $p$-group. Because $H$ centralizes no non-trivial section of the $p(A_i)$-group $A_i$, for $i = 1, \ldots, t$, the primes $p$ and $p(A_i)$ must be distinct. Hence Theorem 2.6 applies to $H$, $P$, and $A_s, \ldots, A_t$, giving us an $H$-invariant Fitting subchain $D_{s+2}, D_{s+3}, \ldots, D_t$ of $A_{s+2}, \ldots, A_t$, which is centralized by $P$. By induction

$$t - (s + 2) + 1 \leq 3(2^{t-1} - 1).$$

So $t \leq s + 3(2^{t-1} - 1) + 1 < 3(2^t - 1)$, which finishes the proof of the theorem.

The second sentence of Theorem 2.8 looks very suspicious. It seems reasonable to make the

**Conjecture 2.9.** There is a function $g$ from the non-negative integers into themselves such that $t \leq g(l(H))$ whenever a finite group $H$ acts on a Fitting chain $A_1, \ldots, A_t$ and centralizes no non-trivial section of any $A_i$, $i = 1, \ldots, t$.

One might even hope that $g$ can be chosen so that $g(l) = O(l)$ as $l \to \infty$.

By an example which is too complicated to give here I can show that Theorem 2.7 does not hold for $p = 3$. So we are forced to consider more complicated chains of groups in order to prove Thompson’s conjecture by this method when $|H|$ is divisible by $6$. The idea is to make the connection between $(A_i$ on $A_{i+1})$ and $(A_{i+1}$ on $A_{i+2})$ stronger when $p(A_{i+1}) = 3$ and to leave everything else alone.

We define an augmented Fitting chain to be a Fitting chain $A_1, \ldots, A_t$ together with certain additional groups, actions, and epimorphisms. We say that an index $i = 1, \ldots, t$ is relevant if $1 \leq i \leq t - 2$ and $p(A_{i+1}) = 3$. For each relevant index $i$, we have an additional group $B_i$, an action of $B_i$ on $A_{i+2}$, and an epimorphism $\eta_i$ of $B_i$ onto $A_i$ (which defines an action of $B_i$ on $A_{i+1}$ via $(A_i$ on $A_{i+1})$) satisfying:

(2.10a) $B_i$ is a $p(A_i)$-group.

(2.10b) $(A_{i+1}$ on $A_{i+2})$ is $B_i$-invariant.
If \( i \leq t - 3 \) then \((A_{i+2} \text{ on } A_{i+3})\) is weakly \(B_i\)-invariant.

We usually write "the augmented Fitting chain \( A_1, \ldots, A_t, \{B_i\} \)" leaving the actions and the epimorphisms \( \eta_i \) to be understood.

Suppose that \( A_1, \ldots, A_t, \{B_i\} \) is an augmented Fitting chain, that \( D_1, \ldots, D_t \) is a Fitting subchain of \( A_1, \ldots, A_t, \) and that \( C_i \) is a section of \( B_i \), for each relevant \( i \). If \( \eta_i \) induces an epimorphism of \( C_i \) onto \( D_i \) and \( C_i \) normalizes \( D_{i+2} \) for each relevant \( i \), and if \( D_1, \ldots, D_t, \{C_i\} \) with these epimorphisms and actions form an augmented Fitting chain, then we call \( D_1, \ldots, D_t, \{C_i\} \) an augmented Fitting subchain of \( A_1, \ldots, A_t, \{B_i\} \). As in Proposition 2.3, we need not verify all the properties of \( D_1, \ldots, D_t, \{C_i\} \).

**Proposition 2.11.** Let \( A_1, \ldots, A_t, \{B_i\} \) be an augmented Fitting chain, \( D_j \) be a section of \( A_j \), for \( j = 1, \ldots, t \), and \( C_i \) be a section of \( B_i \), for all relevant \( i \). Then \( D_1, \ldots, D_t, \{C_i\} \) form an augmented Fitting subchain if and only if they satisfy:

1. \((2.12a) \quad D_1 \neq \{1\}.\)
2. \((2.12b) \quad D_j \text{ normalizes } D_{j+1}, \text{ for } j = 1, \ldots, t - 1.\)
3. \((2.12c) \quad \eta_i \text{ induces an epimorphism of } C_i \text{ onto } D_i, \text{ for all relevant } i.\)
4. \((2.12d) \quad C_i \text{ normalizes } D_{i+2}, \text{ for all relevant } i.\)
5. \((2.12e) \quad \ker (D_j \text{ on } D_{j+1}) = \{1\}, \text{ for } j = 1, \ldots, t - 1.\)
6. \((2.12f) \quad (D_{i+1} \text{ on } D_{i+2}) \text{ is weakly } D_i^{-}\text{invariant, if } i = 1, \ldots, t - 2 \text{ and } p(A_{i+1}) \neq 3.\)
7. \((2.12g) \quad (D_{i+2} \text{ on } D_{i+3}) \text{ is weakly } C_i^{-}\text{invariant for all relevant } i \leq t - 3.\)

**Proof.** It is clear that (2.12) holds whenever \( D_1, \ldots, D_t, \{C_i\} \) is an augmented Fitting subchain.

Suppose that (2.12) holds. We have enough groups, epimorphisms, and actions to form an augmented Fitting subchain \( D_1, \ldots, D_t, \{C_i\} \). So we need only check the various axioms.

Let \( i = 1, \ldots, t - 2 \) with \( p(A_{i+1}) = 3 \). The \( B_i\)-invariance of \((A_{i+1} \text{ on } A_{i+2})\), together with (2.12c, d), implies that \((D_{i+1} \text{ on } D_{i+2})\) is \( C_i\)-invariant. So (2.10b) holds for our subchain. Since \( D_{i+1} \text{ centralizes } \Phi(D_{i+2}) \) by (2.2e), this clearly implies that \((D_{i+1} \text{ on } D_{i+2})\) is weakly \( D_i\)-invariant. So (2.12f) is satisfied for all \( i = 1, \ldots, t - 2 \). This and (2.12a, b, e) are conditions (2.4). Therefore \( D_1, \ldots, D_t \) is a Fitting subchain by Proposition 2.3. Obviously \( p(C_i) = p(B_i) = p(A_i) = p(D_i) \), for all relevant \( i \). And (2.12g) is (2.10c) for the subchain. Hence \( D_1, \ldots, D_t, \{C_i\} \) satisfies (2.10) and the proposition is true.

A group \( H \) acts on an augmented Fitting chain \( A_1, \ldots, A_t, \{B_i\} \) if it acts on each group \( A_j, j = 1, \ldots, t \), and on \( B_i \), for each relevant \( i \), so that all the actions and epimorphisms of the chain are \( H\)-invariant. An augmented Fitting subchain \( D_1, \ldots, D_t, \{C_i\} \) is then \( H\)-invariant if each \( D_j, j = 1, \ldots, t \), and \( C_i \), for each relevant \( i \), is an \( H\)-invariant section. In that case \( H \) acts on the augmented Fitting chain \( D_1, \ldots, D_t, \{C_i\} \) in the natural manner.
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The third basic theorem, whose proof will follow later (see §7) is

**Theorem 2.13.** Let $H$ be a group acting on an augmented Fitting chain $A_1, \ldots, A_t, \{B_i\}$. Suppose that $P$ is a normal subgroup of order 3 in $H$ such that $[A_1, P] \neq \{1\}$. If $t \geq 6$, then there is an $H$-invariant augmented Fitting subchain $D_6, \ldots, D_t, \{C_i\}$ of $A_6, \ldots, A_t, \{B_i\}$ such that $P$ centralizes each $D_j$ and $C_i$.

Assuming the three Theorems 2.6, 2.7 and 2.13, we now prove the following result from which we shall later derive a proof of Thomson’s conjecture (see §8).

**Theorem 2.14.** Let a finite group $H$ act on an augmented Fitting chain $A_1, \ldots, A_t, \{B_i\}$ so that $H$ centralizes no non-trivial section of any $A_i, j = 1, \ldots, t$. Assume further that $H$ is a supersolvable group with a normal 3-Sylow subgroup $M$. Then $t \leq 5(2^{l(H)} - 1)$.

**Proof.** We use induction on $|M|$. If $|M| = 1$, then $H$ and $A_1, \ldots, A_t$ satisfy the hypotheses of Theorem 2.8. That theorem tells us that $t \leq 3(2^{l(H)} - 1) \leq 5(2^{l(H)} - 1)$. So this theorem is true if $|M| = 1$.

Now we assume that $|M| > 1$ and that this theorem is true for all smaller values of $|M|$. Since $H$ is supersolvable it has a normal subgroup $P$ of prime order $p$. We may even choose $P$ to be contained in the normal 3-Sylow subgroup $M$ of $H$. So $p = 3$.

Suppose that $P$ centralizes $A_1, \ldots, A_s$, for some integer $s = 1, \ldots, t$. If $p(A_{i+1}) = 3$, for some $i = 1, \ldots, s - 2$, then the $P$-invariance of $(B_i$ on $A_{i+2})$, together with the fact that $P$ centralizes $A_{i+2}$, implies that $[B_i, P]$ centralizes $A_{i+2}$. Furthermore, the facts that $P$ centralizes $\eta_i(B_i) = A_i$ and leaves $A_i$ invariant imply that $\eta_i([B_i, P]) = \{1\}$. So there is a natural action of $B_i/[B_i, P]$ on $A_{i+2}$ and a natural epimorphism of $B_i/[B_i, P]$ onto $A_i$. Since $P$ is normal in $H$, the subgroup $[B_i, P]$ is $H$-invariant. Hence so are the action $(B_i/[B_i, P]$ on $A_{i+2})$ and the epimorphism of $[B_i, P]$ onto $A_i$. It follows that $A_1, \ldots, A_s, [B_i/[B_i, P]]$ is an augmented Fitting chain on which $H/P$ acts. Since $H/P, M/P$ and $A_1, \ldots, A_s, [B_i/[B_i, P]]$ satisfy our hypothesis with $|M/P| < |M|$, we know by induction that $s \leq 5(2^{l-1} - 1)$, where $l = l(H) = l(H/P) + 1$.

If $t \leq 5(2^{l-1} - 1) + 5$, the theorem is certainly true. So assume that $t > 5(2^{l-1} - 1) + 5$. The preceding paragraph proves that there exists an $s = 1, \ldots, 5(2^{l-1} - 1) + 1$ such that $P$ does not centralize $A_s$. The length $t - s + 1$ of the augmented Fitting chain $A_s, A_{s+1}, \ldots, A_t, \{B_i\}$ is at least $5(2^{l-1} - 1) + 6 - 5(2^{l-1} - 1) + 1 + 1 = 6$.

So Theorem 2.13 gives us an augmented Fitting subehain $D_{s+5}, D_{s+6}, \ldots, D_t, \{C_i\}$ of $A_{s+5}, \ldots, A_t, \{B_i\}$ such that $P$ centralizes each $D_j$ and $C_i$. Applying the present theorem by induction to $H/P, M/P$ and $D_{s+5}, \ldots, D_t, \{C_i\}$, we see that

$$t - (s + 5) + 1 \leq 5(2^{l-1} - 1).$$
Hence
\[ t \leq (s + 5) - 1 + 5(2^{i-1} - 1) \leq 5(2^{i-1} - 1) + 6 - 1 + 5(2^{i-1} - 1) = 5(2^i - 1). \]

This completes the proof of the theorem.

### 3. Ample representations

We begin with some elementary observations about the situation in which

1. PA is the semi-direct product of a group P of prime order p acting on a group A ∈ Q,
2. F is a field of prime characteristic q ≠ p(A) which is a splitting field for all subgroups of PA,
3. V is an irreducible F[PA]-module.

The first observation is

**Proposition 3.2.** If (3.1) holds and \([Z(A), P] \neq 1\), then V is induced from an irreducible F[A]-module U.

**Proof.** Apply Clifford’s theorem (see Theorem V, 17.3 of [4]) to V and the inverse image B of Z(A ;,) in A. Since F is a splitting field for B and B_v = Z(A_v) is abelian, there is a linear F-character } of B and an irreducible F[C_{PA}(λ)]-module U such that V is induced from U and any } ∈ B acts on U as scalar multiplication by λ(σ) ∈ F. Because A centralizes B_v, it fixes λ. By (3.1a), A is maximal in PA. Therefore C_{PA}(λ) is either A or PA. The latter possibility implies that U = V and that P centralizes B/Ker λ = B_v = Z(A_v), contradicting our hypotheses. Hence C_{PA}(λ) = A and the proposition is true.

**Corollary 3.3.** Let C = C_A(P). Then both C_v(P) and \([V, P]^{p-1}\) are non-zero F[C]-submodules of V. Furthermore, both of them are weakly F[C]-equivalent to V.

**Proof.** From V = U^{PA} we conclude that V_{F×C} is F[P × C]-isomorphic to the outer Kronecker product F[P] ⊗ U_c of the regular F[P]-module with U_c. We always have C_{F[P]}(P) ≠ {0} and \([F[P], P]^{p-1} ≠ {0}\). It follows that both

\[ C_v(P) ∼ C_{F[P]}(P) ⊗ U_c \quad \text{and} \quad [V, P]^{p-1} ∼ [F[P], P]^{p-1} ⊗ U_c \]

are non-zero, and that all of V_c, C_v(P), [V, P]^{p-1} are multiples of U_c as F[C]-modules. So they are weakly F[C]-equivalent to each other, which proves the corollary.

Another observation which we shall use repeatedly is

**Proposition 3.4.** Suppose that (3.1) holds, that p ≠ p(A), and that P centralizes Z(A_v) but not A_v. Then [A_v, P] is an extra-special group and Φ([A_v, P]) = Φ(A_v) is centralized by P. The group PA_v is the central product
of its subgroups $P[A_v, P]$ and $C_{A_v}(P)$, which intersect in $\Phi(A_v)$. Regarding $PA_v$ as the natural image of $P[A_v, P] \times C_{A_v}(P)$, the module $V$ is the outer Kronecker product $W \otimes U$ of an irreducible $F[P[A_v, P]]$-module $W$ and an irreducible $F[C_{A_v}(P)]$-module $U$. Furthermore $([A_v, P] \text{ on } W)$ is faithful.

Proof. Our hypotheses insure that $A_v$ is not abelian. So (1.5) tells us that $A_v \not\in \mathfrak{a}$. Since $P$ centralizes $Z(A_v)$, it centralizes $\Phi(A_v)$ (by 1.4b)), which is non-trivial since it contains $A_v$. Therefore $\Phi(A_v) \leq Z(PA_v)$. Because $V$ is an irreducible $F[PA_v]$-module on which $\Phi(A_v)$ is faithfully represented, this implies that $\Phi(A_v)$ is cyclic. Hence $\Phi(A_v)$ has order $p(A) = p(A_v)$ by (1.4c).

From $p \neq p(A_v)$ and $[A_v, P] \neq [1]$ we conclude that

\[ \hat{A}_v = [\hat{A}_v, P] \oplus C_{\hat{A}_v}(P) \quad \text{and} \quad [\hat{A}_v, P] \neq \{0\}. \]

Since $P$ centralizes $\Phi(A_v)$ and leaves the form $f_{A_v}$ of (1.6) invariant, the two subspaces $[\hat{A}_v, P]$ and $C_{\hat{A}_v}(P)$ of $\hat{A}_v$ must be $f_{A_v}$-perpendicular. Therefore $A_v$ is the central product of the inverse images $L$ of $[\hat{A}_v, P]$ and $K$ of $C_{\hat{A}_v}(P)$ with $L \cap K = \Phi(A_v)$.

The radical $[Z(A_v)/\Phi(A_v)]^+$ of $f_{A_v}$ is contained in $C_{\hat{A}_v}(P)$ by hypothesis. So the restriction of $f_{A_v}$ to $[\hat{A}_v, P] \times [\hat{A}_v, P]$ is non-singular. Since $\Phi(A_v)$ is cyclic of order $p(A)$ and $[\hat{A}_v, P] \neq [0]$, we conclude that $L$ is extra-special with $\Phi(L) = \Phi(A_v)$. Obviously $L$ contains $[A_v, P]$. But $[A_v, P]$ covers $\tilde{L} \simeq [\hat{A}_v, P]$. Therefore $L = [A_v, P]$ and the first statement of the proposition is true.

Since $p \neq p(A)$ and $P$ centralizes both $\Phi(A_v)$ and $K/\Phi(A_v) = C_{\hat{A}_v}(P)$, the group $P$ centralizes $K$. It follows that $K = C_{A_v}(P)$. So $A_v$ is the central product of $[A_v, P]$ and $C_{A_v}(P)$. Because $P$ normalizes $[A_v, P]$ and centralizes $C_{A_v}(P)$, the second statement of the proposition follows directly from this.

The third statement of the proposition comes immediately from the second, since $F$ is a splitting field for all the groups involved. Finally, any $\sigma \in [A_v, P] - [1]$ acts non-trivially on $W$ since $\sigma \times 1 \in P[A_u, P] \times C_{A_v}(P)$ has the image $\sigma \in PA_v$ which acts non-trivially on $V$. So the entire proposition is true.

The following fact is well known (see [3] or Theorem (IV.9) of [2]):

(3.5) Under the hypotheses of Proposition 3.4, there is a regular $F[P]$-submodule of $W$ unless $p$ is a Fermat prime, $p(A) = 2$, and $[\hat{A}_v, P] \simeq [A_v, P]$ is an irreducible $Z_2[P]$-module.

We use this to prove the following corollary to Proposition 3.4.

Corollary 3.6. Let $C = C_\lambda(P)$. Unless $p$ is a Fermat prime, $p(A) = 2$, and $[\hat{A}_v, P]$ is an irreducible $Z_2[P]$-module, the subspaces $C_{v}(P)$ and $[V, P]^{p-1}$ are both non-zero $F[C]$-submodules of $V$ and are both weakly $F[C]$-equivalent to $V$. 

Proof. Assume we are not in the exceptional case. Then (3.5) gives us an $F[P]$-submodule of $W_\rho$ isomorphic to $F[P]$. Clearly $C_\rho \leq C_{A_\rho}(P)$. So the proposition tells us that $V_{\rho < C}$ is $F[P \times C]$-isomorphic to the outer Kronecker product $W_\rho \otimes U_\rho$, which contains a submodule isomorphic to $F[P] \otimes U_\rho$. Since neither $C_{F[P]}(P)$ nor $[F[P], P]^{p-1}$ is $\{0\}$, we conclude that $C_{V}(P) \simeq C_{W}(P) \otimes U_\rho$ and $[V, P]^{p-1} \simeq [W, P]^{p-1} \otimes U_\rho$ are both non-zero. Furthermore all the modules $V_\rho$, $C_{V}(P)_C$, and $(W, P]^{p-1}_C$ are isomorphic to positive multiples of $U_\rho$, and hence are weakly $F[C]$-equivalent to each other. So the corollary is true.

When $p = p(A)$, we use a different approach to get a result similar to Corollary 3.6.

Proposition 3.7. Suppose that (3.1) holds with $p = p(A) \geq 3$ and $[A_\rho, P]^{p-1} \neq \{1\}$. Let $C \leq C_{A_\rho}(P)$. Then $C_{V}(P)$ is a non-zero $F[C]$-submodule of $V$ and is weakly $F[C]$-equivalent to $V$.

Proof. If $[Z(A_\rho), P] \neq \{1\}$, the result follows immediately from (1.1) and Corollary 3.3, since $C \leq C_A(P)$. So we may assume that $[Z(A_\rho), P] = \{1\}$.

Under this assumption we first prove that

\begin{equation}
[A_\rho, P]^{p-1} \leq Z([A_\rho, P]^{p-2}).
\end{equation}

For this it suffices by (1.6) to show that

$f_{A_\rho}([A_\rho, P]^{p-1}, [A_\rho, P]^{p-2}) = \{0\}$.

But $[A_\rho, P]^n \leq A_\rho (\pi - 1)^n$, for all $n \geq 0$, where $\pi$ is any generator of $P$. If $\alpha, \beta \in A_\rho$, we use $p \geq 3$ and the fact that $P$ centralizes $f_{A_\rho}(A_\rho, A_\rho) \leq Z(A_\rho)^+$ to compute

\[
f_{A_\rho}(\alpha(\pi - 1)^{p-1}, \beta(\pi - 1)^{p-2}) = f_{A_\rho}(\alpha(\pi - 1)^{p-1}(\pi^{-1} - 1), \beta(\pi - 1)^{p-3}) = f_{A_\rho}(-\alpha(\pi - 1)^{p-1}, \beta(\pi - 1)^{p-3}).\]

But $A_\rho$ is a vector space over a field $Z_p$ of characteristic $p = p(A)$. So $\alpha(\pi - 1)^p = \alpha(\pi^p - 1) = 0$. Therefore (3.8) holds.

Let $U$ be any non-trivial irreducible $F[C]$-submodule of $V$. Since $C \leq [A, P]^{p-1}$, there exists some irreducible $F[[A, P]^{p-1}]$-submodule $W$ of $V$ containing an $F[C]$-submodule isomorphic to $U$. So we may assume that $U \leq W$. Since $C$ is non-trivial on $U$, it is non-trivial on $W$. Therefore $\ker (\rho \rho^{-1} on W) < [A, P]^{p-1}$.

So there must exist some element $s \in [A_\rho, P]^{p-2}$ and some $\pi \in P$ such that

$s, \pi \in [A_\rho, P]^{p-1} - \ker ([A_\rho, P]^{p-1} on W)$.

It follows from (3.8) that $\langle s, [A_\rho, P]^{p-1} \rangle$ is a $P$-invariant abelian subgroup of $A_\rho$. Let $B$ be its inverse image in $A$. Then $[A, P]^{p-1} \leq B$. So there is
some irreducible $F[P_B]$-submodule $Y$ of $V$ containing an $F[A, P]^{|p-1|}$-submodule isomorphic to $V$. As before, we may assume that $W \leq Y$. Clearly $B_Y$ is a homomorphic image of $B_V = \langle \sigma, [A_V, P]^{|p-1|} \rangle$ and hence is abelian. If $P$ centralized $B_Y$, then $[\sigma, \pi]$ would lie in Ker $(B_Y$ on $Y)$, contradicting the fact that $[\sigma, \pi]$ acts nontrivially on $W \leq Y$. Hence $[B_Y, P] \neq \{1\}$. So Corollary 3.3 applies to $PB$ and $Y$. It tells us that $Y$ and $C_Y(P)$ are weakly $F[C_B(P)]$-equivalent. Since $C \leq C_B(P)$, the modules $Y$ and $C_Y(P)$ are weakly $F[C]$-equivalent (by (1.1)). Hence there is an irreducible $F[C]$-submodule of $C_Y(P) \leq C_Y(P)$ which is isomorphic to the non-trivial $F[C]$-submodule $U$ of $Y$.

We have shown that any non-trivial irreducible $F[C]$-submodule $U$ of $V$ is $F[C]$-isomorphic to a submodule of $C_Y(P)$. The converse being obvious, this proves that $C_Y(P)$ and $V$ are weakly $F[C]$-equivalent.

By hypothesis there exists a non-trivial irreducible $F[(A, P)^{|p-1|}]$-submodule $W$ of $V$. Constructing $B$ and $Y$ as above we see from Corollary 3.3 that $C_Y(P) \neq \{0\}$. So $C_Y(P) \neq \{0\}$ and the proposition is proved.

In practice we must consider modules over fields which need not satisfy (3.1b). A simple ground field extension quickly reduces this more general case to the one we have been considering.

Suppose that (3.1a) holds, that $E$ is any field of prime characteristic $q \neq p(A)$, and that $V$ is any irreducible $E[A_P]$-module. We call $V$ ample if one of the following conditions holds:

(3.9a) $p \neq p(A)$ and $[Z(A_V), P] \neq \{1\}.$
(3.9b) $p \neq p(A),$ $[Z(A_V), P] = \{1\}, [A_V, P] \neq \{1\}$ and we are not in the exceptional case in which $p(A) = 2$, $p$ is a Fermat prime, and $[A_V, P]$ is an irreducible $Z_2[P]$-module.
(3.9c) $p = p(A) \geq 3$ and $[A_V, P]^{|p-1|} \neq \{1\}.$

These are, of course, just the hypotheses of Corollaries 3.3 and 3.6 and Proposition 3.7 made into axioms. From these results we easily prove

**Proposition 3.10.** Let $V$ be an ample irreducible $E[A_P]$-module. Let $C$ be $C_A(P)$, if $p \neq p(A)$, or $C_{[A,P]^{|p-1|}}(P)$, if $p = p(A)$. Then $C_Y(P)$ is a non-zero $E[C]$-submodule of $V$ and is weakly $E[C]$-equivalent to $V$. If $p \neq p(A)$, then $[V, P]^{|p-1|}$ is also a non-zero $E[C]$-submodule of $V$ weakly $E[C]$-equivalent to $V$.

**Proof.** Since $PA$ has only a finite number of subgroups, we may choose a finite algebraic extension field $F$ of $E$ so that it is a splitting field for all subgroups of $PA$. Let $U$ be an irreducible $T[PA]$-submodule of the extension $F \otimes_E V$ of $V$ to an $F[PA]$-module. Clearly $F \otimes_E V$, considered as a module over $E[PA]$, is isomorphic to $[F:E] \times V$. Since $V$ is an irreducible $E[PA]$-module, we conclude that the restriction $U_{[F]}$ of $U$ to an $E[PA]$-module satisfies

$$U_{[F]} \cong n \times V$$ (as $E[PA]$-modules),

for some positive integer $n$. 

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some irreducible $F[P_B]$-submodule $Y$ of $V$ containing an $F[A, P]^{|p-1|}$-submodule isomorphic to $V$. As before, we may assume that $W \leq Y$. Clearly $B_Y$ is a homomorphic image of $B_V = \langle \sigma, [A_V, P]^{|p-1|} \rangle$ and hence is abelian. If $P$ centralized $B_Y$, then $[\sigma, \pi]$ would lie in Ker $(B_Y$ on $Y)$, contradicting the fact that $[\sigma, \pi]$ acts nontrivially on $W \leq Y$. Hence $[B_Y, P] \neq \{1\}$. So Corollary 3.3 applies to $PB$ and $Y$. It tells us that $Y$ and $C_Y(P)$ are weakly $F[C_B(P)]$-equivalent. Since $C \leq C_B(P)$, the modules $Y$ and $C_Y(P)$ are weakly $F[C]$-equivalent (by (1.1)). Hence there is an irreducible $F[C]$-submodule of $C_Y(P) \leq C_Y(P)$ which is isomorphic to the non-trivial $F[C]$-submodule $U$ of $Y$.

We have shown that any non-trivial irreducible $F[C]$-submodule $U$ of $V$ is $F[C]$-isomorphic to a submodule of $C_Y(P)$. The converse being obvious, this proves that $C_Y(P)$ and $V$ are weakly $F[C]$-equivalent.

By hypothesis there exists a non-trivial irreducible $F[(A, P)^{|p-1|}]$-submodule $W$ of $V$. Constructing $B$ and $Y$ as above we see from Corollary 3.3 that $C_Y(P) \neq \{0\}$. So $C_Y(P) \neq \{0\}$ and the proposition is proved.

In practice we must consider modules over fields which need not satisfy (3.1b). A simple ground field extension quickly reduces this more general case to the one we have been considering.

Suppose that (3.1a) holds, that $E$ is any field of prime characteristic $q \neq p(A)$, and that $V$ is any irreducible $E[A_P]$-module. We call $V$ ample if one of the following conditions holds:

(3.9a) $p \neq p(A)$ and $[Z(A_V), P] \neq \{1\}.$
(3.9b) $p \neq p(A),$ $[Z(A_V), P] = \{1\}, [A_V, P] \neq \{1\}$ and we are not in the exceptional case in which $p(A) = 2$, $p$ is a Fermat prime, and $[A_V, P]$ is an irreducible $Z_2[P]$-module.
(3.9c) $p = p(A) \geq 3$ and $[A_V, P]^{|p-1|} \neq \{1\}.$

These are, of course, just the hypotheses of Corollaries 3.3 and 3.6 and Proposition 3.7 made into axioms. From these results we easily prove

**Proposition 3.10.** Let $V$ be an ample irreducible $E[A_P]$-module. Let $C$ be $C_A(P)$, if $p \neq p(A)$, or $C_{[A,P]^{|p-1|}}(P)$, if $p = p(A)$. Then $C_Y(P)$ is a non-zero $E[C]$-submodule of $V$ and is weakly $E[C]$-equivalent to $V$. If $p \neq p(A)$, then $[V, P]^{|p-1|}$ is also a non-zero $E[C]$-submodule of $V$ weakly $E[C]$-equivalent to $V$.

**Proof.** Since $PA$ has only a finite number of subgroups, we may choose a finite algebraic extension field $F$ of $E$ so that it is a splitting field for all subgroups of $PA$. Let $U$ be an irreducible $F[PA]$-submodule of the extension $F \otimes_E V$ of $V$ to an $F[PA]$-module. Clearly $F \otimes_E V$, considered as a module over $E[PA]$, is isomorphic to $[F:E] \times V$. Since $V$ is an irreducible $E[PA]$-module, we conclude that the restriction $U_{[F]}$ of $U$ to an $E[PA]$-module satisfies

$$U_{[F]} \cong n \times V$$ (as $E[PA]$-modules),

for some positive integer $n$. 

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If \( p \neq p(A) \), let \( Z = \mathcal{C}_V(P) \) or \( [V, P]^{p-1} \). If \( p = p(A) \), let \( Z = \mathcal{C}_V(P) \).

We must prove that \( Z \) is a non-zero \( E[C] \)-submodule of \( V \) weakly \( E[C] \)-equivalent to \( V \). Let \( Y = \mathcal{C}_U(P) \), if \( Z = \mathcal{C}_V(P) \), and \( Y = [U, P]^{p-1} \), if \( Z = [V, P]^{p-1} \). Evidently the isomorphism (3.11) always carries \( Y \) onto \( n \times Z \). Therefore it suffices to prove that \( Y \) is a non-zero \( F[C] \)-submodule of \( U \) weakly \( F[C] \)-equivalent to \( U \).

One conclusion from (3.11) is that \( \text{Ker } (PA \text{ on } U) = \text{Ker } (PA \text{ on } V) \). Hence \( A_U = A_V \), as factor groups of \( A \). Therefore \( PA, F, \) and \( U \) satisfy the hypotheses of Corollary 3.3 or Corollary 3.6 or Proposition 3.7, if (3.9a) or (3.9b) or (3.9c), respectively, hold. The fact that \( V \) is ample says that one of (3.9a, b, c) is satisfied. So the cited results tell us that \( Y \) is a non-zero \( F[C] \)-submodule of \( U \) which is weakly \( F[C] \)-equivalent to \( U \). Since the weak \( F[C] \)-equivalence of \( Y \) and \( U \) implies their weak \( E[C] \)-equivalence by (1.2), the proposition is proved.

**Corollary 3.12.** If \( V \) is an irreducible \( E[PA] \)-module, \( \mathcal{C}(A_V, P) = \{1\} \), and \( [A_V, P] \neq \{1\} \), then \( [A_V, P] \) is an extra-special group, \( \Phi([A_V, P]) = \Phi(A_V) \) is centralized by \( P \), and \( PA_V \) is the central product of \( P[A_V, P] \) and \( C_{A_V}(P) \) which intersect in \( \Phi(A_V) \).

**Proof.** Apply the first paragraph of the above argument to \( V \). Then (3.11) implies that \( A_V = A_U \). Since \( F, PA \) and \( U \) satisfy (3.1), Proposition 3.4 applies to them, giving this corollary.

The following proposition sometimes helps to prove a module ample. As before, (3.1a) holds, \( E \) is any field of prime characteristic \( q \neq p(A) \), and \( V \) is any irreducible \( E[PA] \)-module. Let \( B \) be some non-trivial \( P \)-invariant subgroup of \( A \). Then \( PB \) also satisfies (3.1a) (by (1.5)).

**Proposition 3.13.** Suppose there is an ample irreducible \( E[PB] \)-component \( W \) of \( V \). Then \( V \) is an ample \( E[PA] \)-module.

**Proof.** Since \( W \) is a component of \( V \), the factor group \( B_W \) is naturally a section of \( A_V \).

Suppose that \( p = p(A) \). Because \( p = p(B) \) and \( W \) is ample, (3.9c) must hold for \( P \) and \( B_W \). Therefore \( p \geq 3 \) and \( [B_W, P]^{p-1} \neq \{1\} \). Since \( B_W \) is a section of \( A_V \), this implies that \( [A_V, P]^{p-1} \neq \{1\} \). So (3.9c) holds for \( P \) and \( A_V \). I.e., \( V \) is ample.

Suppose that \( p \neq p(A) \) and that \( V \) is not ample. Since \( W \) is ample, \( [B_W, P] \) is not \( \{1\} \). Because \( B_W \) is a section of \( A_V \), this implies that \( [A_V, P] \neq \{1\} \). Neither (3.9a) nor (3.9b) can hold for \( PA_V \). So \( [A_V, P] \neq \{1\} \) forces \( PA_V \) to lie in the exceptional case of (3.9b).

From (3.9b) and Corollary 3.12 we know that \( p(A) = 2 \), that \( p \) is a Fermat prime, that \( [A_V, P] \) is extra-special with \( \Phi([A_V, P]) \) centralized by \( PA_V \), and that \( [A_V, P] \cong [A_V, P] \) is an irreducible \( Z_2[P] \)-module. It follows that \( [A_V, P] \) is the only non-trivial subgroup \( D \) of \( A_V \) satisfying \( D = [D, P] \).
But \([B_w, P] \neq \{1\}\) implies \([B_r, P] \neq \{1\}\). Since \(p \neq p(A) = p(B_r)\), we have \([[B_r, P], P] = [B_r, P]\). Therefore \([B_r, P] = [A_r, P]\).

The subgroup \(\Phi([A_r, P]) = \Phi([B_r, P])\) is central in \(PA_r\) and is a non-trivial subgroup of \(A_r\). Since \(V\) is an irreducible \(E[PA_r]\)-module, this implies that \([V, \Phi([A_r, P])] = V\). It follows that \([W, \Phi([B_r, P])] = W\).

Because \([B_r, P] = [A_r, P]\) is extra-special, \(\Phi([B_r, P])\) is its center and has prime order \(p(B) = 2\). It follows that \(\text{Ker} ([B_r, P] \text{ on } W) = \{1\}\). I.e., \([B_w, P] = [B_r, P]\) as sections of \(B\). Therefore \([B_w, P] = [B_r, P]\) is extra-special, with \(\Phi([B_w, P])\) centralized by \(P\), and \([B_w, P]\) is an irreducible \(Z_p[P]\)-module.

Since \(p(B_w) = 2 \neq p\), we have \([Z(B_w), P, P] = [Z(B_w), P]\). But \([Z(B_w), P] = [B_w, P]\) as sections of \(B\).

The last group is centralized by \(P\). Therefore \([Z(B_w), P] = [1]\).

Now we know that \(p(B) = p(A) = 2\), that \(p\) is a Fermat prime, that \(P\) centralizes \(Z(B_w)\) but not \(B_w\), and that \([B_w, P] \simeq [B_r, P]\) is an irreducible \(Z_p[P]\)-module, i.e. \(PB_w\) lies in the exceptional case of \((3.9b)\). This contradicts the hypothesis that \(W\) is ample. The contradiction proves that \(V\) is ample in all cases, which is the proposition.

Let \(E\) and \(PA\) be as above. Now, however, we take \(V\) to be an arbitrary finite-dimensional \(E[PA]\)-module. Since the characteristic of \(E\) is different from \(p(A)\), the restriction \(V_A\) of \(V\) to an \(E[A]\)-module is completely reducible. Let \(\mathfrak{K} = \mathfrak{K}(V)\) be the family of all kernels \(\text{Ker} (A \text{ on } W)\), where \(W\) runs over all irreducible \(E[A]\)-components of \(V_A\). For each \(K \in \mathfrak{K}\), let \(V_A(K)\) be the sum of all those irreducible \(E[A]\)-submodules \(W\) of \(V\) such that \(\text{Ker} (A \text{ on } W) = K\). Then the complete reducibility of \(V_A\) implies that

\[
(3.14a) \quad V_A(K) \text{ is a non-trivial } E[A]\text{-submodule of } V, \text{ for each } K \in \mathfrak{K},
\]

\[
(3.14b) \quad V_A = \oplus \sum_{K \in \mathfrak{K}} V_A(K).
\]

If \(K\) is any normal subgroup of \(A\), let \(K(P) = \bigcap_{\pi \in P} K^\pi\). Then \(K(P)\) is the largest normal subgroup of \(PA\) contained in \(K\). We define \(\mathfrak{K}_{\text{ample}} = \mathfrak{K}_{\text{ample}}(V)\) to be the set of all \(K \in \mathfrak{K}\) such that \((3.9)\) holds with \(A/K(P)\) in place of \(A_r\).

Finally, we set

\[
(3.15) \quad V_{\text{ample}} = \oplus \sum_{K \in \mathfrak{K}_{\text{ample}}} V_A(K).
\]

Then we have

**Proposition 3.16.** \(V_{\text{ample}}\) is an \(E[PA]\)-submodule of \(V\) whose irreducible \(E[PA]\)-components are precisely the ample irreducible \(E[PA]\)-components of \(V\).

*Proof.* Since \(V\) is an \(E[PA]\)-module we have

\[
(3.17) \quad V_A(K^\pi) = V_A(K) \cdot \pi \text{ for all } K \in \mathfrak{K}, \pi \in P.
\]

In particular, \(\mathfrak{K}\) is a \(P\)-invariant family of subgroups of \(A\). From the definition of \(\mathfrak{K}_{\text{ample}}\) it is clear that it is a \(P\)-invariant subfamily of \(\mathfrak{K}\). So \((3.15)\) and \((3.17)\) imply that \(V_{\text{ample}}\) is an \(E[PA]\)-submodule of \(V\).
Let $U$ be any irreducible $E[PA]$-component of $V$, and $W$ be an irreducible $E[A]$-component of $U$. If $K = \text{Ker } (A \text{ on } W)$, then Clifford’s theory (see Theorem V, 17.3 of [4]) says that

$$\text{Ker } (A \text{ on } U) = \bigcap_{\tau \in \mathcal{P}} K^\tau = K(P).$$

Since $K \in \mathfrak{K}$, we conclude that $U$ is ample if and only if $K \in \mathfrak{K}_{\text{ample}}$.

If $U$ is a component of $V_{\text{ample}}$, then (3.15) implies that $K \in \mathfrak{K}_{\text{ample}}$. So $U$ is ample. If $U$ is a component of $V/V_{\text{ample}}$, then (3.14) and (3.15) imply that $K \notin \mathfrak{K} = \mathfrak{K}_{\text{ample}}$. So $U$ is not ample. Therefore the ample irreducible $E[PA]$-components of $V$ are precisely the irreducible $E[PA]$-components of $V_{\text{ample}}$, which proves the proposition.

4. Finding one ample representation

The following situation occurs repeatedly in Fitting chains on which our group $P$ acts:

- (4.1a) $PB$ is the semi-direct product of a group $P$ of prime order $p$ acting on a group $B \in \mathfrak{A}$.
- (4.1b) $PBA$ is the semi-direct product of $PB$ acting on a group $A \in \mathfrak{A}$.
- (4.1c) $V$ is an irreducible $Z_q[PA]$-module, for some prime $q$.
- (4.1d) $p, p(B)$ and $q$ are all different from $p(A)$.
- (4.1e) $\text{Ker } (\Phi(A) \text{ on } V) = \{1\}$.

We shall prove the following consequence of (4.1) under weaker hypotheses because of future applications.

**Proposition 4.2.** Let the semi-direct $PA$ of a group $P$ acting on a group $A \in \mathfrak{A}$ itself act on a group $V$. If $\text{Ker } (\Phi(A) \text{ on } V) = \{1\}$, then the natural epimorphism of $A/Z(A)$ onto $A/\Phi(A)$ is a $P$-isomorphism.

**Proof.** Obviously this epimorphism preserves the actions of $P$. So we need only show it to be an isomorphism, i.e., that $Z(A)$ is the inverse image of $Z(A)$. Suppose that $\sigma \in A - Z(A)$. Then there exists $\tau \in A$ such that $[\sigma, \tau] \neq 1$. Since $[\sigma, \tau] \in \Phi(A)$, our hypotheses say that the image of $[\sigma, \tau]$ in $A/\Phi(A)$ is not $1$. Hence the image of $\sigma$ does not lie in $Z(A)$. It follows that $Z(A)$ contains the inverse image of $Z(A)$. The opposite inclusion is obvious. So the proposition is true.

In our case the value of Proposition 4.2 is that $[A/Z(A)]^+$ is a $Z_{p(A)}[PB]$-module, while $[A/\Phi(A)]^+$ is only a $Z_{p(A)}[P]$-module. We exploit this fact to prove

**Proposition 4.3.** Suppose that (4.1) holds and $V$ is not an ample $Z_q[PA]$-module. Then $[B, P]$ centralizes $A/Z(A)$ unless the following exceptional conditions all occur:

- (4.4a) $p(A) = 2$.
- (4.4b) $p$ is a Fermat prime.
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(4.4c) \(|\Phi(A)| = 2.\)

(4.4d) \(U = [A/Z(A)]^+, [B, P]\) is an irreducible \(Z_2[PB]\)-module.

(4.4e) \([U, P]\) is an irreducible \(Z_2[P]\)-module.

(4.4f) The function \(g: \sigma Z(A) \times r Z(A) \rightarrow [\sigma, r]\) is a well-defined, PB-invariant, non-singular, alternating bilinear map of \(U \times U\) into \(\Phi(A)^+\).

**Proof.** Since \(p \neq p(A)\) (by (4.1d)) and \(V\) is not ample, both (3.9a) and (3.9b) must fail. So there are two possibilities: either \(P\) centralizes \(A\) or the exceptional case in (3.9b) occurs.

If \(P\) centralizes \(A\), then it centralizes \(A/Z(A)\) and hence centralizes \(A/Z(A)\), by Proposition 4.2. But \(A/Z(A)\) is a PB-group. So \([B, P]\) must centralize it, and the proposition is true in this case.

Assume that we are in the exceptional case of (3.9b) and that \([B, P]\) does not centralize \(A/Z(A)\). Then (4.4a, b) hold and \([A', P]\) is an irreducible \(Z_2[P]\)-module. Since \(A'/Z(A')\) is a natural epimorphic image of \(A\) (by (1.4b)), we conclude that \([A'/Z(A')]^+, P\) is either \(\{0\}\) or an irreducible \(Z_2[P]\)-module. By Proposition 4.2 the same holds for \([A/Z(A)]^+, P\).

Therefore \([U, P]\) is either \(\{0\}\) or an irreducible \(Z_2[P]\)-module. But \([U, P]\) cannot be \(\{0\}\), since \(U \neq \{0\}\) by assumption and

\[(4.5) \quad [U, [B, P]] = U,\]

which follows from the definition of \(U\) and the fact that \(p(B) \neq p(A)\) (by (4.1d)). Therefore (4.4e) holds.

Hypothesis (4.1d) says that \(p(A)\) does not divide \(|PB|\). Therefore \(U\) is a completely reducible \(Z_2[PB]\)-module. If \(U\) is reducible, then \(U = U_1 \oplus U_2\), where \(U_1, U_2\) are non-trivial \(Z_2[PB]\)-submodules. Clearly (4.5) implies \([U_i, [B, P]] = U_i\) and hence \([U_i, P] \neq \{0\}\), for \(i = 1, 2\). So \([U, P] = [U_1, P] \oplus [U_2, P]\) is reducible, contradicting (4.4e). Therefore (4.4d) holds.

Since we are in a case of (3.9b), Corollary 3.12 tells us that \([A', P]\) is extra special with \(\Phi([A', P]) = \Phi(A')\). Because \(p(A) = 2\), this implies that \(|\Phi(A')| = 2\). By (4.1e) the natural epimorphism of \(\Phi(A)\) onto \(\Phi(A')\) is an isomorphism. Therefore (4.4c) holds.

By (1.4b), the function \(g\) of (4.4f) is a well-defined, non-singular, alternating bilinear map of \([A/Z(A)]^+ \times [A/Z(A)]^+\) into \(\Phi(A)^+\). It is obviously PB-invariant. From (4.4c) we conclude that \(PB\) centralizes \(\Phi(A)\). Since \(p(B) \neq 2\), this implies that

\([A/Z(A)]^+ = U \oplus C_{[A/Z(A)]^+}([B, P]),\]

where these two subspaces are \(g\)-perpendicular. It follows that the restriction of \(g\) to \(U \times U\) is non-singular, which proves (4.4f) and completes the proof of the proposition.

Now we investigate the exceptional case in Proposition 4.3. I.e., we assume that (4.1) and (4.4) hold. We choose a finite algebraic extension \(F\)
of \( Z_2 \) so that \( F \) is a splitting field for all subgroups of \( PB \). Then the extension \( F \otimes x_1 U \) of \( U \) to an \( F[PB] \)-module has the decomposition

\[
F \otimes U = U_1 \oplus \cdots \oplus U_t,
\]

where \( U_1, \ldots, U_t \) are absolutely irreducible \( F[PB] \)-submodules. From (4.5) and the equation corresponding to (3.11), we know that

\[
\begin{align*}
(4.6a) \quad [U_i, [B, P]] &= U_i \text{ for } i = 1, \ldots, t, \\
(4.6b) \quad [U_i, P] &\neq \{0\}, \text{ for } i = 1, \ldots, t, \\
(4.6c) \quad B_{U_i} &= B_{U_1} \text{ for } i = 1, \ldots, t \text{ (as factor groups of } B). 
\end{align*}
\]

The first step in the investigation is

**Lemma 4.8.** \( p = p(B) \).

**Proof.** Suppose that \( p \neq p(B) \). Let \( i = 1, \ldots, t \). By (4.7a), \( P \) does not centralize \( B_{U_i} \). If \( P \) does not centralize \( Z(B_{U_i}) \), then \( U_i \) is induced from some \( F[B] \)-module by Proposition 3.2. Hence \( U_i \) contains a regular \( F[P] \)-submodule. If \( P \) does centralize \( Z(B_{U_i}) \), then Proposition 3.4 says that \( U_i \) is isomorphic to an outer Kronecker product \( W \otimes Y \) of an \( F[P_{B_{U_i}}, P] \)-module \( W \) and an \( F(C_{B_{U_i}}(P)) \)-module \( Y \). The exceptional case in (3.5) does not hold here, since \( p(B) \neq 2 \). Therefore \( W \) contains a regular \( F[P] \)-submodule, which implies that \( U_i \) does also.

The above argument shows that each \( U_i, i = 1, \ldots, t \), contains a regular \( F[P] \)-submodule. It follows from (4.6) that the multiplicity of any non-trivial irreducible \( F[P] \)-module \( Z \) as a component of \( F \otimes U \) is at least \( t \). But condition (4.4e) forces the multiplicity of \( Z \) as a component of \( [F \otimes U, P] \) to be at most one. Since \( Z \) is non-trivial, these two multiplicities are equal. Therefore \( t = 1 \).

The function \( g \) of (4.4f) has a natural extension to a \( PB \)-invariant, non-singular, alternating, \( F \)-bilinear map \( g' \) of \( (F \otimes U) \times (F \otimes U) \) into \( F \otimes \Phi(A)^+ \). By (4.4c), \( F \otimes \Phi(A)^+ \) is \( F \)-isomorphic to \( F \) as a trivial \( F[PB] \)-module. So the non-singularity of \( g' \) gives us an \( F[PB] \)-isomorphism of \( F \otimes U = U_1 \) onto its dual module \( \text{Hom}_F(F \otimes U, F) = \text{Hom}_F(U_1, F) \). This is impossible since \( U_1 \) is a non-trivial irreducible \( F[PB] \)-module and \( |PB| \) is odd (by (4.1d) and (4.4a)). This contradiction proves the lemma.

Let \( e \) be the smallest positive integer such that \( 2^e \equiv 1 \) (mod \( p \)). Then we have

**Lemma 4.9.** \( t = e \) and \( \dim_F[U_i, P] = 1 \), for \( i = 1, \ldots, e \).

**Proof.** Since \( P \) is cyclic of order \( p \), every non-trivial irreducible \( Z_2[P] \)-module has dimension \( e \). So (4.4e) and (4.7b) imply

\[
e = \dim_F[F \otimes U, P] = \dim_F[U_1, P] + \cdots + \dim_F[U_t, P] \geq t.
\]

Furthermore, equality holds if and only if the lemma is true. Hence we need only show that \( t \geq e \).
Lemma 4.8 tells us that $PB_v$ is a $p$-group. Its normal subgroup $B_v$ is non-trivial by (4.5). So there must exist a subgroup $P_1$ of order $p$ satisfying $P_1 \leq B_v \cap Z(PB_v)$. The group $P_1$ acts faithfully on $U$. Hence there is some non-trivial irreducible $Z_2[P_1]$-submodule $W$ of $U$. Since $P_1$ is also cyclic of order $p$, the $F[P_1]$-submodule $F \otimes W$ is the direct sum $W_1 \oplus \cdots \oplus W_t$ of $t$ distinct irreducible $F[P_1]$-submodules $W_i$. Because $P_1$ is central in $PB_v$, distinct $W_i$ must be submodules of distinct absolutely irreducible $F[BP]$-modules $U_i$. Therefore $t \geq e$, which proves the lemma.

The condition that $\dim_F[U_t, P] = 1$ is very stringent. E.g., it implies

**Lemma 4.10.** $P$ centralizes every $P$-invariant abelian subgroup of $B_v$.

**Proof.** Let $D$ be a $P$-invariant abelian subgroup of $B_v$ such that $[D, P] \neq \{1\}$. By (4.7c), $[D, P]$ acts faithfully on $U_1$. So there must be some irreducible $F[PD]$-submodule $W$ of $U_1$ such that $[D_w, P] \neq \{1\}$. Because $D = Z(D)$ is abelian, Proposition 3.2 tells us that $W$ is induced from an irreducible $F[D]$-module. Hence $W$ contains a regular $F[P]$-submodule. By Lemma 4.9 this implies

$$1 = \dim_F[U_1, P] \geq \dim_F[W, P] \geq \dim_F[F[P], P] = p - 1.$$ 

Therefore $p = 2$, which contradicts (4.1d) and (4.4a). This proves the lemma.

Some judicious choices of abelian subgroups of $B_v$ give us a string of consequences from Lemma 4.10.

**Lemma 4.11.** $B_v$ is extra-special, with $[\Phi(B_v), P] = \{1\}$.

**Proof.** Lemma 4.10 forces $P$ to centralize $Z(B_v)$. So $Z(B_v) \leq Z(PB_v)$. Because $Z(B_v)$ acts faithfully on the irreducible $Z_2[PB_v]$-module $U$, this implies that $Z(B_v)$ is cyclic. We know from (4.1d) and (4.4a) that $p(B)$ is odd. Hence (4.1a) and (1.4d) say that $B_v$ has exponent $p(B)$. Therefore $|Z(B_v)| = p(B)$.

Since (4.4d) holds, $P$ cannot centralize $B_v$. So Lemma 4.10 says that $B_v' \neq \{1\}$. From the inclusion

$$\{1\} < B_v' \leq \Phi(B_v) \leq Z(B_v)$$

(by (1.4b)) and the fact that $|Z(B_v)| = p(B)$, we conclude that $B_v' = \Phi(B_v) = Z(B_v)$ is cyclic of order $p(B)$. I.e., $B_v$ is extra-special. Furthermore, $P$ centralizes $\Phi(B_v) = Z(B_v)$. So the lemma is true.

Fix a generator $\pi$ of the cyclic group $P$.

**Lemma 4.12.** $[\tilde{B_v}, P] = \tilde{B_v}(\pi - 1)$ is $f_{B_v}$-perpendicular to $C_{\tilde{B_v}}(P)$. Hence $[B_v, P]$ centralizes $C_{B_v}(P)$.

**Proof.** Since $P = \langle \pi \rangle$ is cyclic and $\tilde{B_v}$ is a vector space, we know that $[\tilde{B_v}, P] = \tilde{B_v}(\pi - 1)$. Suppose that $\sigma \in \tilde{B_v}$ and $\tau \in C_{\tilde{B_v}}(P)$. Then

$$f_{B_v}(\sigma(\pi - 1), \tau) = f_{B_v}(\sigma, \tau(\pi^{-1} - 1)),$$
since $P$ centralizes $\Phi(B_V)$ (by Lemma 4.11) and leaves $f_{B_V}$ invariant. But
\[ \tau \in C_{B_V}(P) \text{ implies } \tau(\pi^{-1} - 1) = 0 \] and hence
\[ f_{B_V}(\tau, \tau(\pi^{-1} - 1)) = 0. \]
Therefore $B_V(\pi - 1)$ and $C_{B_V}(P)$ are $f_{B_V}$-perpendicular.
The images in $B_V$ of $[B_V, P]$ and $C_{B_V}(P)$ are certainly contained in $[B_V, P]$ and
$C_{B_V}(P)$, respectively. So the above facts and (1.6) imply that $[B_V, P]$ centralizes $C_{B_V}(P)$. This proves the lemma.

**Lemma 4.13.** $[B_V, P, P] = \{1\}.$

**Proof.** By Lemma 4.8 there must be some positive integer $n$ such that
$[B_V, P]^n = \{1\}$. Let $n$ be the least such integer. Assume that $n \geq 3$.
Then we may choose an element $\sigma$ in $[B_V, P]^{n-2} - C_{B_V}(P)$. We have
\[ \sigma^{x^i} \in [B_V, P]^{n-2} \leq [B_V, P] \text{ for } i = 1, \ldots, p, \]
and
\[ [\sigma^{x^i}, \pi^j] \in [B_V, P]^{n-1} \leq C_{B_V}(P) \text{ for } i, j = 1, \ldots, p. \]
So Lemma 4.12 tells us that $\sigma^{x^i}$ commutes with $[\sigma^{x^i}, \pi^j]$, for all $i, j = 1, \ldots, p$.
Hence $\sigma^{x^i}$ commutes with $\sigma^{x^j} = \sigma^{y^i[\sigma^{x^i}, \pi^{i-1}]}$, for all $i, j = 1, \ldots, p$. We
conclude that $(\sigma^{x^i}, \sigma^{x^j}, \ldots, \sigma^{x^p})$ is a $P$-invariant abelian subgroup of $B_V$.
Lemma 4.10 says that it is centralized by $P$. But $P$ does not centralize the
element $\sigma^{x^p} = \sigma$ of this subgroup. Therefore $n \leq 2$, which is the lemma.

**Lemma 4.14.** $\dim_{\mathbb{Z}_p}[B_V, P] \leq 2$.

**Proof.** The endomorphism $\sigma \rightarrow \sigma(\pi - 1)$ of $B_V$ defines a $\mathbb{Z}_p$-isomorphism
of $W = \bar{B}_V/C_{\bar{B}_V}(P)$ onto $[B_V, P]$. It follows from Lemma 4.12 that the function $g$ given by
\[ g(\sigma + C_{\bar{B}_V}(P), \tau + C_{\bar{B}_V}(P)) = f_{B_V}(\sigma, \tau(\pi - 1)) \text{ for } \sigma, \tau \in \bar{B}_V, \]
is a well-defined, $\mathbb{Z}_p$-bilinear map of $W \times W$ into $\Phi(B_V)^{\perp}$.
If $\sigma, \tau \in \bar{B}_V$, we compute
\[
\begin{align*}
g(\sigma + C_{\bar{B}_V}(P), \tau + C_{\bar{B}_V}(P)) &= f_{B_V}(\sigma, \tau(\pi - 1)) \\
&= f_{B_V}(\sigma(\pi^{-1} - 1), \tau) \quad \text{(by Lemma 4.11)} \\
&= -f_{B_V}(\tau, \sigma(\pi^{-1} - 1)) \quad \text{($f_{B_V}$ is alternating)} \\
&= f_{B_V}(\tau, \sigma(1 - \pi^{-1})) \quad \text{($f_{B_V}$ is bilinear)} \\
&= f_{B_V}(\tau, (\pi^{-1})(\pi - 1)) \\
&= g(\tau + C_{\bar{B}_V}(P), (\sigma\pi^{-1}) + C_{\bar{B}_V}(P)) \\
&= g(\tau + C_{\bar{B}_V}(P), \sigma + C_{\bar{B}_V}(P)),
\end{align*}
\]
since $\sigma\pi^{-1} - \sigma \in [B_V, P] \leq C_{B_V}(P)$ by Lemma 4.13. So $g$ is symmetric.
But $\Phi(B_V)^{\perp}$ is $\mathbb{Z}_p$-isomorphic to $\mathbb{Z}_p$ by Lemma 4.11. Therefore $g$ is just an
ordinary quadratic form on the vector space $W$ over $\mathbb{Z}_p$. 

Suppose that \( \dim_{\mathbb{Z}(W)} \geq 3 \). Since \( Z_p \) is a finite field of odd characteristic, there must exist some element \( w \neq 0 \) in \( W \) so that \( g(w, w) = 0 \) (see page 144 of [1]). Let \( \sigma \in \hat{B}_U \) satisfy \( w = \sigma + C_{\hat{B}_U}(P) \). Then \( Z_{\sigma \sigma} + Z_{\sigma}(\pi - 1) \) is totally isotropic with respect to \( f_{\hat{B}_U} \), since \( f_{\hat{B}_U}(\sigma, \sigma(\pi - 1)) = g(w, w) = 0 \). This subspace is \( P \)-invariant by Lemma 4.13 and is not centralized by \( P \) since \( \sigma + C_{\hat{B}_U}(P) = w \neq 0 \). Therefore its inverse image is a \( P \)-invariant abelian subgroup of \( B_U \) which is not centralized by \( P \). This contradicts Lemma 4.10. Hence \( \dim_{\mathbb{Z}(W)} \leq 2 \), which is equivalent to the lemma by the first line of the proof.

We must reach back to Lemma 4.9 to prove

**Lemma 4.15.** \( p = 3 \).

**Proof.** Since (4.4d) holds, \( P \) does not centralize \( B_U \). So we may choose \( \sigma \in B_U \) such that \( [\pi, \sigma] \neq 1 \). If \( \sigma \) centralizes \( [\pi, \sigma] \), then Lemma 4.13 implies that \( \langle \sigma, [\pi, \sigma] \rangle \) is a \( P \)-invariant abelian subgroup of \( B_U \). It is not centralized by \( P \) since \( [\pi, \sigma] \neq 1 \). This contradicts Lemma 4.10. Therefore \( [\pi, \sigma] = [\pi, \sigma, \sigma] \neq 1 \). From Lemmas 4.11 and 4.13 we now conclude that \( D = \langle \sigma, [\pi, \sigma] \rangle \) is a \( P \)-invariant extra-special subgroup of order \( p^3 \) in \( B_U \) with \( \Phi(D) = \Phi(B_U) \).

The group \( P \) centralizes \( \Phi(D) = \langle [\pi, \sigma, \sigma] \rangle \) by Lemma 4.11 and centralizes \( [\pi, \sigma] \) by Lemma 4.13. If follows that \( E = \langle [\pi, [\pi, \sigma], [\pi, \sigma, \sigma] \rangle \) is an abelian subgroup of order \( p^4 \) in \( PD \). Since \( |PD| = p^4 \), \( E \) is normal in \( PD \).

The group \( \Phi(D) = \langle [\pi, \sigma, \sigma] \rangle \) is clearly the center of \( PD \). Since \( \Phi(D) \) acts faithfully on \( U_1 \) (by (4.7c)), there must be an irreducible \( F[PD] \)-submodule \( W \) of \( U_1 \) on which \( \Phi(D) \) acts non-trivially. From Lemma 4.9 we know that \( \dim_{\mathbb{F}^p}[W, P] \) is 0 or 1. So the restriction \( \psi_P \) to \( P \) of the modular character \( \psi \) (see [5] for definitions) of the \( F[PD] \)-module \( W \) has the form:

\[
\psi_P = \lambda + (\psi(1) - 1) \cdot 1,
\]

for some linear character \( \lambda \) of \( P \).

Because \( |PD| = p^4 \) is odd, the modular character \( \psi \) is an ordinary irreducible character of \( PD \) (see [5]). The non-triviality of \( \Phi(D) \) on \( W \) implies that \( \psi_{\Phi(D)} = \psi(1) \cdot \nu \) for some non-trivial ordinary linear character \( \nu \) of \( \Phi(D) \). Since \( \nu \) is faithful and \( \Phi(D) = \langle [\pi, \sigma, \sigma] \rangle \leq [E, \langle \sigma \rangle] \), no extension \( \mu \) of \( \nu \) to a linear character of \( E \) can be fixed by \( \langle \sigma \rangle \). It follows that \( \psi = \mu^{PD} \), for some such extension \( \mu \). Therefore

\[
\psi_E = \mu + \mu^\sigma + \cdots + \mu^{\sigma^{p-1}}.
\]

But \( P \leq E \). Hence

\[
\psi_P = \mu_P + (\mu^\sigma)_P + \cdots + (\mu^{\sigma^{p-1}})_P.
\]

Comparing this with (4.16), we see that \( (\mu^i)_p \) must be trivial for all but one \( i = 0, 1, \ldots, p - 1 \). We may assume that the exceptional value of \( i \)
(if any exists) is \( i = 0 \), so that \( \mu^{e^i}(\pi) = 1 \), for \( i = 1, \ldots, p - 1 \). Hence
\[
1 = \mu^{e^i}(\pi) = \mu(\pi^{-i}) = \mu(\pi(\pi, \sigma)^{-i}(\pi, \sigma, \sigma)^{C(i, 2)})
\]
where, of course, \( C(i, 2) \) is the binomial symbol \( i(i - 1)/2 \). Taking \( i = 1 \) in this we get \( \mu(\pi(\pi, \sigma)) = \mu(\pi) \). Taking \( i = 2 \), we then get \( \mu(\pi(\pi, \sigma, \sigma)) = \mu(\pi) \). If \( p > 3 \), we may take \( i = 3 \), getting \( 1 = \mu(\pi^{-3+3}) = \mu(\pi) = \mu(\pi(\pi, \sigma, \sigma)) \).
This contradicts the fact that \( \mu(\pi(\pi, \sigma, \sigma)) = \nu(\pi(\pi, \sigma, \sigma)) \neq 1 \). So the lemma is true.

We collect the results of the last eight lemmas and one further consequence in

**Proposition 4.17.** If both (4.1) and (4.4) hold, then

\[
\begin{align*}
(4.18a) & \quad p = p(B) = 3, \\
(4.18b) & \quad \text{U is not an ample } Z_2[B]-\text{module,} \\
(4.18c) & \quad B_U \text{ is extra-special,} \\
(4.18d) & \quad \dim_{Z_2} B_{U, P} \leq 2, \\
(4.18e) & \quad \dim_{Z_2} C_U(P) \geq 4.
\end{align*}
\]

**Proof.** Conclusion (4.18a) is Lemmas 4.8 and 4.15.

Conclusion (4.18b) is Lemma 4.13, since (4.18a) holds (compare (3.9c)).

Conclusion (4.18c) is Lemma 4.11.

Conclusion (4.18d) is Lemma 4.14, since \( p = 3 \).

By (4.18c) and (4.7c), each \( U_i \) has dimension at least \( p(B) = 3 \). Since \( U_i = [U_i, P] \oplus C_{U_i}(P) \) and \( \dim_{\mathbb{F}} [U_i, P] = 1 \), by Lemma 4.9, each \( C_{U_i}(P) \) has dimension at least 2. So (4.6) gives
\[
\dim_{Z_2} C_U(P) = \dim_{\mathbb{F}} C_{U \otimes U}(P) = \sum_{i=1}^{t} \dim_{\mathbb{F}} C_{U_i}(P) \geq 2t.
\]
But \( t = e = 2 \) by Lemma 4.9, since \( p = 3 \). Therefore (4.18e) holds and the proposition is true.

We shall apply the above propositions to the situation in which our group \( P \) of arbitrary prime order \( p \) acts on a Fitting chain \( A_1, \ldots, A_t \) with \( [A_1, P] \neq \{1\} \). We wish to show that some representation \((PA_i on A_{i+1})\) has an ample irreducible component, provided the Fitting chain is long enough. To specify the necessary length, we define an integer \( i_0 \) by

\[
\begin{align*}
(4.19a) & \quad i_0 = 2 \text{ if } p \text{ does not divide } \prod_{i=1}^{t-1} |A_i|, \\
(4.19b) & \quad i_0 = 3 \text{ if } p \text{ divides } \prod_{i=1}^{t-1} |A_i| \text{ and } p \neq 3, \\
(4.19c) & \quad i_0 = 5 \text{ if } p \text{ divides } \prod_{i=1}^{t-1} |A_i| \text{ and } p = 3.
\end{align*}
\]

Then we have

**Theorem 4.20.** If \( t > i_0 \), then there is some \( i = 1, \ldots, i_0 \) such that \( p(A_i) \neq p \) and \((PA_i on A_{i+1})\) has an ample irreducible component.

**Proof.** Define the groups \( B_1, \ldots, B_t \) inductively by
\[
B_1 = A_1, \quad B_{i+1} = [A_{i+1}, [B_i, P]], \quad \text{for } i = 1, \ldots, t - 1.
\]
We first prove that

\[(4.21a) \quad B_i \text{ is a } P\text{-invariant subgroup of } A_i, \text{ for } i = 1, \cdots, t,\]

\[(4.21b) \quad B_i \text{ is } PB_{i-1}\text{-invariant, for } i = 2, \cdots, t,\]

\[(4.21c) \quad [B_i, P] \neq \{1\}, \text{ for } i = 1, \cdots, t,\]

\[(4.21d) \quad B_i \in \mathfrak{a}, \text{ for } i = 1, \cdots, t.\]

Statement (4.21a) is obvious from the definition of the $B_i$. It implies that $[B_{i-1}, P]$ is a normal subgroup of $PB_{i-1}$, for $i = 2, \cdots, t$. Therefore

$$B_i = [A_i, [B_{i-1}, P]]$$

is $PB_{i-1}$-invariant, which is statement (4.21b).

Statement (4.21c) is proved by induction on $i$. For $i = 1$ it is true since $[A_1, P] \neq \{1\}$ by hypothesis. Suppose that $i > 1$ and that $[B_{i-1}, P] \neq \{1\}$. By (2.2d), $[B_{i-1}, P]$ acts faithfully on $A_{i-1}$. Therefore

$$B_i = [A_i, [B_{i-1}, P]] \neq \{1\}. \quad (4.21c)$$

Since $p([B_{i-1}, P]) = p(A_{i-1}) \neq p(A_i)$ (by (2.2b)), we have $\{1\} \neq B_i = [B_i, [B_{i-1}, P]]$. If $P$ centralizes $B_i$, then so does $[B_{i-1}, P]$, (by (4.21b)) which contradicts the preceding statement. Therefore $[B_i, P] \neq \{1\}$ and (4.21c) holds. This implies that $B_i \neq \{1\}$, for $i = 1, \cdots, t$. So (4.21d) follows from (1.5) and (2.2a).

Suppose we can prove that

\[(4.22) \text{ there is some } i = 1, \cdots, i_0 \text{ such that } p \neq p(A_i) \text{ and } (PB_i \text{ on } \tilde{A}_{i+1}) \text{ has an ample irreducible component.}\]

Then the theorem will be true. To see this, let $W$ be an ample irreducible component of $(PB_i$ on $\tilde{A}_{i+1})$. Then there must be some irreducible component $V$ of $(PA_i$ on $\tilde{A}_{i+1})$ such that $W$ is $PB_i$-isomorphic to an irreducible component of $(PB_i$ on $V)$. Proposition 3.13, applied to $Z_{p(A_{i+1})}, P, A_i, B_i, V$ and $W$, tells us that $V$ is ample. So the theorem will follow from (4.22).

From now on we assume that (4.22) is false, i.e., that no irreducible component of any $(PB_i$ on $\tilde{A}_{i+1})$ is ample, for any $i = 1, \cdots, i_0$ such that $p(A_i) \neq p$.

Suppose that $2 \leq i \leq i_0$ and $p(A_i) \neq p$. Since $[B_i, P]$ acts faithfully on $A_{i+1}$ (by (2.2d)) and $p(A_{i+1}) \neq p(A_i) = p([B_i, P])$ (by (2.2b)), it acts faithfully on $\tilde{A}_{i+1}$ (see Theorem III, 3.18 of [4]). So (4.21c) implies that $Y_{i+1} = [\tilde{A}_{i+1}, [B_i, P]]$ is a non-zero $Z_{p(A_{i+1})}[PB_i]$-submodule of $\tilde{A}_{i+1}$. Furthermore, $[Y_{i+1}, [B_i, P]] = Y_{i+1}$, since $p([B_i, P]) \neq p(A_{i+1})$. Therefore there is some irreducible component $W_{i+1}$ of $(PB_i$ on $Y_{i+1})$ and any such $W_{i+1}$ satisfies $[W_{i+1}, [B_i, P]] = W_{i+1}$.

Let $K_i = \text{Ker}(\Phi(B_i)$ on $W_{i+1})$. Since $K_i \leq \Phi(B_i) \leq \Phi(A_i)$, it is centralized by $B_{i-1} \leq A_{i-1}$ (by (2.2c)). Therefore $PB_{i-1}$ acts on $B_i/K_i$. Now we see that $P, B = B_{i-1}, A = B_i/K_i, W_{i+1}$ and $q = p(A_{i+1})$ satisfy (4.1).
Suppose that \((B_i)_{w_{i+1}}\) is abelian. Since \([W_{i+1}, [B_i, P]] = W_{i+1}\), we have \([Z((B_i)_{w_{i+1}}), P] = [(B_i)_{w_{i+1}}, P] \neq \{1\}\). But \(p(B_i) = p(A_i) \neq p\). So (3.9a) holds and \(W_{i+1}\) is ample, contradicting our assumptions. Therefore \((B_i)_{w_{i+1}} < (B_i)_{w_{i+1}}\). By Proposition 4.2 this implies that \(Z(A) < A\).

By construction \([B_1, [B_{i-1}, P]] = B_{i-1}\). Therefore \([A/Z(A), [B, P]] = A/Z(A)\). Since \(A/Z(A) \neq \{1\}\) and \(W_{i+1}\) is not ample, Proposition 4.3 says that (4.4) holds. So Proposition 4.17 tells us that (4.18) holds. In particular, \(3 = p = p(B) = p(A_{i-1})\).

Suppose that (4.19a) holds. Then \(p(A_2) \neq p\). The above argument shows that \(p(A_1) = p\), contradicting (4.19a). So (4.22) cannot be false, and the theorem is true in this case.

Suppose that (4.19b) holds. By (2.2b) there is some \(i = 2, 3\) such that \(p(A) \neq p\). The above argument shows that \(p = 3\), contradicting (4.19b). So the theorem is true in this case.

We must be in the case (4.19c). There is some \(i = 4, 5\) such that \(p(A) \neq p\). The above argument shows that \(p(A_1) = p = 3\) and \(p(A_i) = 2\). Furthermore, since (4.18) holds, we have an irreducible \(Z_2[PB_{i-1}]-\)module \(U_i\) so that \((B_{i-1})_{U_i}\) is extra-special and \(\dim_{Z_2} [(B_{i-1})_{U_i}, P] \leq 2\).

Let

\[K_{i-1} = \text{Ker}(\Phi(B_{i-1}) \text{ on } U_i)\]

By (2.2c), \(B_{i-2}\) centralizes \(K_{i-1}\). Therefore \(PB_{i-2}\) acts on \(B_{i-1}/K_{i-1}\). Evidently \(P, B_{i-2}, B_{i-1}/K_{i-1}\), and \(U_i\) satisfy (4.1a, b, c, e). So Proposition 4.2 says that

\[[(B_{i-1}/K_{i-1})/Z(B_{i-1}/K_{i-1})]^+ = Y_{i-1}\]

is \(Z_3[P]\)-isomorphic to \([(B_{i-1})_{U_i}/Z((B_{i-1})_{U_i})]^+\). The latter group is just \((B_{i-1})_{U_i}\), since \((B_{i-1})_{U_i}\) is extra-special. Hence \(Y_{i-1} \neq \{0\}\) and \(\dim_{Z_2} [Y_{i-1}, P] \leq 2\).

Because \(PB_{i-2}\) acts on \(B_{i-1}/K_{i-1}\), it acts on \(Y_{i-1}\). By (2.2b) and the definition of \(B_{i-1}\), we have \([B_{i-1}, [B_{i-2}, P]] = B_{i-1}\). It follows that

\[[Y_{i-1}, [B_{i-2}, P]] = Y_{i-1}\]

Since \(Y_{i-1} \neq \{0\}\), there is some irreducible component \(W_{i-1}\) of \((PB_{i-2} \text{ on } Y_{i-1})\). Clearly \([Y_{i-1}, [B_{i-2}, P]] = Y_{i-1}\) and \(\dim_{Z_2} [Y_{i-1}, P] \leq 2\) imply that

\[\text{(4.23a)}\] \([W_{i-1}, [B_{i-2}, P]] = W_{i-1}\].
\[\text{(4.23b)}\] \(\dim_{Z_2} [W_{i-1}, P] \leq 2\).

Since \(W_{i-1}\) is a section of \(A_{i-1}\), it follows from (2.2e) and (4.23a) that it is isomorphic to some irreducible component of \((PB_{i-2} \text{ on } A_{i-1})\). So \((PB_{i-2} \text{ on } W_{i-1})\) is not ample.

Now we repeat our earlier argument with \(i = 2\) in place of \(i\). It tells us that \(P, B_{i-2}, B_{i-2}/K_{i-2}\) and \(W_{i-2}\) satisfy (4.1), (4.4), and (4.18), where \(K_{i-2} = \text{Ker}(\Phi(B_{i-2}) \text{ on } W_{i-1})\). The definition (4.4d) of the \(Z_2[PB_{i-2}]\)-
module $U_{i-2}$, the definition of $B_{i-2}$, and (2.2b) imply that
\[ U_{i-2} = [(B_{i-2}/K_{i-2})/Z(B_{i-2}/K_{i-2})]^+. \]
From Proposition 4.2 we conclude that $U_{i-2}$ is $P$-isomorphic to
\[ [(B_{i-2})_{w_{i-1}}/Z((B_{i-2})_{w_{i-1}})]^+ = \tilde{B}_{i-2}. \]
Therefore (4.18e) gives
\[ (4.24) \quad \dim_{\mathbb{Z}} C_{B_{i-2}}(P) \geq 4. \]
Let $F$ be a finite algebraic extension field of $\mathbb{Z}$ which is a splitting field for every subgroup of $PB_{i-2}$. Let $X$ be an irreducible $F[PB_{i-2}]$-submodule of the extension $F \otimes W_{i-1}$ of $W_{i-1}$ to an $F[PB_{i-2}]$-module. Then $(B_{i-2})_X = (B_{i-2})_{w_{i-1}}$ as in (4.7e). Because $(PB_{i-2}$ on $W_{i-1}$) lies in the exceptional case of (3.9b), Proposition 3.4 applies to $P$, $B_{i-2}$, $F$ and $X$. It tells us that $X$ is the outer Kronecker product $X = S \otimes T$ of an irreducible $F[P[(B_{i-2})_X, P]]$-module $S$ and an irreducible $F[C_{(B_{i-2})_X}(P)]$-module $T$. Since $Z((B_{i-2})_X) \leq C = C_{(B_{i-2})_X}(P)$, and $(B_{i-2})_X$ is the central product of $[(B_{i-2})_X, P]$ and $C$, we know that $Z((B_{i-2})_X) = Z(C)$. This acts faithfully on $T$ since it acts faithfully on $X$. It follows from (4.24) and Theorem (III.2) of [2] that
\[ \dim_{\mathbb{F}} T = [C:Z(C)]^{1/2} = |C_{B_{i-2}}(P)|^{1/2} \geq (2^4)^{1/2} = 2^2 = 4. \]
Since $P$ acts faithfully on $[(B_{i-2})_X, P]$ and the latter acts faithfully on $S$, we know that $P$ acts faithfully on $S$. So $\dim_{\mathbb{F}} [S, P] \geq 1$. It follows from (4.23b) that
\[ 2 \geq \dim_{\mathbb{F}} [F \otimes W_{i-1}, P] \geq \dim_{\mathbb{F}} [X, P] = \dim_{\mathbb{F}} [S, P] \cdot \dim_{\mathbb{F}} T \geq 4. \]
This contradiction proves the theorem.

5. Finding enough ample representations

We now turn to the problem of going from the ample components of $(PA_{i-1}$ on $A_i)$ to those of $(PA_i$ on $A_{i+1}$) in our $P$-invariant Fitting chain $A_1, \ldots, A_i$. The critical case is the following situation:

(5.1a) $PB$ is the semi-direct product of a group $P$ of prime order $p$ acting on a group $B \in \mathfrak{A}$.
(5.1b) $D$ is a subgroup of $C_p(P)$.
(5.1c) $PBA$ is the semi-direct product of $PB$ acting on a group $A \in \mathfrak{A}$.
(5.1d) $V$ is a finite-dimensional $\mathbb{Z}_q[PA]$-module, for some prime $q$.
(5.1e) $p, p(B)$ and $q$ are all different from $p(A)$.
(5.1f) $[\Phi(A), B] = \{1\}$.
(5.1g) Each irreducible component of $(PB$ on $A)$ is ample.
(5.1h) The representation $(A$ on $V)$ is faithful and weakly $B$-invariant.
(5.1i) If $p = p(B)$ then $D \leq [B, P]^{p-1}$.

One immediate consequence of these hypotheses is

PROPOSITION 5.2. $A' = \Phi(A)$. 

Proof. By (5.1g) and (3.9), $B$ acts non-trivially on each irreducible component $U$ of $(PB \circ A)$. Hence $[U, B] = U$. It follows that $[\bar{A}, B] = \bar{A}$. Since $\bar{A}$ is naturally isomorphic to $(A/A')/\Phi(A/A')$, we conclude that $[A/A', B] = A/A'$. The map $\sigma \mapsto \sigma(A)$ is a $B$-invariant epimorphism of the abelian group $A/A'$ onto $\Phi(A/A') = (A/A')$. Therefore $[\Phi(A)/A', B] = \Phi(A)/A'$. By (5.1f), this implies that $\Phi(A)/A' = \{1\}$. So the proposition holds.

We let $\mathcal{K} = \mathcal{K}(V)$ and $\mathcal{K}_{\text{ample}} = \mathcal{K}_{\text{ample}}(V)$ be the families of (3.14) and (3.15). Define $\mathcal{L}$ to be the subfamily of all $K \in \mathcal{K}$ such that $K \geq \Phi(A)$.

**Proposition 5.3.** Both $\mathcal{K}$ and $\mathcal{L}$ are $PB$-invariant families of normal subgroups of $A$. Furthermore, $\mathcal{K} \cap \mathcal{L} \subseteq \mathcal{K}_{\text{ample}}$.

Proof. Since $V$ is a $PA$-module, $\mathcal{K}$ is $P$-invariant by (3.17). The weak $B$-invariance of $(A on V$) (by (5.1h)) clearly implies that $\mathcal{K}$ is $B$-invariant. So $\mathcal{K}$ is $PB$-invariant. Because $\Phi(A)$ is a characteristic subgroup of $A$, it is $PB$-invariant. So $\mathcal{L}$ is a $PB$-invariant subfamily of $\mathcal{K}$, which finishes the proof of the first statement of the proposition.

Suppose that $K \in \mathcal{K} \cap \mathcal{L}$. Then there exists some irreducible component $W$ of $(A on V)$ such that $K = \text{Ker}(A on W)$. There must be some irreducible component $X$ of $(PA on V)$ such that $W$ is isomorphic to a component of $(A on X)$. Clifford's theory (see Theorem V, 17.3 of [4]) tells us that

$$\text{Ker}(A on X) \cap K = K(P).$$

Therefore $K \in \mathcal{K}_{\text{ample}}$ if and only if $P$ and $A_x = A/K(P)$ satisfy (3.9), i.e., if and only if $(PA on X)$ is ample.

Let $N = \text{Ker}(\Phi(A) on X)$. Then $N$ is a $P$-invariant normal subgroup of $A$. By (5.1f) it is also $B$-invariant. So (5.1e) implies that $P, B, A/N$ and $X$ satisfy (4.1).

Proposition 4.2 says that $Y = [(A/N)/Z(A/N)]^+$ is isomorphic to $[A_x/Z(A_x)]^+$. Since $K \in \mathcal{L}$, we have $K \subset \Phi(A) < \Phi(A)$. By Proposition 5.2, this implies that $K \cap A' < A'$. Therefore

$$K(P) \cap A' \leq K \cap A' < A' \text{ and } A_x' \cong A'/K(P) \cap A' \neq \{1\}.$$ 

So $A_x$ is non-abelian. Hence $[A_x/Z(A_x)]^+ \neq \{0\}$. We conclude from this and (1.4b) that $Y$ is a non-trivial $Z_{p(d)}[PB]$-factor module of $\bar{A}$. In particular, $(PB on Y)$ has at least one irreducible component and, by (5.1g), each such component is ample.

Suppose that $(PA on X)$ is not ample. Then neither is $(P(A/N) on X)$. By Proposition 4.3, either $[B, P]$ centralizes $Y$ or (4.4) holds. But $(PB on Y)$ has an ample irreducible component, which, by (3.9), cannot be centralized by $[B, P]$. Hence (4.4) holds. In particular, (4.4d) says that $U$ is an irreducible component of $(PB on Y)$. So $(PB on U)$ is ample. This contradicts (4.18b). Therefore $X$ is ample and the proposition is true.
Let $L$ be the subgroup of $A$ defined by

$$L = \cap_{K \in \mathfrak{K}} K.$$  \hfill (5.5)

If $\mathfrak{K} = \emptyset$, this intersection is taken to be $A$.

**Proposition 5.6.** $L$ is $PB$-invariant normal subgroup of $A$ such that $L \cap \Phi(A) = \{1\}$. Hence $L$ is elementary abelian and the natural $Z_{p(A)}[PB]$-homomorphism $\varphi$ of $L^+$ into $\bar{A}$ is a monomorphism.

**Proof.** By the definition of $\mathfrak{K}$, each $K \in \mathfrak{K}$ is a normal subgroup of $A$. Proposition 5.3 implies that $\mathfrak{K}$ is a $PB$-invariant subfamily of $\mathfrak{K}$. So $L$ is a $PB$-invariant normal subgroup of $A$ by (5.5).

Let $a$ be a non-trivial element of $\Phi(A)$. By (5.1h) $a$ acts non-trivially on $V$. Since $q \neq p(A)$ (by (5.1e)), the representation $(A$ on $V)$ is completely reducible. So $a$ must act non-trivially on some irreducible component $W$ of $(A$ on $V)$. Hence $K = \text{Ker}(A$ on $W) \in \mathfrak{K}$ and $a \in K$. It follows that $\Phi(A) \leq K$, i.e., that $K \in \mathfrak{K}$. Therefore $a \in L \leq K$ (by (5.5)). This proves that $L \cap \Phi(A) = \{1\}$. The other statements follow directly from this.

Define families $\mathfrak{M}, \mathfrak{N}$ of subgroups of $L$ and a subgroup $N$ of $L$ by

$$\mathfrak{M} = \{K \cap L \mid K \in \mathfrak{K}, L \trianglelefteq K\},$$  \hfill (5.7a)

$$\mathfrak{N} = \{M \in \mathfrak{M} \mid [L, P] \trianglelefteq M\},$$  \hfill (5.7b)

$$N = \cap_{M \in \mathfrak{M}} M.$$  \hfill (5.7c)

Let $V_{ample}$ be the $\mathbb{Z}_q[PA]$-submodule of $V$ defined by (3.15) and $Q$ be the subgroup $\text{Ker}(A$ on $V_{ample})$ of $A$.

**Proposition 5.8.** $\mathfrak{M}$ is a $PB$-invariant family of maximal subgroups of $L$ satisfying

$$\cap_{M \in \mathfrak{M}} M = \{1\}.$$  \hfill (5.9)

The subfamily $\mathfrak{M}$ and the subgroups $N$ and $Q$ are $P \times D$-invariant. Furthermore,

$$Q \leq N \leq L.$$  \hfill (5.10)

**Proof.** Proposition 5.3 says that $\mathcal{E}$ is $PB$-invariant. Proposition 5.6 says that $L$ is $PB$-invariant. This and (5.7a) imply that $\mathfrak{M}$ is $PB$-invariant.

Suppose that $M \in \mathfrak{M}$. Then there exists $K \in \mathcal{E}$ such that $L \trianglelefteq K$ and $M = K \cap L$. Since $\mathcal{E} \subseteq \mathfrak{K}$, there is some irreducible component $W$ of $(A$ on $V)$ such that $K = \text{Ker}(A$ on $W)$. Because $K \in \mathcal{E}$, we have $\Phi(A) \leq K$. So the elementary abelian group $A/\Phi(A)$ acts irreducibly on $W$. This implies that $A_w = [A/\Phi(A)]_w$ is cyclic, and hence has order 1 or $p(A)$. Therefore

$$[L:M] = [L:K \cap L] \leq [A:K] \leq p(A).$$

But $L$ is a $p(A)$-group and $L \trianglelefteq K$. So $[L:K \cap L] = p(A)$. Hence any $M \in \mathfrak{M}$ is a maximal subgroup of $L$. 


Since \( g \neq p(A) \) (by (5.1e)), the representation \((A \text{ on } V)\) is fully decomposable. It follows that

\[
1 = \text{Ker}(A \text{ on } V) \quad \text{(by } (5.1h))
\]

\[
= \bigcap_{K \in \mathcal{K}} K = \bigcap_{K \in \mathcal{K}} K \cap \bigcap_{K \in \mathcal{K}} K \quad \text{(by } (5.5))
\]

\[
= \bigcap_{K \in \mathcal{K}} [L \cap K] \quad \text{(by } (5.7))
\]

\[
= \bigcap_{M \in \mathcal{M}} M \quad \text{(by } (5.7a))
\]

So (5.9) holds.

By (5.1a, b), \( PD = P \times D \leq PB \). Since \( \mathfrak{K} \), \( L \) and \( P \) are \( P \times D \)-invariant, so are \( \mathfrak{A} \) and \( N \) (by (5.7b, c)). It follows from the definition of \( \mathcal{K}_{\text{ample}} \) that it is \( P \times D \)-invariant. Since \( Q \) is the intersection of the members of \( \mathcal{K}_{\text{ample}} \) (by (3.15)), it is \( P \times D \)-invariant.

Suppose that \( K \in \mathcal{K}_{\text{ample}} \). Then \( K \geq \Phi(A) \). Hence \( K(P) = \cap_{t \in P} K^t \geq \Phi(A) \). So \( A/K(P) \) is abelian. Since \( K \in \mathcal{K}_{\text{ample}} \), \( P \) and \( A/K(P) \) cannot satisfy (3.9). Therefore (5.9) and (3.9a) imply that

\[
[A/K(P), P] = [Z(A/K(P)), P] = \{1\}.
\]

Hence \( [A/K, P] = \{1\} \). It follows that \( [L, P] \leq [A, P] \cap L \leq K \cap L \). From this and (5.7b) we conclude that \( K \cap L \in \mathfrak{A} \). So we have

\[
\mathfrak{A} \subseteq \{K \cap L \mid K \in \mathcal{K} \cap \mathcal{K}_{\text{ample}}\}.
\]

This implies

\[
N = \bigcap_{M \in \mathcal{M}} M \quad \text{(by } (5.7c))
\]

\[
\geq \bigcap_{K \in \mathcal{K} \cap \mathcal{K}_{\text{ample}}} (K \cap L)
\]

\[
\geq \bigcap_{K \in \mathcal{K} \cap \mathcal{K}_{\text{ample}}} K \cap \bigcap_{K \in \mathcal{K} \cap \mathcal{K}_{\text{ample}}} K
\]

\[
= \bigcap_{K \in \mathcal{K} \cap \mathcal{K}_{\text{ample}}} K \quad \text{(by Proposition 5.3)}
\]

\[
= Q \quad \text{(by } (3.15))
\]

Therefore (5.10) holds, which completes the proof of the proposition.

Define the section \( C \) of \( A \) by

\[
C = [C_A(P)]_{\text{ample}} = C_A(P)/C_A(P) \cap Q.
\]

Since \( D \) centralizes \( P \) (by (5.1b)), the subgroup \( C_A(P) \) is \( D \)-invariant. By Proposition 5.8, \( Q \) is \( D \)-invariant. Hence

\[
C \text{ is } D \text{-invariant}.
\]

We are interested in conditions which will guarantee the following property:

\[
C \neq \{1\} \text{ and } (D \text{ on } \tilde{A}) \text{ is weakly equivalent to } (D \text{ on } \tilde{C}).
\]
One set of such conditions is given by

**Proposition 5.14.** If \( L = \{1\} \), then (5.13) holds. If \( C_L(P) > C_N(P) \) and \((D \text{ on } C_L(P)^+)\) is weakly equivalent to \((D \text{ on } C_L(P)^+/C_N(P)^+)\), then (5.13) holds.

**Proof.** It follows from (5.1g) and Proposition 3.10 that \( C_A(P) \not\cong \{1\} \) (see Theorem I, 18.6 of [4]). Therefore \( L = \{1\} \) gives \( Q = \{1\} \) (by (5.10)) and \( C \not\cong \{1\} \) (by (5.11)).

If \( C_L(P) > C_N(P) \), then (5.10) implies that \( C_L(P) > C_q(P) \). By (5.11), \( C_L(P)/C_q(P) \) is isomorphic to a subgroup of \( C \). Hence \( C \not\cong \{1\} \) in both cases.

Because \( L = \{1\} \) trivially implies the condition "\((D \text{ on } C_L(P)^+)\) is weakly equivalent to \((D \text{ on } C_L(P)^+/C_N(P)^+)\)" we are reduced to deducing from this condition that \((D \text{ on } C_A(P))\) is weakly equivalent to \((D \text{ on } C_A(P))\).

Since \( Q \) is \( P \times D\)-invariant (by Proposition 5.8), so are \( A/Q \) and the natural epimorphism \( \psi \) of \( A \) onto \( A/Q \). It follows that \( \psi \) induces a \( \mathbb{Z}(A)[P \times D]\)-epimorphism \( \overline{\psi} \) of \( \bar{A} \) onto \( (A/Q) \). From Proposition 5.6 and (5.10) we conclude that \( \varphi(Q^+) = \ker \psi \). Hence

\[
\text{(5.15) } (D \text{ on } \overline{\psi}(C_A(P))) \text{ is equivalent to } (D \text{ on } C_A(P)/C_A(P) \cap \varphi(Q^+)).
\]

We know from (5.1e) that \( p(A) \) does not divide \( |P| \). It follows (see Theorem I, 18.6 of [4]) that \( C_A(P) \) is the natural image in \( \bar{A} \) of \( C_A(P) \).

By (5.11), this implies that \( \overline{\psi}(C_A(P)) \) is the natural image of \( \bar{C} \) in \( (A/Q) \).

The kernel of the natural map of \( \bar{C} \) onto \( \overline{\psi}(C_A(P)) \) is

\[
[(C \cap \Phi(A \text{ ample}))/\Phi(C)]^+,
\]

which \( D \) centralizes by (5.1f). From this we conclude that \( (D \text{ on } \bar{C}) \) is weakly equivalent to \((D \text{ on } \overline{\psi}(C_A(P))) \). In view of (5.15), this gives

\[
\text{(5.16) } (D \text{ on } \bar{C}) \text{ is weakly equivalent to } (D \text{ on } C_A(P)/C_A(P) \cap \varphi(Q^+)).
\]

Since \( p(A) \) does not divide \( |PB| \) (by (5.1e)), there are irreducible \( \mathbb{Z}(p(A))[PB]\)-submodules \( U_1, \ldots, U_r \) of \( \bar{A} \) such that \( \bar{A} = \oplus \sum_{i=1} U_i \). By (5.1g), each \( U_i \) is an ample \( \mathbb{Z}(p(A))[PB]\)-module. If \( p \neq p(B) \), then \( D < C_{p(A)} \) by (5.1b). If \( p = p(B) \), then \( D \leq C_{(b,p)^{-1}} \) (by (5.1b, i)). So Proposition 3.10 and (1.3) tell us that \((D \text{ on } U_i)\) is weakly equivalent to \((D \text{ on } C_{U_i}(P))\), for \( i = 1, \ldots, r \). Since \( \bar{A} = \oplus \sum_{i=1} U_i \) and \( C_A(P) = \oplus \sum_{i=1} C_{U_i}(P) \), it follows that

\[
\text{(5.17) } (D \text{ on } \bar{A}) \text{ is weakly equivalent to } (D \text{ on } C_A(P)).
\]

By assumption \((D \text{ on } C_L(P)^+)\) is weakly equivalent to \((D \text{ on } C_L(P)^+/C_N(P)^+)\).

It follows from this and (5.10) that \((D \text{ on } C_L(P)^+)\) is weakly equivalent to \((D \text{ on } C_L(P)^+/C_q(P)^+)\).
Since \( \varphi \) is an isomorphism, this says that \((D \circ C_{A}(P) \cap \varphi(L^{+}))\) is weakly equivalent to \\
\((D \circ [C_{A}(P) \cap \varphi(L^{+})]/[C_{A}(P) \cap \varphi(Q^{+})])\).

Clearly this implies that \((D \circ C_{A}(P))\) is weakly equivalent to \\
\((D \circ [C_{A}(P)]/[C_{A}(P) \cap \varphi(Q^{+})])\).

Combined with (5.16) and (5.17), this shows that \((D \circ \tilde{A})\) is weakly equivalent to \((D \circ \tilde{C})\), which completes the proof of the proposition.

To establish the hypotheses of Proposition 5.14 it is convenient to pass to the dual \(Z_{p(A)}[PB]\)-module \(U = \text{Hom}_{Z_{p(A)}}(L^{+}, Z_{p(A)})\) the family \(\mathcal{S}\) of all perpendicular subspaces \(M^{\perp} = \{u \in U \mid u(M^{+}) = \{0\}\}\) to the members \(M\) of the family \(\mathcal{R}\), the subfamily \(\mathcal{G}\) of all \(M^{\perp}\), \(M \in \mathcal{R}\), and the subgroup \(J = N^{\perp}\). These satisfy

**Proposition 5.18.** Every irreducible component of \((PB \circ U)\) is ample. \(\mathcal{S}\) is a PB-invariant family of non-trivial \(Z_{p(A)}\)-subspaces of \(U\). Furthermore

\begin{align*}
(5.19a) & \quad U = \sum_{I \in \mathcal{S}} I, \\
(5.19b) & \quad \mathcal{G} = \{I \in \mathcal{S} \mid [I, P] \neq \{0\}\}, \\
(5.19c) & \quad J = \sum_{I \in \mathcal{G}} I.
\end{align*}

If \(U = \{0\}\) or if

\begin{align*}
(5.20) & \quad C_{J}(P) \neq \{0\} \text{ and } (D \circ C_{V}(P)) \text{ is weakly equivalent to } (D \circ C_{J}(P)),
\end{align*}

then (5.13) holds.

**Proof.** Any irreducible component \(W\) of \((PB \circ U)\) is obviously \(Z_{p(A)}[PB]\)-isomorphic to the dual of an irreducible component \(Y\) of \((PB \circ L^{+})\). By Proposition 5.6, \(Y\) is \(Z_{p(A)}[PB]\)-isomorphic to an irreducible component of \((PB \circ \tilde{A})\). So (5.1g) implies that \((PB \circ Y)\) is ample. Since \(B_{Y} = B_{W}\), it follows from this and (3.9) that \((PB \circ W)\) is ample, which proves the first statement of the proposition.

The second statement and (5.19a) come directly from the first statement of Proposition 5.8 and (5.9) by duality. Equations (5.19b, c) come from (5.7b, c) by duality.

Since \(p(A) \neq p\) (by (5.1e)) we have \\
\[U = C_{V}(P) \oplus [U, P],\]

where \([U, P] = C_{L}(P)^{\perp}\). Similarly

\[J = C_{J}(P) \oplus [J, P]\]

with \([J, P] = J \cap [U, P] = (C_{L}(P)N)^{\perp} = (C_{L}(P) \times [N, P])^{\perp}\). Since \(J = N^{\perp} = (C_{N}(P)) \times [N, P]^{\perp}\), it follows that
(5.21a) \( (D \text{ on } C_U(P)) \) is equivalent to the dual of \( (D \text{ on } C_L(P)^+) \),
(5.21b) \( (D \text{ on } C_J(P)) \) is equivalent to the dual of \( (D \text{ on } C_L(P)^+/C_N(P)^+) \).

If \( U = \{0\} \), then \( L = \{1\} \) and (5.13) holds by Proposition 5.14. If \( U \neq \{0\} \) and (5.20) is true, then (5.21) gives \( C_L(P) > C_N(P) \) and the weak equivalence of \( (D \text{ on } C_L(P)^+) \) with \( (D \text{ on } C_L(P)^+/C_N(P)^+) \). So Proposition 5.14 also gives (5.13) in this case, and the proof is complete.

When \( L \neq \{1\} \), Proposition 5.18 and (5.1) say that \( P, B, D, U, \sigma \) satisfy:

(5.22a) \( PB \) is the semi-direct product of a group \( P \) of prime order \( p \) acting on a group \( B \in \mathfrak{a} \).
(5.22b) \( D \) is a subgroup of \( C_B(P) \).
(5.22c) \( U \) is a non-zero finite-dimensional \( Z_r [PB] \)-module, for some prime \( r \neq p, p(B) \).
(5.22d) \( \sigma \) is a \( PB \)-invariant family of non-zero \( Z_r \)-subspaces of \( U \).
(5.22e) \( U = \sum_{I \in \mathfrak{a}} I \).

Now we consider arbitrary \( P, B, D, U, \sigma \) satisfying (5.22). We define \( g \) and \( J \) by (5.19b, c). From (5.22b, d) it is clear that

(5.23a) \( g \) is a \( P \times D \)-invariant subfamily of \( \sigma \),
(5.23b) \( J \) is a \( Z_r [P \times D] \)-module of \( U \).

Of course, we are looking for situations in which (5.20) holds.

By (5.22a, c), \( U \) is a completely reducible \( Z_r [PB] \)-module. Let \( U_1, \ldots, U_s \) be irreducible \( Z_r [PB] \)-submodules of \( U \) so that

(5.24) \( U = \bigoplus_{i=1}^s U_i \).

Fix \( i = 1, \ldots, s \). Then the projection \( \pi_i \) of \( U \) onto \( U_i \) determined by the decomposition (5.24) is a \( Z_r [BP] \)-epimorphism. Define \( \mathfrak{s}_i \) to be the family

\[ \{ \pi_i(I) \mid I \in \mathfrak{s}, \pi_i(I) \neq \{0\} \} \]

Then we have

**Lemma 5.25.** \( P, B, D, \mathfrak{s}_i, U_i, \sigma_i \) satisfy (5.22), for each \( i = 1, \ldots, s \). If \( P, B, D, U_i, \sigma_i \) satisfy (5.20), for all \( i = 1, \ldots, s \), then \( P, B, D, U, \sigma \) satisfy (5.20).

**Proof.** Fix \( i = 1, \ldots, s \). Conditions (5.22a, b, c) are satisfied by \( P, B, D, U_i, \sigma_i \) by hypothesis. Condition (5.22d) for them comes from the original (5.22d) and the \( PB \)-invariance of \( \sigma_i \). The original (5.22e) gives:

\[ U_i = \pi_i(U) = \sum_{I \in \mathfrak{s}_i} \pi_i(I) = \sum_{I \in \mathfrak{s}_i, \pi_i(I) \neq \{0\}} \pi_i(I) = \sum_{I \in \mathfrak{s}_i} I_i. \]

So the first statement of the proposition holds.

For each \( i = 1, \ldots, s \), we define \( \mathfrak{g}_i \) and \( J_i \) by (5.19b, c) with \( \mathfrak{s}_i \) in place of \( \mathfrak{g} \).

If \( I \in \mathfrak{g}_i \) then \( I_i \in \mathfrak{g}_i \) and \( [I_i, P] \neq \{0\} \). By definition of \( \mathfrak{s}_i \), there is some \( I \in \mathfrak{s} \) such that \( I_i = \pi_i(I) \). Because \( \pi_i \) is \( P \)-invariant we have \( \{0\} \neq [I_i, P] = \{0\} \).
\[\pi_i([I, P]) \neq \{0\}, \text{ i.e. } I \in \mathfrak{g}. \] It follows that

\[J_i = \sum_{i \in \mathfrak{g}_i} I_i \leq \sum_{i \in \mathfrak{g}} \pi_i(I) = \pi_i(J), \text{ for } i = 1, \ldots, s.\]

Suppose that \(P, B, D, U_i, \mathfrak{s}_i\) satisfy \((5.20)\), for all \(i = 1, \ldots, s\). By \((5.22c)\) we must have \(s \geq 1\). So \(C_{i_1}(P) \neq \{0\}\). Since \((P \text{ on } J)\) is completely reducible (by \((5.22c)\)), it follows from this and \((5.26)\) that \(C_{i}(P) \neq \{0\}\).

Let \(W\) be any non-trivial irreducible component of \((D \text{ on } C_{\mathfrak{u}}(P))\). By \((5.24)\) there is some \(i = 1, \ldots, s\) such that \(W\) is \(Z\)-isomorphic to an irreducible component of \((D \text{ on } C_{i}(P))\). Since \((5.20)\) holds for \(P, B, D, U, \mathfrak{s}_i\), the module \(W\) is \(Z\)-isomorphic to an irreducible component of \((D \text{ on } C_{i}(P))\). Then \((5.26)\) and the complete reducibility of \((P \text{ on } J)\) imply that \(W\) is \(Z\)-isomorphic to an irreducible component of \((D \text{ on } C_{i}(P))\).

Obviously any irreducible component of \((D \text{ on } C_{i}(P))\) is an irreducible component of \((D \text{ on } C_{\mathfrak{u}}(P))\). Therefore \((D \text{ on } C_{\mathfrak{u}}(P))\) is weakly equivalent to \((D \text{ on } C_{i}(P))\), which proves the lemma.

Now we study the elements \(I \in \mathfrak{g}_-\mathfrak{g}\) and their translates.

**Lemma 5.27.** Let \((5.22)\) hold. If \(I \in \mathfrak{g}^{-}\mathfrak{g}\), then \(C_{B}(I)\) is a \(P\)-invariant subgroup of \(B\). Furthermore, if \(\sigma \in B\), then \(I \in \mathfrak{g}_-\mathfrak{g}\) if and only if \(\sigma^{r-1} \in C_{B}(I)\) for all \(\pi \in P\).

**Proof.** By \((5.19b)\), \(P\) centralizes \(I\). Therefore \(C_{B}(I)\) is \(P\)-invariant.

If \(\sigma \in B\), then \(I \in \mathfrak{g}\) by \((5.22d)\). By \((5.19b)\), \(I \in \mathfrak{g}^{-}\mathfrak{g}\) if and only if \(P\) centralizes \(I\), i.e., if and only if \(y\sigma^{r} = y\sigma\), for all \(y \in I, \pi \in P\). Since \(I \in \mathfrak{g}^{-}\mathfrak{g}\), we know that \(y = y^{r-1}\). Hence \(I \sigma = I\) if and only if \(y^{r-1} \sigma^{r} = y\sigma\), for all \(y \in I, \pi \in P\), i.e., if and only if \(y^{r-1} = y\), for all \(y \in I, \pi \in P\). So the lemma is true.

The following lemma is the key to proving \((5.20)\):

**Lemma 5.28.** Let \((5.22)\) hold. Suppose that \(U_{D}\) is a primary \(Z[D]\)-module. Assume that there is some \(I \in \mathfrak{g}_-\mathfrak{g}\) and some \(\sigma \in B\) satisfying

\[(5.29) \quad \text{for each } \pi \in P - \{1\}, \text{ there exists } \rho \in P - \{1\} \text{ such that } \sigma^{(r-1)(r-1)} \in C_{B}(I).\]

Then \((5.20)\) holds and \(C_{i}(P) \neq \{0\}\).

**Proof.** By \((5.22d)\) we may choose some \(y \in I - \{1\}\). Our hypotheses give us an element \(\sigma \in B\) satisfying \((5.29)\). We define \(u \in U\) by

\[u = y\sigma \left( \sum_{\pi \in P} \pi \right).\]

Clearly \(u\) is centralized by \(P\).

By \((5.19b)\), \(P\) centralizes \(I\). Therefore \(y\sigma^{r} = y^{r-1}\sigma^{r} = y\sigma^{r}\), for all \(\pi \in P\). If \(\sigma^{r-1} \in C_{B}(I)\), for any \(\pi \in P - \{1\}\), then the \(P\)-invariance of \(C_{B}(I)\) (by Lemma 5.27) implies that \(\sigma^{(r-1)(r-1)} \in C_{B}(I)\), for all \(\rho \in P - \{1\}\). This contradicts \((5.29)\). So \(\sigma^{r-1} \notin C_{B}(I)\), for all \(\pi \in P - \{1\}\). Since \(P\) is abelian and \(C_{B}(I)\) is \(P\)-invariant, we have \(\sigma^{(r-1)} \notin C_{B}(I)\), for all \(\rho \in P, \pi \in P - \{1\}\). Be-
cause \( P \neq \{1\} \), this and Lemma 5.27 imply that \( I \sigma^\pi \in \mathfrak{g} \), for all \( \rho \in P \). Hence \( u = \sum_{\pi \in P} y\sigma^\pi \in J \) (by (5.19c)). If \( u \neq 0 \), we conclude that \( C_I(P) \neq \{0\} \).

Suppose that \( u = 0 \). Then
\[
0 = u\sigma^{-1} = y + \sum_{\pi \in P \setminus \{1\}} y\sigma^{\pi^{-1}}
\]
Condition (5.29) and Lemma 5.27 tell us that \( I \sigma^{\pi^{-1}} \in \mathfrak{g} \), for all \( \pi \in P \setminus \{1\} \). Hence
\[
y = -\sum_{\pi \in P \setminus \{1\}} y\sigma^{\pi^{-1}} \in J.
\]
But \( y \) is non-zero and centralized by \( P \). Therefore \( C_I(P) \neq \{0\} \) in all cases.

Since \( (D \text{ on } U) \) is primary, any two non-trivial \( Z[D] \)-submodules of \( U \) are weakly \( Z[D] \)-equivalent. Clearly \( C_I(P) \neq \{0\} \) implies \( C_U(P) \neq \{0\} \). So \( C_I(P), C_U(P) \) are two non-zero \( Z[D] \)-submodules of \( U \) and the lemma is true.

To obtain some information about \( C_B(I) \) we use the following technical lemma:

**Lemma 5.30.** Suppose that (5.22a, c) holds and that \( (PB \text{ on } U) \) is irreducible. Let \( E \) be a \( P \)-invariant subgroup of \( Z(B) \) and \( Y \) be any non-trivial \( Z[E] \)-submodule of \( U \). Then \( \ker (E \text{ on } U) = \ker (E \text{ on } Y) \).

**Proof.** Clifford’s theory (see Theorem V, 17.3 of [4]) and (5.22a) give us a primary \( Z[B] \)-submodule \( W \) of \( U \) such that
\[
U = \sum_{\pi \in P} W\pi.
\]
Let \( X \) be an irreducible \( Z[E] \)-submodule of \( W \). Since \( E \) is central in \( B \) and \( W \) is \( Z[B] \)-primary, the module \( W_\pi \) is \( Z[E] \)-primary. Because \( P \) normalizes \( E \), each \( W_\pi, \pi \in P \), is \( Z[E] \)-primary with \( X\pi \) as an irreducible \( Z[E] \)-submodule. It follows from this and (5.31) that every irreducible component of \( (E \text{ on } U) \) is \( Z[E] \)-isomorphic to \( X\pi \), for some \( \pi \in P \).

Since \( Y \neq \{0\} \), it has an irreducible \( Z[E] \)-component which, by the above argument, must be isomorphic to \( X\pi \), for some \( \pi \in P \). By \( Y \) is a \( Z[PE] \)-submodule. Therefore it must contain irreducible \( Z[E] \)-components isomorphic to \( X\pi_0 \), for all \( \pi_0 \in P \). By the above argument it can contain no other irreducible \( Z[E] \)-components. Since \( (E \text{ on } Y) \) is completely reducible (by (5.22c)), we conclude that:
\[
\ker (E \text{ on } Y) = \cap_{\pi_0 \in P} \ker (E \text{ on } X\pi_0).
\]
Obviously this expression is independent of \( Y \), which proves the lemma.

We use this to prove

**Lemma 5.32.** Suppose that (5.22) holds and that \( (PB \text{ on } U) \) is irreducible. Then \( C_B(I) \cap Z(B) = \ker (Z(B) \text{ on } U), \) for any \( I \in \mathfrak{i} - \mathfrak{g} \).

**Proof.** By Lemma 5.27, \( C_B(I) \) is \( P \)-invariant. Hence \( E = C_B(I) \cap Z(B) \) is a \( P \)-invariant subgroup of \( Z(B) \) which centralizes \( I \). It follows from this
and (5.22d) that $I$ is a non-trivial $Z_r[PE]$-submodule of $U$ such that $E = \ker (E \mid I)$. Since $(PB \mid U)$ is irreducible, Lemma 5.30 implies that $E = \ker (E \mid U) \leq \ker (Z(B) \mid U)$. Since $\ker (Z(B) \mid U)$ is obviously contained in $E = C_\nu(I) \cap Z(B)$, this proves the lemma.

Now we can establish (5.20) in a substantial case:

**Lemma 5.33.** If (5.22) holds, $(PB \mid U)$ is irreducible, and

$$[Z(B_\nu), P, P] \not\cong \{1\},$$

then (5.20) is true.

**Proof.** Obviously $B_\nu$ is non-trivial. Therefore it lies in $\zeta$ (by (1.5)). It follows easily that $P, B_\nu, D_\nu, U, \zeta$ satisfy (5.22). Clearly (5.20) holds for $P, B, D, U, \zeta$ if and only if it holds for $P, B_\nu, D_\nu, U, \zeta$. So we may replace $B, D$ by $B_\nu, D_\nu$, respectively, and assume that $(B \mid U)$ is faithful.

Let $B_1 = Z(B) \cdot D$. This is a $P$-invariant non-trivial subgroup of $B$. So it lies in $\zeta$. It follows that $P, B_1, D, U, \zeta$ satisfy (5.22). Of course, $(PB \mid U)$ need not be irreducible. Let $U_1, \ldots, U_s$ be irreducible $Z_r[PB_1]$-submodules of $U$ so that (5.24) holds. Lemma 5.25 tells us that we need only verify (5.20) for $P, B_1, D, U_i, \zeta_i$, for each $i = 1, \ldots, s$.

Let $\rho$ be a generator of the cyclic group $P$. Since $Z(B)$ is abelian, we have $\{1\} \not\cong [Z(B), P, P] = Z(B)^{(\rho^{-1})\cdot}$.

So we may choose an element $\sigma \in Z(B)$ such that $\sigma^{(\rho^{-1})\cdot} \not\cong 1$. Because $\rho = |P|$ is a prime, this condition implies that

$$\sigma^{(\rho^{-1})\cdot} \not\cong 1, \quad \text{for all } \pi \in P - \{1\}. \quad (5.34)$$

Fix $i = 1, \ldots, s$. The group $PB_1$ is the central product of $PZ(B)$ and $D$ (by (5.22b))). Since $U_i$ is an irreducible $Z_r[PB_1]$-module, its restriction $(U_i)_{PB_1}$ must be a primary $Z_r[D]$-module. Lemma 5.30 with $E = Z(B)$ and $Y = U_i$ tells us that $Z(B)$ acts faithfully on $U_i$.

Suppose that $\zeta_i = \zeta_i$. Then $J_i = U_i$. So (5.20) for $P, B_1, D, U_i, \zeta_i$ reduces to the condition $C_{U_i}(P) \not\cong \{0\}$. Since $[Z(B), P] \not\cong \{1\}$ and $Z(B)$ acts faithfully on $U_i$, we have $[Z(B_i)_{U_i}, P] \cong [\zeta(B)_{U_i}, P] \supseteq \{1\}$. As in the proof of Proposition 3.10, it follows from this and Corollary 3.3 that $C_{U_i}(P) \not\cong \{0\}$. Hence (5.20) for $P, B_1, D, U_i, I_i$ holds in this case.

Suppose that $\zeta_i \subset \zeta_i$. Choose $I \in \zeta_i \setminus \zeta_i$. Since $Z(B)$ acts faithfully on $U_i$, Lemma 5.32 implies that $C_{U_i}(I) \cap Z(B) = \{1\}$. But $\sigma \in Z(B)$. So none of the elements on the left in (5.34) can lie in $C_{U_i}(I)$. Now Lemma 5.28 proves (5.20) for $P, B_1, D, U_i, I_i$ in this case.

We conclude that (5.20) holds for $P, B_1, D, U_i, \zeta_i$ in all cases. As noted above, this is enough to prove the lemma.

We now have enough information to handle the case in which $p \neq p(B)$.

**Proposition 5.35** Assume that (5.22) holds with $p \neq p(B)$. If each irreducible component of $(PB \mid U)$ is ample, then (5.20) is true.
Proof. In view of Lemma 5.25, it suffices to prove this proposition under the additional hypothesis that \((PB \text{ on } U)\) is irreducible. As in the first paragraph of the proof of Lemma 5.33, we may also replace \(B, D\) by \(B_U, D_U\) and assume that \((B \text{ on } U)\) is faithful.

If \([Z(B), P] \neq \{1\}\), then \(p \neq p(B)\) implies that \([Z(B), P, P] = [Z(B), P] \neq \{1\}\). Since \((B \text{ on } U)\) is faithful and \((PB \text{ on } U)\) is irreducible, Lemma 5.33 gives (5.20).

Now suppose that \([Z(B), P] = \{1\}\). By hypothesis, \([B, P] \neq \{1\}\). So Corollary 3.12 tells us that \(PB\) is the central product of \(P[B, P]\) and \(C_B(P)\). In view of (1.3) and (5.22b) it suffices to prove (5.20) under the additional hypothesis that \(D = C_B(P)\). Then \(U_D\) is a primary \(Z, [D]\)-module.

If \(\bar{g} = \bar{g}\), then \(J = U\). Since \((PB \text{ on } U)\) is ample and irreducible, Proposition 3.10 gives \(C_J(P) = C_{U}(P) \neq \{0\}\), which proves (5.20). Hence we may assume that \(\bar{g} \subset \bar{g}\).

Let \(I\) be an element of \(\bar{g} - \bar{g}\). By Lemma 5.32, \(C_B(I) \cap Z(B) = \{1\}\). It follows that \(C_{[B, P]}(I) \cap Z([B, P]) = \{1\}\). So \(C_{[B, P]}(I)\) is a proper \(P\)-invariant subgroup of \([B, P]\). Let \(B_1\) be a maximal \(P\)-invariant subgroup of \([B, P]\) containing \(C_{[B, P]}(I)\). Then \([B, P]/B_1\) is an irreducible \(Z_{P(B)}[P]\)-module such that

\[
[[B, P]/B_1]^{+}, P] = [[B, P]/B_1]^{+}.
\]

Since \(P\) is cyclic of prime order \(p\), it follows that \(\bar{g}(\pi - 1) \neq 0\), for all \(\bar{g} \in [[B, P]/B_1]^{+}, P\) and all \(\pi \in P - \{1\}\). Choose \(\sigma \in [B, P]\) so that its image \(\bar{g}\) in \([B, P]/B_1]^{+}\) is non-zero. Then none of \(\sigma^{(\pi - 1)}\), where \(\pi, \rho \in P - \{1\}\), has a nonzero image in \([B, P]/B_1]^{+}\). In particular, none of them can lie in \(C_B(I)\). So \(\sigma\) satisfies (5.29). Now the hypotheses of Lemma 5.28 are all satisfied. So that lemma completes the proof of this proposition.

For the case \(p = p(B)\), we need a few routine technical lemmas.

**Lemma 5.36.** Suppose that (5.22a) holds with \(p = p(B)\). Then

\[
(5.37) \quad [[B, P]_i, [B, P]_j] \leq [\Phi(B), P]^{\max(i+j+1-p, 0)}, \text{ for all } i, j \geq 0.
\]

**Proof.** Suppose that \(i = 0\). If \(j \leq p - 1\), then \(\max(i + j + 1 - p, 0) = 0\) and (5.37) holds. If \(j \geq p\), then \([\bar{B}, P]^j = \{0\}\) (since \(p = p(B)\)), and (5.37) follows from (1.6). So (5.37) is true for \(i = 0\).

Suppose that \(i > 0\) and that (5.37) is true for all smaller values of \(i\). Let \(\sigma\) be an element of \([B, P]_{i-1}\), \(\tau\) be an element of \([B, P]_i\), and \(\pi\) be an element of \(P\). Let \(\bar{\sigma}, \bar{\tau}\) be the images of \(\sigma, \tau\), respectively, in \(\bar{B}\). From (1.6) and the bilinearity and \(P\)-invariance of \(f_B\) we compute

\[
[[\sigma, \pi], \tau]^{+} = f_B(\bar{\sigma}(\pi - 1), \bar{\tau})
\]

\[
= f_B(\bar{\sigma}, \bar{\tau}) - f_B(\bar{\sigma}, \bar{\tau})
\]

\[
= f_B(\bar{\sigma}, \bar{\tau}^{-1}\pi - f_B(\bar{\sigma}, \bar{\tau})
\]
By induction the first term lies in
\[[\Phi(B)^+,\ P]^{\text{Max}(i+j-p,0)},\ P] \leq [\Phi(B)^+,\ P]^{\text{Max}(i+j+1-p,0)}\]
and the second lies in
\[[\Phi(B)^+,\ P]^{\text{Max}(i+j+1-p,0)}\].

Therefore \([\sigma,\ \tau] \in [\Phi(B)^+,\ P]^{\text{Max}(i+j+1-p,0)}\), for all \(\sigma \in [B,\ P]^{i-1},\ \tau \in [B,\ P]^j\), \(j \geq 0\). Evidently this proves (5.37) for \(i\) and finishes the inductive proof of the lemma.

**Lemma 5.38.** Suppose that (5.22a) holds, that \(p = p(B) \geq 5\), and that \([\Phi(B),\ P]^2 = \{1\}\). For any \(\rho,\ \pi \in P - \{1\}\), the map
\[
\lambda_{\rho,\pi}: \sigma \rightarrow (\sigma^p)(\rho^{-1})
\]
is a \(P\)-epimorphism of \([B,\ P]^{p-3}\) onto \([B,\ P]^{p-1}\). If \(B_1\) is a \(P\)-invariant subgroup of \([B,\ P]^{p-3}\), then the image \(\lambda_{\rho,\pi}(B_1)\) is independent of the choice of \(\rho,\ \pi \in P - \{1\}\).

**Proof.** First we show that \(\lambda_{\rho,\pi}\) is a homomorphism. If \(a,\ \tau \in [B,\ P]\), we compute
\[
(\sigma\tau)^{\rho^{-1}}(\rho^{-1}) = [(\sigma\tau)^\rho(\sigma\tau)^{-1}]^{\rho^{-1}}
= [\sigma^\rho\tau^{\rho^{-1}}\sigma^{-1}]^{\rho^{-1}}
= (\sigma^{\rho^{-1}}\tau^{\rho^{-1}})(\tau^{-1},\ \sigma^{-1})^{\rho^{-1}}
= (\sigma^{\rho^{-1}}\tau^{\rho^{-1}}\tau^{-1},\ \sigma^{-1})^{\rho^{-1}}(\sigma^{\rho^{-1}}\tau^{\rho^{-1}})^{-1}.
\]
Since \(\tau^{\rho^{-1}} \in [B,\ P]^{p-2}\) and \(\sigma^{-1} \in [B,\ P]^{p-3}\), it follows from (5.37) that their commutator lies in
\([\Phi(B),\ P]^{(p-2)+(p-3)+1-p} = [\Phi(B),\ P]^{p-4}\).
This is contained in \([\Phi(B),\ P]\), since \(p \geq 5\). From \([\Phi(B),\ P]^2 = \{1\}\), we conclude that \([\tau^{\rho^{-1}},\ \sigma^{-1}]^{\rho^{-1}} = 1\). So this term may be dropped from the above expression, giving
\[
(\sigma\tau)^{\rho^{-1}}(\rho^{-1}) = (\sigma^{\rho^{-1}}\tau^{\rho^{-1}})^{\rho^{-1}}\tau^{\rho^{-1}}(\sigma^{\rho^{-1}}\tau^{\rho^{-1}})^{-1}
= \sigma^{(p-1)}\tau^{(\rho^{-1})(\rho^{-1})}\tau^{(\rho^{-1})^{-1}}.
\]
Now \((\sigma^{\rho^{-1}})^{-1} \in [B,\ P]^{p-2}\) and \(\tau^{(\rho^{-1})(\rho^{-1})} \in [B,\ P]^{p-1}\). So (5.37) says that their commutator lies in
\([\Phi(B),\ P]^{(p-2)+(p-1)+1-p} = [\Phi(B),\ P]^{p-2}\).
This is \(\{1\}\), since \(p \geq 5\) and \([\Phi(B),\ P]^2 = \{1\}\). Therefore the terms \(\tau^{(\rho^{-1})(\rho^{-1})}\)
and \((\sigma^{p-1})^{-1}\) commute in the above expression and we have

\[
(\sigma^\tau)^{(p-1)(\tau-1)} = \sigma^{(p-1)(\tau-1)}\tau^{(p-1)(\tau-1)},
\]

which proves that \(\lambda_{p,\pi}\) is a homomorphism.

Since \(P\) is abelian, \(\lambda_{p,\pi}\) is \(P\)-invariant. Clearly it sends \([B, P]^{p-1}\) into \([B, P]^{p-1}\). Before showing that it is onto (and hence proving the first statement of the proposition), we prove the last statement of the proposition.

Fix \(\pi, \rho \in P - \{1\}\). If \(i = 1, \ldots, p - 2\), then \(\pi^i\) and \(\pi^{i+1}\) lie in \(P - \{1\}\), since \(p \mid |P|\) is a prime. For \(\sigma \in B_1\), we compute:

\[
\lambda_{p,\pi^{i+1}}(\sigma) = (\sigma^p)^{\pi^{i+1}-1} = (\sigma^{p-1})^{\pi^{i+1}-1} - 1
\]

\[
= \sigma^{(p-1)(\pi^{i+1}-1)}\sigma^{(p-1)(\pi^{i+1}-1)} = \lambda_{p,\pi^i}(\sigma^p)\lambda_{p,\pi}(\sigma).
\]

Using the \(P\)-invariance of \(B_1\), we conclude by induction on \(i\) that

\[
\lambda_{p,\pi^i}(B_1) \leq \lambda_{p,\pi^i}(B_1), \text{ for all } i = 1, \ldots, p - 1.
\]

Since \(|P| = p\) is a prime, there exists, for each \(i = 1, \ldots, p - 1\), some \(j = 1, \ldots, p - 1\) such that \(\pi = \pi^j\). The above inclusion for \(\pi^i, j\) in place of \(\pi, i\) respectively is just:

\[
\lambda_{p,\pi^i}(B_1) \leq \lambda_{p,\pi^j}(B_1), \text{ for all } i = 1, \ldots, p - 1.
\]

Combined with the original inclusion and the fact that \(P\) is cyclic, this proves that \(\lambda_{p,\pi}(B_1)\) is independent of the choice of \(\pi \in P - \{1\}\).

Now we vary \(\rho\). For \(i = 1, \ldots, p - 2\) we have

\[
\lambda_{p^i+1,\pi}(\sigma) = (\sigma^{p^{i+1}-1})^{\pi-1} = \frac{[\sigma^{p^{i+1}-1}, \sigma^{p-1}]}{\sigma^{p^{i+1}-1}}
\]

\[
= \sigma^{(p-1)(\pi-1)}\sigma^{(p^{i+1}-1)(\pi-1)}\sigma^{(p^{i+1}-1)}\sigma^{(p^{i+1}-1)}\frac{[\sigma^{p^{i+1}-1}, \sigma^{p-1}]}{\sigma^{p^{i+1}-1}}.
\]

We know that \(\sigma^{(p^{i+1}-1)(\pi-1)} \in [B, P]^{p-1}\) and \([\sigma^{p^{i+1}-1}]^{-1} \in [B, P]^{p-2}\). By (5.37), their commutator lies in

\[
[\Phi(B), P]^{(p-1)+(p-2)+1-p} = [\Phi(B), P]^{p-2} = \{1\},
\]

since \(p \geq 5\) and \([\Phi(B), P]^2 = \{1\}\). So they commute and the above expression becomes

\[
\lambda_{p^i+1,\pi}(\sigma) = \sigma^{(p^{i+1})(\pi-1)}\sigma^{(p-1)(\pi-1)}
\]

\[
= \lambda_{p^i,\pi}(\sigma^p)\lambda_{p,\pi}(\sigma), \text{ for all } i = 1, \ldots, p - 2.
\]

As in the above case of \(\pi\), this is enough to show that \(\lambda_{p,\pi}(B_1)\) is independent of the choice of \(\rho \in P - \{1\}\). So the last statement of the lemma is true.

Obviously \([B, P]^{p-1}\) is generated by its subgroups \(\lambda_{p,\pi}([B, P]^{p-2})\), where \(\rho, \pi \in P - \{1\}\). By the preceding argument all these subgroups are equal to each other and hence to \([B, P]^{p-1}\). Therefore \(\lambda_{p,\pi}([B, P]^{p-2}) = [B, P]^{p-1}\), for all \(\rho, \pi \in P - \{1\}\), which finishes the proof of the lemma.
Now we can handle the case \( p = p(B) \geq 5 \).

**Proposition 5.39.** Suppose that (5.22) holds with \( p = p(B) \geq 5 \). If each irreducible component of \((PB on U)\) is ample and \( D \leq [B, P]^{p^{-1}} \), then (5.20) is true.

**Proof.** In view of Lemma 5.25, we may assume that \((PB on U)\) is irreducible. Since \((PB on U)\) is ample, (3.9c) implies that \( B_U \neq \{1\} \). So \( B_U \in \mathcal{A} \) (by (1.5)), and we may replace, \( B \), \( D \) by \( B_U \), \( D_U \) without disturbing our hypotheses, assumptions, or conclusions. I.e., we may assume that \((B on U)\) is faithful.

Since \((PB on U)\) is faithful, irreducible, and ample, (3.9c) implies that \([B, P]^{p^{-1}} \neq \{1\}\). In view of (1.1), we may assume that \( D = C_{[B, P]^{p^{-1}}}(P) \neq \{1\} \).

If \([\Phi(B), P, P] \neq \{1\}\), then \([Z(B), P, P] \neq \{1\}\) by (1.4b). Since \((B on U)\) is faithful and \((PB on U)\) is irreducible, Lemma 5.33 gives (5.20) in this case. So we may assume that \([\Phi(B), P, P] = \{1\}\).

Let \( \rho \) be any element of \( P - \{1\} \). Our assumptions and Lemma 5.35 tell us that \( \lambda_{\rho, \sigma} : \sigma \mapsto \sigma^{(p^{-1})^3} \) is a \( P \)-epimorphism of \([B, P]^{p^{-3}}\) onto \([B, P]^{p^{-1}}\). Since \((PB on U)\) is ample, (3.9c) implies that \([B, P]^{p^{-1}} \neq \{1\}\). Hence \([B, P]^{p^{-3}} \neq \{1\}\). It follows that \( P, [B, P]^{p^{-3}}, D, U, \delta \) satisfy (5.22) and that we only need prove (5.20) for this quintuple.

Decompose \( U \) as in (5.24) into a direct sum of irreducible \( \mathcal{Z} \) submodules \( U_i \) where \( i = 1, \ldots, s \). We first consider such an \( i \) for which \( \mathfrak{g}_i \subset \mathfrak{s}_i \) and \([B_{U_i}, P]^{p^{-1}} \neq \{1\}\).

It follows from (5.37) that

\[
[B, P]^{p^{-3}} \leq [\Phi(B), P]^{(p^{-3} + (p^{-1})^3) = [\Phi(B), P]^{p^{-3}}.
\]

This is \( \{1\}\), since \( p \geq 5 \) and \([\Phi(B), P]^{p} = \{1\}\). Hence \([B, P]^{p^{-1}} \) is central in \([B, P]^{p^{-3}}\). If \( I \in \mathfrak{g}_i \setminus \mathfrak{s}_i \), we conclude from Lemma 5.32 that

\[
C_{[B, P]^{p^{-1}}}(I) = \text{Ker} ([B, P]^{p^{-1}} on \ U_i).
\]

By the choice of \( U_i \), the last group is not equal to \([B, P]^{p^{-1}}\). Hence there exists some \( \sigma \in [B, P]^{p^{-3}} \) such that

\[
\lambda_{\rho, \rho}(\sigma) \in [B, P]^{p^{-1}} - C_{[B, P]^{p^{-1}}}(I).
\]

It follows from the last statement of Lemma 5.38 that the inverse image in \([B, P]^{p^{-3}}\) of \( C_{[B, P]^{p^{-1}}}(I)\) under \( \lambda_{\rho, \rho}\) is independent of the choice of \( \pi \in P - \{1\}\). Therefore

\[
\lambda_{\rho, \rho}(\sigma) = \sigma^{(p^{-1})^3} \notin C_{[B, P]^{p^{-1}}}(I),
\]

for all \( \pi \in P - \{1\} \), i.e., \( \sigma, I \) satisfy (5.29).

Because \( D \) is a subgroup of \([B, P]^{p^{-1}}\), it is central in \([B, P]^{p^{-3}}\). Because \( D \) centralizes \( P \), it is central in \( [P[B, P]^{p^{-3}} on \ U_i]\). Since \((P[B, P]^{p^{-3}} on \ U_i)\) is irreducible, this implies that \((D on U_i)\) is primary. So Lemma 5.28 says that (5.20) holds for \( P, [B, P]^{p^{-3}}, D, U_i, \mathfrak{s}_i \).
If $g_i = s_i$, for some $i = 1, \ldots, s$, then $J_i = U_i$. So $(D \circ C_j(P)) = (D \circ C_{v_i}(P))$. If $[B_{v_i, P}]^{p-1} = \{1\}$, for some $i = 1, \ldots, s$, then $D$ centralizes $U_i$. So $(D \circ C_j(P))$ is trivially weakly equivalent to $(D \circ C_{v_i}(P))$. This and the above arguments tell us that $(D \circ C_j(P))$ is weakly equivalent to $(D \circ C_{v_i}(P))$, for all $i = 1, \ldots, s$. As in Lemma 5.25, we conclude that $(D \circ C_j(P))$ is weakly equivalent to $(D \circ C(P))$.

Since $(PB \circ U)$ is simple and irreducible, Proposition 3.10 implies that $(D \circ U)$ is weakly equivalent to $(D \circ C_{v_i}(P))$, and hence to $(D \circ C_j(P))$. But $D = D_{v_i} \neq \{1\}$. Therefore $D$ acts non-trivially on $C_j(P)$, which implies $C_j(P) \neq \{0\}$ and completes the proof of the proposition.

We collect the results of this section in

**Theorem 5.40.** If (5.1) holds with either $p \neq p(B)$ or $p = p(B) \geq 5$, then, (5.13) is true.

**Proof.** Define $U, \mathcal{S}$ as in Proposition 5.18. By that proposition we may assume that $U \neq \{0\}$. Then we only need prove (5.20).

Evidently $P, B, D, U, \mathcal{S}$ satisfy (5.22). Furthermore, each irreducible component of $(PB \circ U)$ is ample (by Proposition 5.18). If $p \neq p(B)$, then Proposition 5.35 proves (5.20). If $p = p(B) \geq 5$, then $D \leq [B, P]^{p-1}$ by (5.11) and Proposition 5.39 proves (5.20). Therefore the theorem is true.

**6. The case $p = 3$**

When $p = p(B) = 3$, Proposition 5.29 does not hold and the arguments of the last section do not suffice. However, in this case our Fitting chain is augmented. So we consider the more complicated situation in which:

(6.1a) \(PE\) is the semi-direct product of a group $P$ of order 3 acting on a non-trivial group $E$ of prime power order.

(6.1b) \(PEB\) is the semi-direct product of $PE$ acting on a group $B \in \mathfrak{A}$.

(6.1c) $F$ is a subgroup of $C_E(P)$.

(6.1d) \(PFBA\) is the semi-direct product of $PFB$ acting on a group $A \in \mathfrak{A}$.

(6.1e) \(V\) is a finite dimensional $Z_{[PA]}$-module, for some prime $q$.

(6.1f) $p(E) \neq p(B) = 3 \neq p(A) \neq q$.

(6.1g) $[\Phi(B), E] = \{1\}$.

(6.1h) $\tilde{B}$ is a completely reducible $Z_{[PE]}$-module.

(6.1i) $[\Phi(A), B] = \{1\}$.

(6.1j) Each irreducible component of $(PB \circ A)$ is ample.

(6.1k) The representation $(A \circ V)$ is faithful and weakly $FB$-invariant.

We define

(6.2) \[D = [[B, P]^2, F]].\]

Then we have

**Proposition 6.3.** It follows from (6.1a, b, c, g) and $p(B) = 3$ that $D \leq C_E(P)$. If all of (6.1) holds, then $P, B, A, D, V$ satisfy (5.1) with $p = 3$. 

Proof. Suppose that (6.1a, b, c, g) hold and \( p(B) = 3 \). Since \( |P| = 3 \), we have \([B, P]^g = \{1\}\). From \( B \in G \) and (1.4b) we conclude that \([B, P]^g \leq Z(B)\). It follows that, for each \( \pi \in P \), the map \( \mu : \sigma \to \sigma^{-1} \) is a homomorphism of \([B, P]^g \) into \( \Phi(B) \). It is clear from (6.1b, c) that \( \mu \) is \( F \)-invariant. By (6.1c, g), \( F \) centralizes \( \mu ([B, P]^g) \leq \Phi(B) \). Therefore \( D \leq \operatorname{Ker} \mu \), i.e., \( P \) centralizes \( D \). This proves the first statement of the proposition.

If all of (6.1) holds, then (5.1a) comes from (6.1a, b), (5.1b) is the first statement of this proposition, (5.1c) comes from (6.1d), (5.1d) from (6.1e), (5.1e) from (6.1f), (5.1f) from (6.1i), (5.1g) from (6.1j), (5.1h) from (6.1k), and (5.1i) from (6.2) since \( p = 3 \). So the proposition is true.

Now we may define the families \( \mathfrak{K}, \mathfrak{L}, \mathfrak{M}, \mathfrak{N} \) and the subgroups \( L, N, Q \) and \( C \) as in §5. Furthermore, we define \( U, \mathfrak{s}, \mathfrak{g}, J \) as in Proposition 6.18.

**Proposition 6.4.** \( D \) is an \( F \)-invariant subgroup of \( CB(P) \). The families \( \mathfrak{K}, \mathfrak{L}, \mathfrak{M} \) and the subgroup \( L \) are \( \operatorname{PFB} \)-invariant. Hence \( U \) is a \( Z()_{\operatorname{PFB}} \)-module and \( \mathfrak{s} \) is a \( \operatorname{PFB} \)-invariant family. The family \( \mathfrak{K} \) and the subgroups \( N, C \) and \( Q \) are \( P \times \operatorname{FD} \)-invariant. Hence so are \( \mathfrak{g} \) and \( J \).

**Proof.** The first statement follows directly from (6.1a, b, c), (6.2), and Proposition 6.3.

The \( F \)-invariance of the families \( \mathfrak{K}, \mathfrak{L} \) follows from their definitions and (6.1d, k). They are \( \operatorname{PB} \)-invariant by Proposition 5.3. Hence they are \( \operatorname{PFB} \)-invariant. By (5.5), this implies that \( L \) is \( \operatorname{PFB} \)-invariant. This and (5.7a) give the \( \operatorname{PFB} \)-invariance of \( \mathfrak{M} \) and complete the proof of the second statement.

The third statement follows from the second by duality and the definitions of \( U \) and \( \mathfrak{s} \) preceding Proposition 5.18.

Both \( F \) and \( D \) centralize \( P \) by (6.1c) and Proposition 6.3. It follows from this and (5.7b, c) that \( \mathfrak{K} \) and \( N \) are \( P \times \operatorname{FD} \)-invariant. Furthermore, it follows that \( \mathfrak{K}_{\text{ample}} \) is \( P \times \operatorname{FD} \)-invariant. Since \( Q \) is the intersection of the members of \( \mathfrak{K}_{\text{ample}} \) (by (3.15)), it is \( P \times \operatorname{FD} \)-invariant. Clearly \( C_A(P) \) is \( P \times \operatorname{FD} \)-invariant. So (5.11) implies that \( C \) is \( P \times \operatorname{FD} \)-invariant, which completes the proof of the fourth statement.

The last statement follows from the fourth by duality and the definitions of \( \mathfrak{g} \) and \( J \). Therefore the proposition holds.

Instead of (5.20) we now try to establish

\[(6.5) \quad (F \text{ on } D_{C_U(P)}) \text{ is weakly equivalent to } (F \text{ on } D_{C_U(P)})\]

This has the following consequence:

**Proposition 6.6.** If (6.5) holds, then \( (F \text{ on } D_{\mathfrak{g}}) \) is weakly equivalent to \( (F \text{ on } D_{\overline{g}}) \).

**Proof.** By (6.1f) the representations \( (D \text{ on } \mathfrak{A}) \) and \( (D \text{ on } \mathfrak{C}) \) are both fully reducible. So (5.16) and (5.17) give

\[(6.7a) \quad \operatorname{Ker} (D \text{ on } \mathfrak{A}) = \operatorname{Ker} (D \text{ on } C_{\mathfrak{K}}(P)), \]

\[(6.7b) \quad \operatorname{Ker} (D \text{ on } \mathfrak{C}) = \operatorname{Ker} (D \text{ on } C_{\mathfrak{K}}(P)/C_{\mathfrak{K}}(P) \cap \varphi(Q^+))).\]
In particular, \( \ker (D \circ A) \leq \ker (D \circ C) \). Therefore we only need to show that a given non-trivial irreducible component \( W \) of \( (F \circ D) \) is \( Z_3 [F] \)-isomorphic to an irreducible component \( W \) of \( (F \circ D) \). To simplify the notation we make the definition:

\[
(6.8) \quad \text{"} W \lesssim D_1 \text{"} \text{ means "} D_1 \text{ is an } F \text{-invariant section of } D \text{ and } W \text{ is } Z_3 [F] \text{-isomorphic to an irreducible component of } (F \circ D_1)\text{".}
\]

The subgroup \( L \) is \( PFB \)-invariant by Proposition 6.4. Hence so are the natural monomorphism of Proposition 5.6 and the submodule \( \varphi (L^+) \) of \( \bar{A} \). Since \( (D \circ \bar{A}) \) is completely reducible, we conclude from this and (6.7) that

\[
(6.9a) \quad \ker (D \circ \bar{A}) = \ker (D \circ C(A)/\varphi (C_L(P)^+)) \cap \ker (D \circ \varphi (C_L(P)^+)).
\]

\[
(6.9b) \quad \ker (D \circ \bar{C}) = \ker (D \circ C(A)/\varphi (C_L(P)^+)) \cap \ker (D \circ \varphi (C_L(P)^+)).
\]

By hypothesis \( W \lesssim D_1 = D/\ker (D \circ A) \). So (6.9a) implies that either \( W \lesssim D/\ker (D \circ C(A)/\varphi (C_L(P)^+)) \) or \( W \lesssim D/\ker (D \circ \varphi (C_L(P)^+)) \). In the former case, \( W \lesssim D/\ker (D \circ \bar{C}) = D_{\bar{C}} \) by (6.9b), and we are done. So we may assume that the latter case holds.

Proposition 5.6 says that \( \varphi \) is a monomorphism. Hence

\[
\ker (D \circ \varphi (C_L(P)^+)) = \ker (D \circ C_L(P)^+).
\]

By (5.21a) this is just \( \ker (D \circ C_L(P)^+) \). Therefore

\[
W \lesssim D/\ker (D \circ C_L(P)^+) = D_{\varphi (P)}.
\]

Now (6.5) implies that \( W \lesssim D_{\varphi (P)} = D/\ker (D \circ C_L(P)^+) \). From (5.21b) we have

\[
\ker (D \circ C_L(P)^+) = \ker (D \circ C_L(P)^+/C_N(P)^+).
\]

This contains \( \ker (D \circ C_L(P)^+/C_N(P)^+) \) by (5.10). Since \( \varphi \) is a monomorphism, the last kernel is just \( \ker (D \circ \varphi (C_L(P)^+)/\varphi (C_N(P)^+)) \). Therefore

\[
W \lesssim D/\ker (D \circ \varphi (C_L(P)^+)/\varphi (C_N(P)^+)).
\]

By (6.9b) this implies that \( W \lesssim D/\ker (D \circ \bar{C}) = D_{\bar{C}} \), which completes the proof of the proposition.

It is clear from (6.1) and Propositions 5.18 and 6.4 that \( P, F, E, B, U, \sigma \) satisfy

\[
(6.10a) \quad PE \text{ is the semi-direct product of a group } P \text{ of order 3 acting on a non-trivial group } E \text{ of prime power order.}
\]

\[
(6.10b) \quad PEB \text{ is a semi-direct product of } PE \text{ acting on a group } B \in \alpha.
\]

\[
(6.10c) \quad F \text{ is a subgroup of } C_{B}(P).
\]
(6.10d) $U$ is a finite-dimensional $\mathbb{Z}_r[PF_B]$-module, for some prime $r$.
(6.10e) $\mathcal{S}$ is a $PF_B$-invariant family of non-trivial $\mathbb{Z}_r$-subspaces of $U$.
(6.10f) $p(E) \neq p(B) = 3 \neq r$.
(6.10g) $|\Phi(B), E| = \{1\}$.
(6.10h) $\mathcal{D}$ is a completely reducible $\mathbb{Z}_3[PE]$-module.
(6.10i) $U = \sum_{i=1}^{t_0} I$.

Now we consider the most general, $P$, $F$, $E$, $B$, $U$, $\mathcal{S}$ satisfying these conditions. We define $D$ by (6.2), $\mathcal{J}$ by (5.19b) and $J$ by (5.19c). Since (6.10a, b, c, g) are (6.1a, b, c, g), Proposition 6.3 and the definitions of $D$, $\mathcal{J}$, $J$ imply

(6.11a) $D$ is an $F$-invariant subgroup of $C_B(P)$,
(6.11b) $\mathcal{J}$ is a $P \times FD$-invariant subfamily of $\mathcal{S}$,
(6.11c) $J$ is a $\mathbb{Z}_r[P \times FD]$-submodule of $U$.

Of course, we are trying to prove (6.5). We first make some preliminary reductions to the "minimal case".

**Lemma 6.12.** Suppose that (6.5) holds whenever we assume, in addition to (6.10), that

(6.13) $\mathcal{D}$ is an irreducible $\mathbb{Z}_3[PE]$-module with $\mathcal{D} = \mathcal{B}, E$.

Then (6.5) always holds when (6.10) does.

**Proof.** Let (6.10) hold. By (6.10h) there exist irreducible $\mathbb{Z}_3[PE]$-submodules $Y_1, \cdots, Y_t$ of $\mathcal{D}$ so that:

(6.14) $\mathcal{D} = \bigoplus_{i=1}^{t} Y_i$ (as $\mathbb{Z}_3[PE]$-modules).

For each $i = 1, \cdots, t$, it follows from the normality of $E$ in $PE$ and the irreducibility of $(PE$ on $Y_i)$ that $Y_i, E$ is either $\{0\}$ or $Y_i$. We choose the notation so that $[Y_i, E] = Y_i$, for $i = 1, \cdots, s$, and $[Y_i, E] = \{0\}$, for $i = s + 1, \cdots, t$. Then

(6.15) the natural map of $\mathcal{D}$ into $\mathcal{D}$ is a $\mathbb{Z}_3[F]$-isomorphism of $\mathcal{D}$ onto $\bigoplus_{i=1}^{t} [Y_i, P]^2, F]$.

Indeed, by (6.2) the image of $\mathcal{D}$ is $[[\mathcal{D}, P]^2, F]$, which is $\bigoplus_{i=1}^{t} [Y_i, P]^2, F]$ by (6.14). Evidently $[[Y_i, P]^2, F] \leq [Y_i, E] = \{0\}$, for $i = s + 1, \cdots, t$. So the image of $\mathcal{D}$ is $\bigoplus_{i=1}^{t} [Y_i, P]^2, F]$.

Since $p(E) \neq 3 = p(B)$ by (6.10f), it follows from (6.2) that $[D, F] = D$. Hence $[\mathcal{D}, F] = \mathcal{D}$ and $C_B(F) = \{0\}$. The kernel of the map in (6.15) is $[D \cap \Phi(B)/\Phi(D)]^+$, which is contained in $C_B(F)$ by (6.10g). Therefore the kernel is $\{0\}$ and the map, which is obviously $F$-invariant, satisfies (6.15).

Let $B_{i+}$ be the inverse image in $B$ of $Y_i$ and $B_i = [B_{i+}, E]$ for $i = 1, \cdots, s$. Obviously each $B_i$ is a $PE$-invariant subgroup of $B$. Furthermore
the natural map of $\hat{B}_i$ into $\hat{B}$ is a $Z_3[P,E]$-isomorphism of $\hat{B}_i$ onto $Y_i$, for $i = 1, \cdots, s$.

Indeed, the image of this map is $[Y_i,E] = Y_i$, by the construction. The kernel is $[B_i \cap \Phi(B)/\Phi(B_i)]^+$ which is contained in $C_{\hat{B}_i}(E)$ by (6.10g). But $p(E) \neq p(B) = 3$ implies that $C_{\hat{B}_i}(E) = \{0\}$. So (6.16) holds.

Fix $i = 1, \cdots, s$. It follows easily from (6.10) and (6.16) that both (6.10) and (6.13) hold with $B_i$ in place of $B$. Let $D_i = [[B_i, P]^2, F]$. Then our hypotheses tell us that

\[(6.17) \quad (F \circ (D_i)c_B(P)) \text{ is weakly equivalent to } (F \circ (D_i)c_B(P)), \text{ for } i = 1, \cdots, s.\]

Since $J \leq U$ we have

$$\text{Ker } (D \circ C_J(P)) \geq \text{Ker } (D \circ C_U(P)).$$

So (6.5) will follow once we prove that any non-trivial irreducible component $W$ of $(F \circ D_{c_B(P)})$ is $Z_3[F]$-isomorphic to an irreducible component of $(F \circ D_{c_U(P)})$. For simplicity we adopt the notation (6.8).

It is clear from their definitions and (6.2) that each $D_i$, $i = 1, \cdots, s$, is a subgroup of $D$. So each $D_i\Phi(D)$ is an $F$-invariant normal subgroup of $D$. The natural image of $D_i$ in $\hat{B}$ is clearly $[[Y_i, P]^2, F]$, for $i = 1, \cdots, s$. It follows from (6.15) that $\prod_{i=1}^s (D_i \Phi(D))$ covers $D/\Phi(D) = \hat{B}$. Hence $D$ is the product of its $F$-invariant normal subgroups $D_i \Phi(D)$, $i = 1, \cdots, s$. Since

$$W \leq D_{c_U(P)} = \prod_{i=1}^s (D_i \Phi(D))_{c_B(P)},$$

we conclude that there is some $i = 1, \cdots, s$ such that $W \leq (D_i \Phi(D))_{c_B(P)}$. By (1.4b), $\Phi(D)$ is central in $D_i \Phi(D)$. By (6.10g) it is centralized by $F$. Since $(F \circ W)$ is non-trivial, we must have $W \leq (D_i)c_B(P)$. Then (6.17) tells us that $W \leq (D_i)c_B(P)$. This implies that $W \leq D_{c_B(P)}$, which proves the lemma.

Having reduced $\hat{B}$, we now simplify $U$.

**Lemma 6.18.** Suppose that (6.5) holds whenever we assume, in addition to (6.10) and (6.13) that

\[(6.19) \quad (PFB \circ U) \text{ is irreducible.}\]

Then (6.5) holds whenever (6.10) is satisfied.

**Proof.** By Lemma 6.12 it suffices to prove (6.5) under the hypotheses that (6.10) and (6.13) hold.

Let $\mathfrak{u}$ be the family of all irreducible $Z_r[PFB]$-factor modules of $U$. We do not know that $(PFB \circ U)$ is completely reducible, since $p(E)$ may equal $r$. However, $PB$ is a normal subgroup of $PFB$ by (6.10a, b, c) and $(PB \circ U)$ is completely reducible, by (6.10f). It follows easily that any irreducible component of $(PB \circ U)$ is $Z_r[PB]$-isomorphic to an irreducible component of $(PB \circ Y)$, for some $Y \in \mathfrak{u}$. Using the complete reducibility of $(P \times D \circ U)$, we
see from this that any irreducible component of \((D \circ C_V(P))\) is \(Z_r[D]\)-isomorphic to an irreducible component of \((D \circ C_Y(P))\), for some \(Y \in \mathfrak{U}\). It follows that

\begin{equation}
(6.20) \quad \text{Ker} (D \circ C_V(P)) = \bigcap_{\mathfrak{U}} \text{Ker} (D \circ C_Y(P)).
\end{equation}

Suppose that \(W\) is a non-trivial irreducible component of \((F \circ D_{C_U(P)})\). We adopt the notation \((6.8)\). Then we need only show that \(W \lesssim D_{C_J(P)}\).

By \((6.20)\) there exists some \(Y \in \mathfrak{U}\) such that \(W \lesssim D_{C_Y(P)}\). Let \(\mathfrak{Y}\) be the family of all non-zero images in \(Y\) of elements \(I \in \mathfrak{S}^r\). Define \(J_Y\) and \(J_{C_J}\) by \((5.19b, c)\) with \(Y, \mathfrak{Y}\) in place of \(U, \mathfrak{S}\), respectively. Then \(P, E, F, B, Y, \mathfrak{Y}\) are easily seen to satisfy \((6.10), (6.13),\) and \((6.19)\). By hypothesis, they then satisfy \((6.5)\). So \(W \lesssim D/Ker (D \circ C_{J_{C_J}}(P))\).

It is obvious from \((5.19b)\) that any \(I_Y \in \mathfrak{Y}\) is the image in \(Y\) of some \(I \in \mathfrak{S}\). It follows that \(J_Y\) is contained in the image of \(J\). Since \((P \circ J)\) is completely reducible (by \((6.10f)\)), this implies that \(C_{J_Y}(P)\) is contained in the image of \(C_J(P)\). Therefore

\[
\text{Ker} (D \circ C_J(P)) \leq \text{Ker} (D \circ C_{J_Y}(P)).
\]

From this and \(W \lesssim D/Ker (D \circ C_{J_Y}(P))\), we conclude that

\[
W \lesssim D/Ker (D \circ C_J(P)) = D_{C_J(P)},
\]

which proves the lemma.

We get rid of the easy cases by

**Lemma 6.21.** Suppose that \((6.10)\) and \((6.19)\) hold. If \([Z(B_U), P, P] \neq \{1\}\), then \((6.5)\) is true.

**Proof.** Evidently \(P, B, D, U\) and \(\mathfrak{s}\) satisfy \((5.22)\). Let \(U_1\) be an irreducible \(Z_r[PB]\)-submodule of \(U\). Since \((PB \circ F \circ U)\) is irreducible (by \((6.19)\)) and \(PB\) is a normal subgroup of \(PFB\), any irreducible component of \((PB \circ U)\) is \(Z_r[PB]\)-isomorphic to \((U_1)\), for some \(\sigma \in F\).

Clearly \([Z(B_U), P, P]\) is an \(F\)-invariant subgroup of \(B_U\). If it centralizes \(U_1\), it therefore centralizes each \((U_1)\), \(\sigma \in F\). Hence it centralizes \(U\), contradicting our hypotheses. So

\[
[Z(B_{U_1}), P, P] \geq [Z(B_U), P, P]_{U_1} > \{1\},
\]

for all irreducible \(Z_r[PB]\)-submodules \(U_1\) of \(U\). Now Lemmas 5.25 and 5.33 tell us that \((5.20)\) holds. This and the complete reducibility of \((D \circ U)\) imply that \(D_{C_{J_Y}}(P) = D_{C_J}(P)\). So \((6.5)\) holds and the lemma is proved.

We now prove several lemmas under the following hypotheses:

\begin{enumerate}
\item[(6.22a)] Conditions \((6.10), (6.13)\) and \((6.19)\) all hold.
\item[(6.22b)] \(\text{Ker} (\Phi(B) \circ U) = \{1\}\).
\item[(6.22c)] \(\Phi(B) \neq \{1\}\).
\item[(6.22d)] \(\Phi(B), P, P\) = \{1\}.
\end{enumerate}

First we draw some routine conclusions.
LEMMA 6.23. Let (6.22) hold. Then \( Z(B) = \Phi(B) = B' \), i.e., \( B \) is non-abelian and special. The representation \((C_{\Phi(B)}(P) \text{ on } U)\) is faithful, primary, and fully decomposable. The groups \( C_{\Phi(B)}(P) \) and \( \Phi(B)/[\Phi(B), P] \) are both cyclic of order 3. Finally, \( D \cap \Phi(B) = \{1\} \).

Proof. By (6.13) we have \([B, E] = B\). So \([B/B', E] = B/B'\). Applying the \(E\)-invariant epimorphism \(z \rightarrow z'\) of \(B/B'\) onto \(\Phi(B)/B'\) we get \([\Phi(B)/B', E] = \Phi(B)/B'\). This and (6.10g) imply that \(\Phi(B) = B'\).

Since \(\Phi(B) \leq Z(B) \leq B\) (by (1.4b)) and \((PE \text{ on } B)\) is irreducible (by (6.13)), either \(Z(B) = B\) or \(Z(B) = \Phi(B)\). But \(Z(B) = B\) implies \(\{1\} = B' = \Phi(B)\), contradicting (6.22c). Hence \(Z(B) = \Phi(B)\), and the first statement is true.

It follows from (1.4b) and (6.10g) that \(C_{\Phi(B)}(P)\) is central in \(PFB\). Since \((PFB \text{ on } U)\) is irreducible (by (6.19)), we conclude that \((C_{\Phi(B)}(P) \text{ on } U)\) is primary and fully decomposable. It is faithful by (6.22b). So the second statement is true.

Since \(\Phi(B)\) is a non-trivial elementary 3-group (by (6.22c) and (1.4c)), the second statement implies that \(|C_{\Phi(B)}(P)| = 3\). The other half of the third statement follows from this since \(P\) is cyclic.

From Lemma 5.36 we obtain
\[
[[B, P]^2, [B, P]^2] \leq [\Phi(B), P]^{2^{2+2+1-3}} = [\Phi(B), P]^2.
\]
This is \(\{1\}\) by (6.22d). Therefore \([B, P]^2\) is an abelian subgroup of \(B\). Since \(p(B) = 3\) does not divide \(|F|\) (by (6.10f)), we have
\[
\]
The first factor is \(D\) by (6.2). The second contains \([B, P]^2 \cap \Phi(B)\) by (6.10g). Therefore the last statement is proved and the lemma is true.

The next lemma is merely an aide to the following one.

LEMMA 6.24. Let (6.22) hold. If \(\delta \in D - \{1\}\) and \(\tau\) is any non-trivial element of \(\langle \delta, C_{\Phi(B)}(P) \rangle\), then there is some irreducible component \(W\) of \((\langle \delta, C_{\Phi(B)}(P) \rangle \text{ on } U)\) such that neither \(\delta\) nor \(\tau\) acts trivially on \(W\).

Proof. Let \(\beta\) be a generator for \(C_{\Phi(B)}(P)\) (which is cyclic by Lemma 6.23). Then \(\beta\) is central in \(B\) (by (1.4b)), so \(\langle \delta, C_{\Phi(B)}(P) \rangle = \langle \delta, \beta \rangle\) is abelian. It follows from the last statement of Lemma 6.23 that \(\delta \notin \langle \beta \rangle\). This and (1.4d) imply that \(\langle \delta, \beta \rangle = \langle \delta \rangle \times \langle \beta \rangle\) is elementary of order 9.

Let \(W_0\) be any irreducible component of \((\langle \delta, \beta \rangle \text{ on } U)\). By Lemma 6.23, \((\langle \beta \rangle \text{ on } U)\) is faithful, primary and fully reducible. Hence
\[
\text{Ker } (\langle \delta, \beta \rangle \text{ on } W_0) \cap \langle \beta \rangle = \{1\}.
\]
But \(\langle \delta, \beta \rangle_{W_0}\) must be cyclic. Therefore \(\text{Ker } (\langle \delta, \beta \rangle \text{ on } W_0) = \langle \delta^i \beta^j \rangle\), for some \(i = 0, 1, 2\).

The image \(\delta\) of \(\delta\) in \(B\) lies in \(C_{\Phi}(P)\) by (6.11a). So the map \(g : \bar{\sigma} \rightarrow f_\beta(\bar{\sigma}, \sigma)\) is a \(Z_3[P]\)-homomorphism of \(\bar{\sigma}\) into \(\Phi(B)^+\). The last statement of Lemma
6.23 implies that \( \delta \neq 0 \). Then the first statement of that lemma says that 
\( g(\tilde{B}) \neq \{0\} \). Since \( |P| = 3 \), we conclude that \( g(\tilde{B}) \cap C_{\Phi(B)}(P)^+ \neq \{0\} \). But 
\( C_{\Phi(B)}(P)^+ \) is a one-dimensional subspace of \( \Phi(B)^+ \), by Lemma 6.23. Hence 
\( C_{\Phi(B)}(P)^+ \leq g(\tilde{B}) \). Therefore there exists some element \( \sigma \in B \) such that 
\( [\delta, \sigma] = \beta \).

For each \( j = 0, 1, 2 \) we have: \([\delta, \sigma^j] = \beta^j, [\beta, \sigma^j] = 1 \). So \( \sigma^j \) normalizes 
\( \langle \delta, \beta \rangle \). Therefore \( W = W_0 \sigma^j \) is also an irreducible component of \( \langle \delta, \beta \rangle \) on \( U \). Furthermore

\[
\text{Ker} \left( \langle \delta, \beta \rangle \text{ on } W \right) = \text{Ker} \left( \langle \delta, \beta \rangle \text{ on } W_0 \right)^{\sigma^j} = \langle [\delta \beta^j \delta^j, \sigma^j] \rangle = \langle \delta \beta^{j+1} \rangle.
\]

Both \( \tau \) and \( \delta \) are non-trivial elements of \( \langle \delta, \beta \rangle \). Hence we may choose \( j = 0, 1, 2 \) so that neither \( \tau \) nor \( \delta \) lies in \( \langle \delta \beta^{j+1} \rangle \). Then \( W \) satisfies the conditions of the lemma.

Now comes the key step.

**Lemma 6.25.** Let \( (6.22) \) hold. Fix a generator \( \pi \) for \( P \). For any \( \delta \in D - \{1\} \), suppose that we can find an element \( \bar{\tau} \in \bar{B} \) satisfying

\[
\begin{align*}
(6.26a) \quad & \bar{\tau}(\pi - 1)^2 = \bar{\delta} \\
(6.26b) \quad & f_B(\bar{\tau}(\pi - 1)^i, \bar{\tau}(\pi - 1)^j) \in C_{\Phi(B)}(P)^+, \quad \text{for all } i, j = 0, 1, 2, \\
& \text{where } \bar{\delta} \text{ is the image of } \delta \text{ in } \bar{B}.
\end{align*}
\]

**Proof.** If \( (D \text{ on } C_J(P)) \) is faithful, then \( D_{C_J(P)} = D \) and \( (6.5) \) holds. So we need only prove that each element \( \sigma \in D - \{1\} \) acts non-trivially on \( C_J(P) \).

Fix \( \delta \in D - \{1\} \). Choose \( \bar{\tau} \in \bar{B} \) satisfying \( (6.26) \). Let \( B_1 \) be the inverse image in \( B \) of the \( Z[P] \)-submodule \( \langle \bar{\tau}, (\pi - 1)^j, \bar{\tau}(\pi - 1)^{\bar{\delta}} \rangle \) of \( \bar{B} \). Then \( B_1 \) is a non-trivial \( P \)-invariant subgroup of \( B \) containing \( \delta \). So \( P, B_1, \langle \delta \rangle, U \) and \( g \) satisfy \( (5.22) \).

Next we prove

\[
(6.27) \quad \langle \delta \rangle \times \Phi(B) \subseteq Z(B_1).
\]

Obviously \( \Phi(B) \leq Z(B) \cap B_1 \subseteq Z(B_1) \). Since \( \delta \in D - \{1\} \), the last statement of Lemma 6.23 implies that \( \langle \delta, \Phi(B) \rangle = \langle \delta \rangle \times \Phi(B) \). So we need only show that \( \delta \) is central in \( B_1 \). It clearly suffices to prove that \( f_B(\bar{\tau}(\pi - 1)^i, \bar{\delta}) = 0 \), for \( i = 0, 1, 2 \). If \( i \geq 1 \), then

\[
f_B(\bar{\tau}(\pi - 1)^i, \bar{\delta}) = f_B(\bar{\tau}, (\pi - 1)^i) = f_B(\bar{\tau}, (\pi - 1)^2)(\pi - 1)^i = 0
\]

by \( (6.26a, b) \). If \( i = 0 \), then \( (6.26) \) implies that

\[
f_B(\bar{\tau}, \bar{\delta}) = f_B(\bar{\tau}, (\pi - 1)^2) = f_B(\bar{\tau}(\pi^{-1} - 1)^2, \bar{\tau}) = f_B(\bar{\tau}(\pi - 1)^2\pi^2, \bar{\tau})
\]

\[
= f_B(\bar{\delta}^{-2}, \bar{\tau}) = f_B(\delta, \bar{\tau}),
\]

since \( \delta \in C_B(P) \). But \( f_B \) is alternating and \( p(B) = 3 \) is odd. So \( f_B(\bar{\tau}, \bar{\delta}) = f_B(\bar{\delta}, \bar{\tau}) \) implies \( f_B(\bar{\tau}, \bar{\delta}) = 0 \), which finishes the proof of \( (6.27) \).
Let \( \tau \) be an element of \( B_1 \) having the image \( \bar{\tau} \) in \( \bar{B} \). By (6.26), 
\[
\tau^{r-1} = \delta \mod \Phi(B).
\]
If \( P \) centralizes \( \tau^{r-1} \), we take \( \sigma = \tau \). Otherwise, we take \( \sigma = \tau^{-1} \). In either case we have
\[
(6.28) \quad \sigma^{r^{-i}(r^{-1})} = (\sigma^{r^{-1}})^i \in \langle \delta, C_{\Phi(B)}(P) \rangle - \{1\}, \text{ for } i = 1, 2.
\]
Indeed, (6.27) implies that \( C_{\langle \delta \rangle \times \Phi(B)}(P) = \langle \delta \rangle \times C_{\Phi(B)}(P) \). If \( P \) centralizes \( \tau^{r-1} \) then \( \tau^{r^{-1}} \delta^{-1} \in \Phi(B) \cap C_{\delta}(P) \) implies (6.28) for \( i = 1 \). If \( P \) does not centralize \( \tau^{r-1} \), then (6.27) gives \( 1 \neq \sigma^{r^{-1}} \in [\Phi(B), P] \). So (6.28) for \( i = 1 \) follows from (6.22d) in this case. Therefore (6.28) always holds for \( i = 1 \).

We compute 
\[
\sigma^{r^{-2}(r^{-1})} = [\sigma^{r^{-1}r^{-1}}](r^{-1}) = (\sigma^{r^{-1}})^{r^{-1}} = \sigma^{(r^{-1})^{2}} = (\sigma^{r^{-1}})^{[\sigma^{r^{-1}}]}.
\]
But \( \sigma^{r^{-1}} \in \langle \tau^{r^{-1}} \rangle, \Phi(B) \rangle = \langle \delta \rangle \times \Phi(B) \leq Z(B_1) \), by (6.26a) and (6.27). Therefore \( \sigma^{r^{-1}} \) commutes with \( [\sigma^{r^{-1}}]^{-1} \), and we have 
\[
\sigma^{r^{-2}(r^{-1})} = \sigma^{r^{-1}} = \sigma^{r^{-1}} = \sigma^{r^{-1}},
\]
since \( \sigma^{r^{-1}} \) is centralized by \( \pi \). Therefore (6.28) always holds.

By Lemma 6.24, there is an irreducible component \( W \) of \( \langle \delta, C_{\Phi(B)}(P) \rangle \) on \( U \). 

We wish to prove that \( C_{\delta}(P) \neq \{0\} \). If \( \delta = \delta_1 \), then \( J_1 = U_1 \) and this follows from the preceding paragraph. So we may assume that \( \delta_1 \subset \delta_1 \).

Fix \( I \in \delta_1 - \delta_1 \). By Lemma 5.32 and (6.27) we have 
\[
C_{B_1}(I) \cap \langle \delta, C_{\Phi(B)}(P) \rangle = \text{Ker} (\langle \delta, C_{\Phi(B)}(P) \rangle \text{ on } U_1)
\]
\[
\leq \text{Ker} (\langle \delta, C_{\Phi(B)}(P) \rangle \text{ on } W).
\]

We choose \( W \) so that \( \sigma^{r^{-1}} \) does not lie in the last group. This and (6.28) imply that \( \sigma, I \) satisfy (5.29) with \( B_1 \) in place of \( B \). So Lemma 5.28 tells us that \( C_{\delta_1}(P) \neq \{0\} \).

Because \( \delta \) acts non-trivially on \( W \), it acts non-trivially on \( U_1 \). Since \( (U_1)_{\delta} \) is a completely reducible primary \( Z_1[\delta] \)-module, this implies that \( \delta \) acts non-trivially on the non-trivial \( Z_1[\delta] \)-submodule \( C_{\delta_1}(P) \). So \( \delta \) acts non-trivially on \( C_{\delta}(P) \), which completes the proof of the lemma.

One possibility in (6.22) is now easy to handle.
LEMMA 6.29. \textit{If (6.22) holds with }$[\Phi(B), P] \neq \{1\}$, then (6.5) is true.

\textbf{Proof.} Let $\delta$ be an element of $D - \{1\}$ and $\delta$ be its image in $\bar{B}$. By Lemma 6.25, we need only find an element $\bar{\delta} \in \bar{B}$ satisfying (6.26) for some generator $\pi$ of $P$. By (6.2), $\delta \in [\bar{B}, P] = \bar{B}([B] - 1)^2$. So there exists some $\bar{\tau} \in \bar{B}$ satisfying (6.21a). Condition (6.26b) obviously holds, since $\Phi(B)^+ = C_{\Phi(B)}(P)^+$. Therefore the lemma is true.

For the other possibility in (6.22) we need more information about the action of $P$ on $\bar{B}$.

LEMMA 6.30. \textit{Let (6.22) hold with }$[\Phi(B), P] \neq \{1\}$. \textit{Then }$\bar{B}$ \textit{is a free }$\mathbb{Z}_3[\bar{P}]$-module.

\textbf{Proof.} We may choose a finite algebraic extension field $Z^*_1$ of $Z_3$ so that $Z^*_1$ is a splitting field for all subgroups of $PE$. Since $Z_3$ is a finite field, $Z^*_1$ is a normal separable extension of $Z_3$. We denote by $G$ the Galois group of $Z^*_1$ over $Z_3$. Then $G$ operates naturally on the extension $Z^*_1 \otimes Z_3 \bar{B}$ of $\bar{B}$ to a $Z^*_1[PE]$-module, by $(z \otimes \beta) \sigma = (z \otimes \beta)$, for all $z \in Z^*_1$, $\beta \in \bar{B}$, $\sigma \in G$. There are absolutely irreducible $Z^*_1[PE]$-submodules $\tilde{B}_1, \ldots, \tilde{B}_t$ of $Z^*_1 \otimes \bar{B}$ so that

\begin{align*}
(6.31a) & \quad Z^*_1 \otimes \bar{B} = \tilde{B}_1 \oplus \cdots \oplus \tilde{B}_t \text{ (as } Z^*_1[PE]-\text{modules}), \\
(6.31b) & \quad \text{for any } i, j = 1, \ldots, t, \text{ there exists } \sigma \in G \text{ such that } (\tilde{B}_i \sigma \text{ is } Z^*_1[PE]-\text{isomorphic to } \tilde{B}_j).
\end{align*}

(See Theorem V, 13.13 of [4].)

The subgroup $E$ is normal of prime index $3 = |P|$ in $PE$. It follows from Clifford's theory, (Theorem V, 17.3 of [4]) that there are two possibilities: either $(Z^*_1[E]$ on $\bar{B}_1)$ is irreducible, or $\bar{B}_1$ is induced from some irreducible $Z^*_1[E]$-submodule.

Suppose that $(Z^*_1[E]$ on $\bar{B}_1)$ is irreducible. By (6.31b), $(Z^*_1[E]$ on $\bar{B}_i)$ is irreducible, for $i = 1, \ldots, t$. These modules are absolutely irreducible by the choice of $Z^*_1$. It follows from this and Schur's Lemma that

\[ \dim_{Z^*_1} (\tilde{B}_i \otimes Z^*_1 \bar{B}_j/(\tilde{B}_i \otimes \bar{B}_j, E)) \leq 1, \text{ for all } i, j = 1, \ldots, t. \]

Since $|P| = 3$, the only one-dimensional $Z^*_1[P]$-module is the trivial one. Therefore $P$ centralizes $(\tilde{B}_i \otimes \bar{B}_j/(\tilde{B}_i \otimes \bar{B}_j, E))$, for $i, j = 1, \ldots, t$. This and (6.31a) imply that $P$ centralizes

\[ (Z^*_1 \otimes \bar{B}) \otimes Z^*_1 (Z^*_1 \otimes \bar{B})/(Z^*_1 \otimes \bar{B} \otimes \bar{B}, E). \]

Hence $P$ centralizes $\bar{B} \otimes Z^*_1 \bar{B}/[\bar{B} \otimes \bar{B}, E]$. But $\Phi(B)^+$ defines a $Z^*_1[PE]$-homomorphism $g$ of $\bar{B} \otimes \bar{B}$ into $\Phi(B)^+$. By (6.10g), the kernel of $g$ contains $[\bar{B} \otimes \bar{B}, E]$. Therefore $P$ centralizes the image of $g$. This image is clearly $(B')^+$, which equals $\Phi(B)^+$, by Lemma 6.23. By hypothesis, $[\Phi(B)^+, P] \neq \{0\}$. The contradiction proves that $(Z^*_1[E]$ on $\bar{B}_1)$ cannot be irreducible.

We now know that $\bar{B}_1$ is $Z^*_1[PE]$-isomorphic to $Y^*_1$, for some irreducible $Z^*_1[E]$-module $Y_1$. Therefore $(\bar{B}_1)^+$ is a free $Z^*_1[P]$-module. Hence so is $Z^*_1 \otimes$
It follows easily that $\hat{B}$ is a free $\mathbb{Z}_3[\mathbb{P}]$-module. Therefore the lemma is true.

We investigate $f_B(\hat{a}, \hat{r}) \pmod{[\Phi(B)^+, \mathbb{P}]}$ more closely.

**Lemma 6.32.** Let (6.22) hold. Then the function

$$h(\hat{a}, \hat{r}) = f_B(\hat{a}, \hat{r}(\pi - 1)) + f_B(\hat{r}, \hat{a}(\pi - 1)) + [\Phi(B)^+, \mathbb{P}]$$

is a symmetric, bilinear map of $\hat{B} \times \hat{B}$ into $\Phi(B)^+/[\Phi(B)^+, \mathbb{P}]$. Its radical is $C_{\hat{B}}(\mathbb{P})$. And the subspace $[\hat{B}, \mathbb{P}]$ is $h$-isotropic.

**Proof.** The first statement is obvious from the definition of $h$.

For the second, notice that the map

$$g(\hat{a}, \hat{r}) = f_B(\hat{a}, \hat{r}) + [\Phi(B)^+, \mathbb{P}]$$

is a PE-invariant (by (6.10g)) alternating bilinear map of $\hat{B} \times \hat{B}$ into $\Phi(B)^+/[\Phi(B)^+, \mathbb{P}]$. Since $B' = \Phi(B)$ (by Lemma 6.23), the map $g$ is not trivial. So the radical of $g$ is a PE-invariant proper subspace of $\hat{B}$. By (6.13), this radical must be $\{0\}$, i.e., $g$ is non-singular.

For any $\hat{a}, \hat{r} \in \hat{B}$ we compute

$$(6.33) \quad h(\hat{a}, \hat{r}) = g(\hat{a}, \hat{r}(\pi - 1)) + g(\hat{r}, \hat{a}(\pi - 1))$$

$$= g(\hat{a}, \hat{r}(\pi - 1)) + g(\hat{r}, \hat{a}(\pi - 1), \hat{a})$$

$$= g(\hat{a}, \hat{r}(\pi - 1)) - g(\hat{a}, \hat{r}(\pi - 1), \hat{a})$$

$$= g(\hat{a}, \hat{r}(\pi^2 - 1)^{(1)}),$$

using the PE-invariance of $G$ and the fact that $PE$ centralizes $\Phi(B)/[\Phi(B), \mathbb{P}]$. Since $g$ is non-singular, we conclude that $\hat{r}$ lies in the radical of $h$ if and only if $\hat{r}(\pi^2 - 1)^{(1)} = 0$, i.e., if and only if $\mathbb{r} \in C_{\hat{B}}(\mathbb{P})$. This is the second statement of the lemma.

If $\hat{a}, \hat{r} \in \hat{B}$, then

$$g(\hat{a}(\pi - 1), \hat{r}(\pi - 1)^3) = g(\hat{a}, \hat{r}(\pi - 1)^3(\pi^{-1} - 1))$$

$$= -g(\hat{a}, \hat{r}(\pi - 1)^3(\pi^{-1} - 1)) = 0,$$

since $\hat{r}(\pi - 1)^3 = \hat{r}(\pi^2 - 1) = \hat{r}(1 - 1) = 0$. The third statement of the lemma follows directly from this and (6.33). So the lemma is true.

At last we can prove

**Lemma 6.34.** Let (6.22) hold with $[\Phi(B), \mathbb{P}] \neq \{1\}$. Then (6.5) is true.

**Proof.** By Lemma 6.30, $\hat{B}$ is a free $\mathbb{Z}_3[\mathbb{P}]$-module. So there is some integer $n > 0$ such that $\hat{B}$ has dimension $3n$, $[\hat{B}, \mathbb{P}]$ has dimension $2n$, and $[\hat{B}, \mathbb{P}]^2 = C_{\hat{B}}(\mathbb{P})$ has dimension $n$. Therefore $\hat{B}_1 = \hat{B}/C_{\hat{B}}(\mathbb{P})$ has dimension $2n$.

Lemma 6.32 says that $h$ induces a non-singular symmetric bilinear form $h_1$ on $\hat{B}_1 \times \hat{B}_1$ to $\Phi(B)^+/[\Phi(B)^+, \mathbb{P}]$. The latter space is one-dimensional, by
Lemma 6.23. So we may apply the ordinary theory of quadratic forms to $h_1$. The subspace $[\bar{B}, P]/[\bar{B}, P]^2$ is $h_1$-isotropic, by Lemma 6.32, and has dimension $n$, which is one half the dimension of $\bar{B}_1$. Therefore there is some complementary $h_1$-isotropic subspace $Y_1$ such that

$$\bar{B}_1 = Y_1 \oplus [\bar{B}, P]/[\bar{B}, P]^2$$

(see Theorem 3.8 of [1]). It follows that the inverse image $Y$ of $Y_1$ in $\bar{B}$ is $h$-isotropic and satisfies $Y + [\bar{B}, P] = \bar{B}$.

Now let $\delta$ be any element of $D$, and $\bar{\delta}$ be the image of $\delta$ in $\bar{B}$. By (6.2), $\bar{\delta} \in [\bar{B}, P]^2$. If $\pi$ is a generator for $P$, then $[\bar{B}, P]^2 = B(\pi - 1)^2 = (Y + [\bar{B}, P])(\pi - 1)^2 = Y(\pi - 1)^2$, since $[\bar{B}, P](\pi - 1)^2 = B(\pi - 1)^2 = \{0\}$. So there exists $\bar{\tau} \in Y$ such that $\bar{\delta} = \bar{\tau}(\pi - 1)^2$, i.e., so that (6.26a) holds. Because $Y$ is $h$-isotropic, we have

$$0 = h(\bar{\tau}, \bar{\tau}) = 2f_b(\bar{\tau}, \bar{\tau}(\pi - 1)) + [\Phi(B)^+, P].$$

Hence $f_b(\bar{\tau}, \bar{\tau}(\pi - 1)) \in [\Phi(B)^+, P] \leq C_{\Phi(B)}(P)^+ \leq C_B(P)$ (by (6.22d)).

We compute

$$f_b(\bar{\tau}, \bar{\tau}(\pi - 1)^2) = f_b(\bar{\tau}(\pi^2 - 1)^2, \pi) \mod [\Phi(B)^+, P]$$

$$= f_b(\bar{\tau}(\pi^2 - 1)^2, \pi) \mod [\Phi(B)^+, P]$$

$$= -f_b(\bar{\tau}, \bar{\tau}(\pi - 1)^2) \mod [\Phi(B)^+, P],$$

since $f_b$ is alternating, $P$-invariant, and bilinear, and $\bar{\tau}(\pi - 1)^2 \in C_B(P)$. Since 3 is odd, we conclude that $f_b(\bar{\tau}, \bar{\tau}(\pi - 1)^2) \in [\Phi(B)^+, P]$.

Finally, $f_b(\bar{\tau}(\pi - 1), \bar{\tau}(\pi - 1)^3) \in [\Phi(B)^+, P]$ by Lemma 5.36. Therefore (6.26b) holds for $0 \leq i < j \leq 2$. Because $f_b$ is alternating, this proves (6.26b) in all cases. So Lemma (6.25) says that (6.5) holds. This finishes the proof of this lemma.

We collect the results of this section in

**Theorem 6.35.** If (6.1) holds, then $(F$ on $D_\bar{\pi})$ is weakly equivalent to $(F$ on $D_{\bar{\alpha}}$).

**Proof.** By Proposition 6.6, it suffices to show that (6.5) holds whenever (6.10) does. Lemma 6.18 says that it is enough to prove (6.5) when (6.10), (6.13) and (6.19) all hold, i.e., when (6.22a) holds.

The subgroup $\text{Ker} \ (\Phi(B)$ on $U)$ is $E$-invariant (by (6.10g)) and $PB$-invariant by (6.10d). It follows that $P, F, E, B/\text{Ker} \ (\Phi(B)$ on $U), U, \mathcal{g}$ also satisfy (6.22a). Clearly $D_{\mathcal{g}(P)}$ and $D_{\mathcal{g}(P)}$ are unchanged when we replace $B$ by $B/\text{Ker} \ (\Phi(B)$ on $U)$. So it suffices to prove (6.5) when (6.22a, b) hold.

If $\Phi(B) = \{1\}$, then $B$ is abelian. So

$$D_U \leq [B_U, P, P] = [Z(B_U), P, P].$$

If $D_U \neq \{1\}$, then (6.5) holds by Lemma 6.21. If $D_U = \{1\}$, then (6.5) is trivial. Therefore it suffices to prove (6.5) when (6.22a, b, c) hold.
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If \([\Phi(B), P, P] \neq \{1\}\), then (6.22b) implies that
\[\{1\} \neq [\Phi(B_v), P, P] \leq [Z(B_v), P, P].\]

Again Lemma 6.21 proves (6.5). So it suffices to prove (6.5) when (6.22) holds.

If \([\Phi(B), P] = \{1\}\), Lemma 6.29 proves (6.5). If \([\Phi(B), P] \neq \{1\}\), Lemma 6.34 proves (6.5). Therefore (6.5) holds in all cases and the theorem is true.

7. Proofs of the basic theorems

We shall carry out a large part of the proofs of Theorems 2.6, 2.7 and 2.13 simultaneously. For the first two theorems we assume that (2.5) holds. For Theorem 2.13, we assume in addition that \(p = 3\) and that \(A_1, \cdots, A_t\) has been extended to an augmented Fitting Chain \(A_1, \cdots, A_t, \{B_i\}\) on which \(H\) also acts.

We may also assume that \(t \geq 3\) for Theorem 2.6, \(t \geq 4\) for Theorem 2.7, and \(t \geq 6\) for Theorem 2.13. In particular, \(t > i_0\), where \(i_0\) is defined by (4.19). In view of (2.5c), Theorem 4.20 gives us an integer \(j\) satisfying:

\[
\begin{align*}
(7.1a) & \quad 1 \leq j \leq i_0 < t. \\
(7.1b) & \quad p(A_j) \neq p. \\
(7.1c) & \quad \{0\} \neq \hat{A}_{j+1,\text{ample}} \text{ (defined with respect to } (PA_j \text{ on } \hat{A}_{j+1})).
\end{align*}
\]

This integer \(j\) will be fixed throughout this section.

It is convenient to add one more term to our chain. Let \(q\) be any prime different from \(p\). Form the semidirect product \(HA_{t-1} \rtimes A_t\). Let \(A_{t+1}\) be the regular \(\mathbb{Z}_q[H A_{t-1} A_t]\)-module written multiplicatively. If \(p(A_t) = 3\) and we are proving Theorem 2.13, let \(B_t = A_{t-1}\) and \(\eta_{t-1}\) be the identity isomorphism. Since \((A_t \text{ on } \hat{A}_{t+1})\) is weakly invariant under \(\text{Aut}(A_t)\), we easily verify that \(H, P\) and \(A_1, \cdots, A_{t+1}\) or \(A_1, \cdots, A_{t+1}, \{B_i\}\) satisfy the hypotheses of our theorems.

We define subspaces \(S_i\) of \(\hat{A}_i\) and subgroups \(E_i\) of \(A_i\) by

\[
\begin{align*}
(7.2a) & \quad S_j = \hat{A}_j, E_j = A_j. \\
(7.2b) & \quad S_i \text{ is the sum of all ample irreducible } Z_{p(A_i)}[P E_{i-1}]\text{-submodules of } \hat{A}_i, \\
& \quad \text{for } i = j + 1, \cdots, t + 1, \\
(7.2c) & \quad E_i = [X, E_{i-1}], \text{ where } X \text{ is the inverse image in } A_i \text{ of } S_i, \\
& \quad \text{for } i = j + 1, \cdots, t + 1.
\end{align*}
\]

Clearly \(S_i\) and \(E_i\) are \(P\)-invariant whenever \(E_{i-1}\) is. So it makes sense to form the group \(P E_{i-1}\) in (7.2b).

**Proposition 7.3.** Let \(i = j + 1, \cdots, t + 1\). Then both \(S_i\) and \(E_i\) are \(HE_{i-1}\)-invariant. The natural map of \(E_i\) into \(\hat{A}_i\) is an \(HE_{i-1}\)-isomorphism of \(E_i\) onto \(S_i\). Hence \(E_i\) is a direct sum of ample irreducible \(Z_{p(A_i)}[P E_{i-1}]\)-submodules. Finally, defining \(A_i,\text{ample}\) by (3.15) with respect to \((P E_{i-1} \text{ on } \hat{A}_i)\), we have that \((E_{i-1} \text{ on } \hat{A}_i,\text{ample})\) is weakly equivalent to \((E_{i-1} \text{ on } \hat{E}_i)\).
Proof. By (7.2a), $S_j$ and $E_j$ are $H$-invariant. Since $HE_{i-1}$ acts on $\tilde{A}_i$ and $P$ is a normal subgroup of $H$, it follows from (3.9) that $H$ permutes the ample irreducible $Z_{p(A_i)}[PE_{i-1}]$-submodules of $\tilde{A}_i$ among themselves. Hence $HE_{i-1}$ leaves $S_i$ invariant. By (7.2e), it also leaves $E_i$ invariant. This completes the inductive proof of the first statement.

By (3.9), $E_{i-1}$ acts non-trivially on any ample irreducible $Z_{p(A_i)}[PE_{i-1}]$-submodule $V$ of $\tilde{A}_i$. Therefore $[V, E_{i-1}] = V$. We conclude that $S_i = [S_i, E_{i-1}]$ is the image in $\tilde{A}_i$ of $\tilde{E}_i$. Since $p(A_{i-1}) \neq p(A_i)$ (by (2.2b)) we have

$$[E_i, E_{i-1}] = [X, E_{i-1}]^g = [X, E_{i-1}] = E_i,$$

where $X$ is as in (7.2c). Furthermore, $\tilde{E}_i = [\tilde{E}_i, E_{i-1}] \oplus C_{\tilde{E}_i}(E_{i-1})$. It follows that $C_{\tilde{E}_i}(E_{i-1}) = \{0\}$. But the kernel of the natural map of $\tilde{E}_i$ into $\tilde{A}_i$ is $[E_i \cap \Phi(A_i)/\Phi(E_i)]^\perp$, which is contained in $C_{\tilde{E}_i}(E_{i-1})$ by (2.2c). Therefore this kernel is $\{0\}$ and the second statement is true.

The third statement follows immediately from the second and (7.2b).

Since $p(A_{i-1}) \neq p(A_i)$, the action $(E_{i-1} \cdot \tilde{A}_i)$ is completely reducible. It follows that any reducible component of $(E_{i-1} \cdot \tilde{A}_i)$ is $E_i$-isomorphic to one of $(E_{i-1} \cdot S_i)$ and hence one of $(E_{i-1} \cdot \tilde{E}_i)$. The converse is obvious. So the last statement is true, which proves the proposition.

We define subgroups $F_i$ by

\begin{align*}
(7.4a) & \quad F_j = \{1\}, \\
(7.4b) & \quad F_i = C_{E_i}(P) \text{ if } p(A_i) \neq p \text{ and } i = j + 1, \ldots, t + 1, \\
(7.4c) & \quad F_{j+1} = C_{(E_{j+1})^p}(P), \text{ if } p(A_{j+1}) = p, \\
(7.4d) & \quad F_i = [[E_i, P]^{p-1}, F_{i-1}], \text{ if } p(A_i) = p \text{ and } i = j + 2, \ldots, t + 1.
\end{align*}

They satisfy

**Proposition 7.5.** For each $k = j, \ldots, t + 1$, the subgroup $F_i$ of $E_i$ is normalized by $H$ and centralized by $P$. If $p(A_i) = p$, then $F_i \leq [E_i, P]^{p-1}$. If $i \geq j + 1$, then $F_{i-1}$ normalizes $F_i$.

**Proof.** Since $H$ normalizes $P$ and also each $E_i$ (by Proposition 7.3), it follows easily from (7.4) and induction that $H$ normalizes each $F_i$.

If $F_i$ is defined by (7.4b, c), then it is clearly centralized by $P$. Assume that (7.4d) holds and that $P$ centralizes $F_{i-1}$. Then $F_{i-1}$ normalizes $[E_i, P]^{p-1}$. Since $P$ is cyclic of order $p = p(A_i)$, we have $[\tilde{E}_i, P]^p = \{0\}$. Hence $[E_i, P]^p \leq [\Phi(E_i), Z(E_i)]$ (by (2.2a) and (1.4b)). For any $\pi \in P$ we easily compute that $\mu: \sigma \rightarrow \sigma^{-1}$ is an $F_{i-1}$-homomorphism of $[E_i, P]^{p-1}$ into $\Phi(E_i)$. Since $F_{i-1}$ centralizes $\Phi(E_i) \leq \Phi(A_i)$ (by (2.2c)), we must have

$$F_i = [[E_i, P]^{p-1}, F_{i-1}] \leq \ker \mu.$$

Therefore $P$ centralizes $F_i$ and the first statement is true.

If (7.4e) holds, then clearly $F_i \leq [E_i, P]^{p-1}$. If (7.4d) holds, then $F_{i-1}$
normalizes $[E_i, P]^{P-1}$ by the above argument. So the second statement is true.

If $i \geq j + 1$, then $F_{i-1}$ centralizes $P$ and normalizes $E_i$ (by Proposition 7.3). The third statement follows directly from this and (7.4). So the proposition is true.

The sections $D_i$ are defined in most cases by

\begin{equation}
D_i = (F_i)_{E_i+1}, \text{ for } i = j + 1, \ldots, t, \text{ unless } p(A_i) = p = 3.
\end{equation}

From Theorem 5.40 we get

**Proposition 7.7.** Suppose that $p(A_i) \neq p$ and $E_i \neq \{1\}$, for some $i = j + 1, \ldots, t$. Then $D_i$ is $F_{i-1}$-invariant. If either $p \neq p(A_{i-1})$ or $p = p(A_{i-1}) \geq 5$, then $D_i \neq \{1\}$ and $(F_{i-1}$ on $E_i)$ is weakly equivalent to $(F_{i-1}$ on $D_i)$.

**Proof.** It is obvious from (7.2) that $E_i \neq \{1\}$ implies $E_{i-1} \neq \{1\}$. From the first statement of Proposition 7.3 we see that $P, B = E_{i-1}, A = E_i$ and $V = A_{i+1}$ satisfy (5.1a, c, d). Proposition 7.5 says that $D_i$ satisfies (5.1b, i). Condition (5.1e) comes from (2.2b), since $p \neq p(A_i)$. Condition (5.1f) comes from (2.2c), condition (5.1g) from Proposition 7.3, and condition (5.1h) from (2.2d, e).

By (7.4b) the section $C$ of (5.11) is

\[ F_i/F_i \cap \text{Ker } (F_i \text{ on } \bar{A}_{i+1, \text{ample}}). \]

Proposition 7.3 implies that $(F_i \text{ on } \bar{A}_{i+1, \text{ample}})$ is weakly equivalent to $(F_i \text{ on } \bar{E}_{i+1})$. Since $p(A_i) \neq p(A_{i+1})$ (by (2.2b)), we conclude that

\[ \text{Ker } (F_i \text{ on } \bar{A}_{i+1, \text{ample}}) = \text{Ker } (F_i \text{ on } \bar{E}_{i+1}). \]

So (7.6) says that $C = D_i$.

Now (5.12) tells us that $D_i$ is $F_{i-1}$-invariant. The last statement of the proposition comes from Theorem 5.40. So the proof is complete.

We can now finish the definition of $D_i$. Suppose that $i = j + 1, \ldots, t - 1$, and $p(A_i) = p = 3$. Then $p(A_{i+1}) \neq p$ by (2.2b). So $D_{i+1}$ is defined by (7.6). If $F_i$ centralizes $E_{i+1}$, then it centralizes $D_{i+1}$. If $F_i$ does not centralize $E_{i+1}$, then $E_{i+1} \neq \{1\}$ and $F_i$ normalizes $D_{i+1}$ by Proposition 7.7. Since $F_i$ normalizes $D_{i+1}$ in both cases, we may define

\begin{align}
(7.8a) & \quad D_i = (F_i)_{D_{i+1}}, \text{ if } i = j + 1, \ldots, t - 1 \text{ and } p(A_i) = p = 3, \\
(7.8b) & \quad D_i = F_i, \text{ if } p(A_i) = p = 3.
\end{align}

The result corresponding to Proposition 7.7 for $p(A_i) = p$ is

**Proposition 7.9.** Suppose that $p(A_i) = p \geq 3$ for some $i = j + 1, \ldots, t$. If $i > j + 1$, we also assume that $D_{i-1} \neq \{1\}$. Then Ker $(F_i \text{ on } \bar{E}_{i+1}) = \{1\}$, $F_i \neq \{1\}$, and $(F_{i-1}$ on $\bar{E}_i)$ is weakly equivalent to $(F_{i-1}$ on $\bar{F}_i)$.

**Proof.** $(E_i \text{ on } \bar{A}_{i+1})$ is faithful by (2.2d). It follows from this and (3.9e)
that \((E_i, P)^{p-1}\) on \(A_{i+1, \text{ample}}\) is faithful. By Proposition 7.3, \((E_i, P)^{p-1}\) on \(A_{i+1, \text{ample}}\) is weakly equivalent to \((E_i, P)^{p-1}\) on \(E_{i+1}\). Since \(p(A_i) \neq p(A_{i+1})\), this implies that \((E_i, P)^{p-1}\) on \(E_{i+1}\) is faithful. Proposition 7.5 says that \(F_i \leq [E_i, P]^{p-1}\). Hence Ker \((F_i \mid E_{i+1}) = \{1\}\).

Suppose that \(i = j + 1\). Then \(E_{j+1, \text{ample}} \neq \{0\}\) by (7.1c). It follows from Proposition 7.3 that \(E_{j+1} \neq \{0\}\). Since \(E_{j+1}\) is a sum of ample irreducible \(PE_{j}\)-submodules (by Proposition 7.3) and \(p(A_j) \neq p\) (by (2.2b)), Proposition 3.10 implies that \([E_{j+1}, P]^{p-1} \neq \{0\}\). This and (7.4c) give \(F_{j+1} \neq \{1\}\). Since \(F_j = \{1\}\) (by (7.4a)), this proves the proposition for \(i = j + 1\).

Suppose that \(i > j + 1\). Since \(E_i\) is a sum of ample irreducible \(PE_{i-1}\)-modules and \(F_{i-1} = C_{B_{i-1}}(P)\), Proposition 3.10 says that \((F_{i-1} \mid E_i)\) is weakly equivalent to \((F_{i-1} \mid [E_i, P]^{p-1})\) and hence to \((F_{i-1} \mid [E_i, P]^{p-1}, F_{i-1})\).

We know that \(F_{i-1}\) centralizes \(F_i \cap \Phi(E_i)\) by (2.2c). It follows from this and (7.4d) that \((F_{i-1} \mid E_i)\) is weakly equivalent to \((F_{i-1} \mid F_i)\). By (2.2b), \(p(A_{i-1}) \neq p(A_i) = p\). Therefore \(D_{i-1} = (F_{i-1} \mid F_i) = (F_{i-1} \mid F_i)\). Hence \(F_i \neq \{0\}\), which finishes the proof of the proposition.

In the case of Theorem 2.13 we must also define sections \(C_i\) of \(B_i\). They are given by

\[(7.10) \quad C_i = C_X(P), \text{ where } X \text{ is the inverse image in } B_i \text{ of } F_i, \text{ for all relevant } i = j + 1, \cdots, t - 2.\]

Since \(\eta_i\) is a \(P\)-epimorphism of \(B_i\) onto \(A_i\) and \(p(A_i) \neq p = 3\), we conclude from (7.4b) that

\[(7.11) \quad \eta_i(C_i) = F_i, \text{ for all relevant } i = j + 1, \cdots, t - 2.\]

Theorem 6.35 will give us

**Proposition 7.12.** Suppose that \(p(A_i) = p = 3\) and \(D_{i-1} \neq \{1\}\), for some \(i = j + 2, \cdots, t - 1\). Then \(F_{i-1}\) normalizes \(D_i\), \(C_{i-1}\) normalizes \(D_{i+1}\), and \((F_{i-1} \mid E_i)\) is weakly equivalent to \((F_{i-1} \mid D_i)\).

**Proof.** Proposition 7.9 tells us that \(F_i \neq \{1\}\) and Ker \((F_i \mid E_{i+1}) = \{1\}\). Hence \(E_i \neq \{1\}\) and \(E_{i+1} \neq \{1\}\). Let \(E\) be the inverse image in \(B_{i-1}\) of \(E_{i-1}\), \(B\) be \(E_i\), \(F\) be \(C_{i-1}\), \(A\) be \(E_{i+1}\), and \(V\) be \(A_{i+2}\). The augmentation tells us that \(PC_{i-1} E_i\) acts on \(A_{i+1}\). Since \(C_{i-1}\) centralizes \(P\), it must leave \(S_{i+1}\) and \(E_{i+1}\) invariant (by (3.9) and (7.2)). Hence \(E_{i+1}\) is \(PC_{i-1} E_i\)-invariant, and (6.1a–e) hold. Condition (6.1f) comes from (2.2b), since \(p(A_i) = 3\). Conditions (6.1g, i) come from (2.2c) Conditions (6.1h, j) come from Proposition 7.3. Finally, condition (6.1k) comes from (2.2e) and (2.10c).

Evidently (7.11) implies that the group \(D\) of (6.2) is the group \(F_i\) defined by (7.4d). It follows from Proposition 7.3 that the section \(C\) of (5.11) is \(D_{i+1}\). (see the proof of Proposition 7.7). So (7.8a) becomes \(D_i = D_{i+1}\).
Proposition 6.4 says that $C_{i-1}$ normalizes $D_{i+1}$ and $F_i$. Hence it normalizes $D_i$. This and (7.11) imply that $F_{i-1}$ normalizes $D_i$.

Since $F_{i-1}$ centralizes $\Phi(D_i)$, the action $(F_{i-1} \cdot \tilde{D}_i)$ is weakly equivalent to $(F_{i-1} \cdot D_i)$. Theorem 6.35 and (7.11) say that $(F_{i-1} \cdot D_i)$ is weakly equivalent to $(F_{i-1} \cdot (\tilde{D}_i \cdot F_{i+1}))$. By Proposition 7.9, the group $(\tilde{D}_i \cdot F_{i+1})$ is just $F_i$. Since $F_{i-1}$ centralizes $\Phi(F_i)$ (by (2.2c)), $(F_{i-1} \cdot F_i)$ is weakly equivalent to $(F_{i-1} \cdot \tilde{F}_i)$. This, in turn, is weakly equivalent to $(F_{i-1} \cdot \tilde{D}_i)$ by Proposition 7.9. So the proposition is true.

We must return to the techniques of §5 to prove

**Proposition 7.13.** If $p(A_{j+1}) = p = 3$, then $D_{j+1} \neq \{1\}$.

**Proof.** If $j + 1 = t$, this is clear from (7.8b) and Proposition 7.9. So we may assume that $t > j + 1$.

Proposition 7.9 says that $F_{j+1} \neq \{1\}$ and Ker $(F_{j+1} \cdot \tilde{E}_{j+2}) = \{1\}$. Hence $E_{j+2} \neq \{1\}$. As in the proof of Proposition 7.7, this implies that $P, B = E_{j+1}, A = E_{j+2}, D = F_{j+1}$, and $V = \tilde{A}_{j+3}$ satisfy (5.1) with $p(B) = p = 3$ and $q = p(A_{j+2})$. Furthermore, the section $C$ of (5.11) is $D_{j+2}$. Since $D_{j+1}$ is given by (7.8a) and $F_{j+1}$ by (7.4c) we are reduced to proving

$$(7.14) \quad \text{Suppose that (5.1) holds with } p = p(B) = 3 \text{ and } D = C_{[u,v]}(P).$$

If $D = D_{2} \neq \{1\}$, then $D$ does not centralize $\tilde{C}$.

Next we pass to the situation (5.22). Let the hypotheses of (7.14) hold with $D_{2} = \{1\}$. Then (5.16) and $p(B) \neq p(A)$ imply that $D$ centralizes $C_{\bar{A}}(P)/C_{\bar{A}}(P) \cap \varphi(Q_{+})$. This and (5.10) imply that $D$ centralizes $C_{\bar{A}}(P)/C_{\bar{A}}(P) \cap \varphi(N_{+})$. On the other hand $D = D_{2}$ is faithfully represented on $\bar{A}$ and hence on $C_{\bar{A}}(P)$ by (5.17). So $D$ acts faithfully on $C_{\bar{A}}(P) \cap \varphi(N_{+})$. Using the fact that $\varphi$ is a monomorphism (by Proposition 5.6), we conclude that $D$ acts faithfully on $C_{\bar{A}}(P)$ and centralizes $C_{\bar{A}}(P)/C_{\bar{A}}(P)$. By (5.21), this implies that $D$ centralizes $C_{\bar{A}}(P)$ and acts faithfully on $C_{\bar{A}}(P)$. So we are reduced to deriving a contradiction from the situation in which

$$(7.15a) \quad \text{conditions (5.22) hold,}$$

$$(7.15b) \quad p = p(B) = 3,$$

$$(7.15c) \quad D = C_{[u,v]}(P) \neq \{1\},$$

$$(7.15d) \quad (D \text{ on } C_{\bar{A}}(P)) \text{ is faithful},$$

$$(7.15e) \quad D \text{ centralizes } C_{\bar{A}}(P).$$

We define $U_{i}, \pi_{i}, s_{i}, \tilde{g}_{i}, J_{i}$, for $i = 1, \ldots, s$, as in (5.24) and Lemma 5.25. Notice that (5.26), $p \neq r$, and (7.15e) imply

$$(7.16) \quad D \text{ centralizes } C_{\bar{A}}(P) \leq \pi_{i}(C_{\bar{A}}(P)), \text{ for } i = 1, \ldots, s.$$ 

The next step is to prove

$$(7.17) \quad \sigma^{(r-1)2} \notin Z(B) - \{1\}, \text{ for all } \sigma \in B, \pi \in P.$$ 

Suppose that $\sigma^{(r-1)2} \in Z(B) - \{1\}$, for some $\sigma \in B, \pi \in P$. If $\pi = 1$, then
\[ \sigma^{r-1} = 1. \] So \( r \neq 1 \) and \( P = \langle \pi \rangle \). Since \( p = p(B) \) there exists an integer \( n \geq 0 \) such that \( \sigma^{r-1+n} \neq 1 \), and \( \sigma^{r-1+n} = 1 \). Replacing \( \sigma \) by \( \sigma^{r-1} \), we may assume that \( \sigma^{r-1} = 1 \). Then

\[ \sigma^{r-1} \in C_{[B, P]^2}(P) = D. \]

Since \( \sigma^{r-1} \in D = \{1\} \), condition (7.15d) and (5.24) give us an integer \( i = 1, \ldots, s \) such that \( \sigma^{r-1} \) does not centralize \( C_{U_i}(P) \). Hence

\[ \sigma^{r-1} \in Z(B) - \text{Ker}(B \text{ on } U_i). \]

Furthermore, this and (7.16) give \( J_i < U_i \). So there is some member \( I_i \) of \( \delta_i - \delta_i \). Now Lemma 5.32 tells us that \( \sigma^{r-1} \notin C_B(I_i) \). Since \( \sigma^{r-1} \in Z(B) \), we easily compute that \( \sigma^{r-1} = [\sigma^{r-1}]^2. \) This does not lie in \( C_B(I_i) \), since \( p(B) = 3 \). Hence \( \sigma \) and \( I_i \) satisfy (5.29). Obviously \( U_i \) is a primary \( Z \![\langle \sigma^{r-1} \rangle] \)-module. So Lemma 5.28, applied to \( \langle \sigma^{r-1} \rangle \), tells us that \( \langle \sigma^{r-1} \rangle \) on \( C_{U_i}(P) \) is weakly equivalent to \( \langle \sigma^{r-1} \rangle \) on \( C_{U_i}(P) \). This is impossible since \( \sigma^{r-1} \) centralizes \( C_{U_i}(P) \) but not \( C_{U_i}(P) \) and \( r \neq p(B) \). Therefore (7.17) holds.

From (7.17) we will conclude that

(7.18) \( D = [B, P]^2 \) is generated by all \( \sigma^{r-1}, \sigma \in B, \pi \in P \).

Suppose that \( \sigma \in B, \pi \in P, \) and \( \sigma^{r-1} \notin [B, P]^2 - D \). Then \( \pi \neq 1 \) and \( P = \langle \pi \rangle \). Hence \( \sigma^{r-1} \neq 1 \) (by (7.15c)). But \( [B, P]^3 = \{0\} \), since \( p = 3 = p(B) \). Therefore

\[ \sigma^{r-1} \in \Phi(B) - \{1\} \subseteq Z(B) - \{1\} \]

(by (1.4b)), which violates (7.17). We conclude that \( D \) contains \( \sigma^{r-1} \), for all \( \sigma \in B, \pi \in P \).

The subgroup \( [B, P]^2 \) is generated by the elements \( \sigma^{r-1} \) and \( \sigma^{r-1} \sigma^{r-1} \), \( \sigma \in B, \pi \in P \). The first elements lie in \( D \). For the second we compute

\[ \sigma^{r-1} \sigma^{r-1} = \sigma^{r-1} \sigma^{r-1} = [\sigma^{r-1}]^2 \in D, \]

since \( \sigma^{r-1} \in D \) is centralized by \( P \). Hence \( [B, P]^2 \leq D \). This and (7.15c) give (7.18).

Next we show that

(7.19) \( D \cap [\langle \sigma^{r-1} \rangle, B] = \{1\}, \) for all \( \sigma \in B, \pi \in P \).

Suppose that \( \delta \) is a non-trivial member of \( D \cap [\langle \sigma^{r-1} \rangle, B] \), for some \( \sigma \in B, \pi \in P \). Clearly \( \langle \pi \rangle = P \). By (7.15d) and (5.24), we may choose some \( i = 1, \ldots, s \) so that \( \delta \) acts non-trivially on \( C_{U_i}(P) \).

Let \( \sigma_i, \delta_i \) be the images of \( \sigma, \delta \), respectively in \( B_i = B_{U_i} \). Since \( \delta \in D \cap B') \) it follows from (7.15c) and (1.4b) that \( \langle \delta_i \rangle \leq Z(PB_i) \cap \Phi(B_i) \). Because \( B_i \) acts faithfully on the irreducible \( Z_1[PB_i] \)-module \( U_i \), the subgroup \( Z(PB_i) \cap B_i \) is cyclic. From this, \( \delta_i \neq 1 \), and (1.4c) we conclude that

(7.20) \( \langle \delta_i \rangle = Z(PB_i) \cap \Phi(B_i) = C_{\Phi(B_i)}(P). \)
Since \( \delta_i \in [\sigma_i^{(x-1)^2}, B_i] \), there exists an element \( \tau \in B_i \) so that \([\sigma_i^{(x-1)^2}, \tau] = \delta_i\). Hence \( \sigma_i^{(x-1)^2} \notin \Phi(B_i) \). It follows from this, (1.4d) and (7.20) that \( Y = \langle \sigma_i^{(x-1)^2} \rangle \times \langle \delta_i \rangle \) is an elementary abelian \( \tau \)-invariant subgroup of order 9 in \( B_i \). Clearly

\[
\sigma_i^{(x^2-1)(x-1)} \equiv [\sigma_i^{(x-1)^2}]^2 \pmod{\Phi(B_i)}.
\]

So \( \sigma_i^{(x^2-1)(x-1)} \neq 1 \). Furthermore, (7.15c), (7.18) and (7.20) give

\[
\sigma_i^{(x^2-1)(x-1)[\sigma_i^{(x-1)^2} - 1]} \epsilon C_{\Phi(B_i)}(P) = \langle \delta_i \rangle.
\]

Therefore both \( \sigma_i^{(x-1)^2} \) and \( \sigma_i^{(x^2-1)(x-1)} \) are non-trivial elements of \( Y \).

Let \( W \) be an irreducible \( \mathbb{Z}[Y] \)-submodule of \( U_i \). Since \( \delta_i \) is central in \( PB_i \) and \( (PB_i \otimes U_i) \) is irreducible, we know that \( \langle \langle \delta_i \rangle, \Phi(B_i) \rangle \) is primary, completely reducible, and non-trivial. Hence \( \langle \langle \delta_i \rangle, \Phi(B_i) \rangle \) is non-trivial and \( \text{Ker}(Y \otimes W) = \langle \sigma_i^{(x-1)^2} \rangle \), for some \( e = 0, 1, 2 \). Because \( \tau \) normalizes \( Y \), the translate \( W \tau^k \) is also an irreducible \( \mathbb{Z} [Y] \)-submodule of \( U_i \), for each \( k = 0, 1, 2 \). But

\[
\text{Ker}(Y \otimes W \tau^k) = \langle \langle \delta_i \rangle \rangle = \langle \sigma_i^{(x^2-1)(x-1)} \rangle.
\]

Therefore we may choose \( W \) so that neither \( \sigma_i^{(x-1)^2} \) nor \( \sigma_i^{(x^2-1)(x-1)} \) lies in \( \text{Ker}(Y \otimes W) \).

By (7.15c), (3.9), and the fact that \( \delta_i \neq 1 \), the action \( (PB_i \otimes U_i) \) is ample. Proposition 3.10 says that we may take \( W \leq C_{U_i}(P) \). The complete reducibility of \( (Y \otimes C_{U_i}(P)) \) gives us a \( \mathbb{Z} [Y] \)-submodule \( X \) of \( C_{U_i}(P) \) so that

\[
C_{U_i}(P) = W \oplus X \quad (as \mathbb{Z} [Y] \text{-modules}).
\]

Since \( \langle \langle \delta_i \rangle \rangle \) is primary, completely reducible, and non-trivial, (7.16) implies that \( C_{I_i}(P) \neq \{0\} \). By (5.22e) and (5.19), this gives

\[
C_{U_i}(P) = C_{I_i}(P) + \sum I_{i_1 \cdots i_k \neq i} I_i = \sum I_{i_1 \cdots i_k \neq i} I_i.
\]

Therefore there is some \( I_i \in \mathcal{I}_i - \mathcal{I}_i \) whose projection \( (I_i + X) \cap W \) is non-trivial. We conclude that \( C_Y(I_i) \leq \text{Ker}(Y \otimes W) \). In particular, neither \( \sigma_i^{(x-1)^2} \) nor \( \sigma_i^{(x^2-1)(x-1)} \) lies in \( C_Y(I_i) \). Hence \( \sigma_i \), \( I_i \) satisfy (5.29).

Lemma 5.28, applied to \( \langle \delta_i \rangle \), tells us that \( C_{I_i}(P) \neq \{0\} \), contradicting a statement above. Therefore (7.19) holds.

Now we finish the proof of Proposition 7.13. By (7.15c) and (7.18) there exists some \( \sigma \in B_i, \pi \in P \) such that \( \sigma^{(x-1)^2} \neq 1 \). We compute

\[
\sigma^{(x^2-1)(x-1)} = \sigma^{(x^2-\tau)x^{(x-1)^2}} \sigma^{(x-1)^2} [\sigma^{(x^2-\tau)^{-1}} = \sigma^{(x-1)^2} \sigma^{(x-1)^2} [\sigma^{(x-1)^2}, \sigma^{(x^2-\tau)^{-1}}].
\]

It follows from this and (7.18) that

\[
[\sigma^{(x-1)^2}, (\sigma^{x^2-\tau})^{-1}] \epsilon D \cap [(\sigma^{(x-1)^2}), B].
\]
So (7.19) gives

\[ (7.21) \quad \sigma^{(\pi^{-1})^3}, \sigma^{\pi^{-1}} = [\sigma^{(\pi^{-1})^3}, (\sigma^{\pi^2 \pi^{-1}})^{-1}, \sigma^{(\pi^{-1})^3}]^{-1} = 1. \]

Next we compute

\[ (\sigma^2)^{(\pi^{-1})^2} = (\sigma^x \sigma^{-1})^{(\pi^{-1})} = ((\sigma^{-1})^3[\sigma^{-1}, \sigma^{-1}])^{(\pi^{-1})} \]
\[ = \sigma^{(\pi^{-1})^2} \sigma^{(\pi^{-1})^2} (\sigma^{-1})^{-1}[\sigma^{-1}, \sigma^{-1}]^{(\pi^{-1})} \]
\[ = \sigma^{(\pi^{-1})^2} \sigma^{(\pi^{-1})^2} [\sigma^{-1}, \sigma^{-1}]^{(\pi^{-1})} \quad \text{(by (7.21)).} \]

This and (7.18) give \([\sigma^{-1}, \sigma^{-1}]^{(\pi^{-1})} \in D. \) But

\[ [\sigma^{-1}, \sigma^{-1}]^{(\pi^{-1})} = [\sigma^{(\pi^{-1})^2}, \sigma^{-1}]^{[\sigma^{-1}, \sigma^{-1}]^{(\pi^{-1})}} \]
\[ = [\sigma^{(\pi^{-1})^2}, \sigma^{-1}]^{[\sigma^{-1}, \sigma^{-1}]} \]
\[ = [\sigma^{(\pi^{-1})^2}, \sigma^{-1}]. \]

Therefore (7.19) gives

\[ 1 = [\sigma^{(\pi^{-1})^2}, \sigma^{-1}]^{-1} = [\sigma^{(\pi^{-1})^2} \sigma^{-1}, \sigma] = [\sigma^{(\pi^{-1})^2}, \sigma]. \]

We conclude from this and (7.21) that \(\sigma^{(\pi^{-1})^2}\) lies in the center of the group

\[ B_i = \langle \sigma, \sigma^{-1}, \sigma^{(\pi^{-1})^2}, \Phi(B) \rangle. \]

Obviously \(\sigma^{(\pi^{-1})^2} \in D_i = C_{[B_i, p]}^2(P) \leq D. \) Therefore \(P, B_i, D_i, U\) and \(S\) also satisfy (7.15). But \(\sigma^{(\pi^{-1})^2} \notin Z(B_i) - \{1\}\) violates (7.17). This final contradiction proves that (7.15) is impossible and that Proposition 7.13 is true.

\textbf{Proofs of Theorems 2.6 and 2.7.} In these cases either \(p(A_i) \neq p\) or \(p(A_i) = p \geq 5\), for all \(i = 1, \cdots, t\). So \(D_i\) is always defined by (7.6).

We know from (7.1c) and Proposition 7.3 that \(E_{j+1} \neq \{1\}. \) If \(p(A_{j+1}) \neq p\), then Proposition 7.7 implies that \(D_{j+1} \neq \{1\}. \) If \(p(A_{j+1}) = p\), then \(D_{j+1} = F_{j+1} \neq \{1\}\) by Proposition 7.9. So \(D_{j+1} \neq \{1\}\) in both cases.

Suppose that \(D_{i-1} \neq \{1\}, \) for some \(i = j + 1, \cdots, t. \) Then \(E_i \neq \{1\}\) by (7.6). If \(p(A_i) \neq p\), then \(D_i \neq \{1\}\) by Proposition 7.7. If \(p(A_i) = p\), then \(D_i = F_i \neq \{1\}\) by Proposition 7.9. So \(D_i \neq \{1\}\) in all cases.

\[ (7.22a) \quad D_i \neq \{1\}, \text{ for all } i = j + 1, \cdots, t. \]
\[ (7.22b) \quad E_i \neq \{1\}, \text{ for all } i = j + 1, \cdots, t. \]

Now Proposition 7.9 implies that \(D_i = F_i\) whenever \(p(A_i) = p. \) From this, Proposition 7.5 and Proposition 7.7 we have

\[ (7.23) \quad F_{i-1} \text{ normalizes } D_i, \text{ for } i = j + 2, \cdots, t. \]

In view of (7.22), Propositions 7.7 and 7.9 also tell us that

\[ (7.24) \quad (F_{i-1} \text{ on } E_i) \text{ is weakly equivalent to } (F_{i-1} \text{ on } D_i), \text{ for } i = j + 2, \cdots, t. \]

We shall use Proposition 2.3 to show that \(D_{j+1}, \cdots, D_i \) is a Fitting sub-
chain of $A_{i+1}, \cdots, A_t$. We must verify (2.4).

Condition (2.4a) comes from (7.22a).

If $i = j + 2, \cdots, t$, then $F_{i-1}$ normalizes $D_i$ by (7.23). Since $p(A_{i-1}) \neq p(A_i)$, it follows from (7.24) that

$$\text{Ker } (F_{i-1} \text{ on } E_i) = \text{Ker } (F_{i-1} \text{ on } D_i) = \text{Ker } (F_{i-1} \text{ on } D_i).$$

This and (7.6) imply that $D_{i-1}$ normalizes $D_i$ and $\text{Ker } (D_{i-1} \text{ on } D_i) = \{1\}$, which are (2.4b, c).

Let $i = j + 3, \cdots, t$. By (2.2e), $(E_i \text{ on } \tilde{A}_i)$ is weakly $F_{i-2}$-invariant.

Since $F_{i-2}$ centralizes $P$ (by Proposition 7.5), it follows easily from (3.9) and (3.15) that $(E_i \text{ on } \tilde{A}_i)$ is weakly $F_{i-2}$-invariant. By Proposition 7.3, $(E_{i-1} \text{ on } \tilde{E}_i)$ is weakly $F_{i-2}$-invariant. Hence $(F_{i-1} \text{ on } \tilde{E}_i)$ is weakly $F_{i-2}$-invariant. By (7.24), $(F_{i-1} \text{ on } \tilde{D}_i)$ is weakly $F_{i-2}$-invariant. Therefore $(D_{i-1} \text{ on } \tilde{D}_i)$ is weakly $D_{i-2}$-invariant, which is (2.4d).

Now $D_{i+1}, \cdots, D_i$ is a Fitting subchain of $A_{i+1}, \cdots, A_t$ by Proposition 2.3. Proposition 7.5 and (7.6) imply that $P$ centralizes each $D_i$. Since $H$ normalizes each $F_i$ (by Proposition 7.5) and $E_{i+1}$ (by Proposition 7.3), it normalizes each $D_i$ (by (7.6)). From (7.1a) and (4.19a, b) we see that $j + 1 \leq 3$ in the case of Theorem 2.6 and $j + 1 \leq 4$ in the case of Theorem 2.7.

So $D_3, \cdots, D_t$ (respectively $D_1, \cdots, D_t$) satisfy the conditions of Theorem 2.6 (Theorem 2.7). This completes the proofs of these theorems.

Proof of Theorem 2.13. In this case $p = 3$.

We know from (7.1c) and Proposition 7.3 that $E_{j+1} \neq \{1\}$. If $p(A_{j+1}) \neq p$, then (7.1b) and Proposition 7.7 give $D_{j+1} \neq \{1\}$. If $p(A_{j+1}) = p = 3$, then $D_{j+1} \neq \{1\}$ by Proposition 7.13. So $D_{j+1} \neq \{1\}$ in both cases.

Suppose that $D_{i-1} \neq \{1\}$, for some $i = j + 2, \cdots, t$. Then $E_i \neq \{1\}$ by (7.6) and (7.8). If $p(A_{i-1}) \neq p \neq p(A_i)$, then $D_i \neq \{1\}$ by Proposition 7.7. If $p(A_{i-1}) = p$, then (7.8a) and $D_{i-1} \neq \{1\}$ imply that $D_i \neq \{1\}$. If $p(A_i) = p$ and $i = t$, then $D_t = F_t \neq \{1\}$ by (7.8b) and Proposition 7.9. If $p(A_i) = p$ and $i < t$, then (7.6) and Proposition 7.12 give

$$\{1\} \neq D_{i-1} = (F_{i-1} \tilde{D}_i) = (F_{i-1} \tilde{E}_i).$$

Hence $D_i \neq \{1\}$ in all cases, so that (7.22) holds.

Now Propositions 7.7, 7.9, and 7.12 tell us that (7.23) holds. In place of (7.24), they now give

$$(F_{i-1} \text{ on } E_i) \text{ is weakly equivalent to } (F_{i-1} \text{ on } D_i), \text{ for all } i = j + 2, \cdots, t \text{ such that } p(A_{i-1}) \neq 3.$$

In addition, Proposition 7.12 says that

$$(C_i \text{ normalizes } D_{i+2}, \text{ for each relevant index } i = j + 1, \cdots, t - 2).$$

We shall use Proposition 2.11 to show that $D_{j+1}, \cdots, D_t, \{C_i\}$ is an augmented Fitting subchain of $A_{j+1}, \cdots, A_t, \{B_i\}$. We must verify (2.12).
Condition (2.12a) comes from (7.22a).

If \( i = j + 2, \ldots, t \) and \( p(A_{i-1}) \neq 3 \), then (2.12b, e) come from (7.25) as (2.4b, c) come from (7.24). If \( p(A_{i-1}) = 3 \), they come from the definition (7.8) and the remarks preceding it.

Condition (2.11c) comes from (7.11) and (7.6).

Condition (2.12d) is (7.26).

Condition (2.12f) comes from (7.25) as (2.4d) came from (7.24).

Let \( i = j + 1, \ldots, t - 3 \) be a relevant index. It follows from (7.11) and Proposition 7.3 that \( C_i \) normalizes \( E_{i+1} \). Since \( C_i \) centralizes \( P \) (by (7.10)) it must permute the ample irreducible \( Z_{p(A_{i+2})}[PE_{i+2}] \)-submodules of \( A_{i+2} \) among themselves. So it leaves \( S_{i+2} \) and \( E_{i+2} \) invariant. It follows from (2.10e) that \( E_{i+2} \) on \( A_{i+2} \) is weakly \( C_i \)-invariant. Since \( C_i \) centralizes \( P \), it follows that \( E_{i+2} \) on \( A_{i+3, \text{ample}} \) is weakly \( C_i \)-invariant. By Proposition 7.3, this implies that \( E_{i+2} \) on \( A_{i+3} \) is weakly \( C_i \)-invariant. Hence so is \( F_{i+2} \) on \( E_{i+3} \). Since \( p(A_{i+2}) \neq p(A_{i+1}) = 3 \), it follows from (7.25) that \( F_{i+2} \) on \( D_{i+2} \) is weakly \( C_i \)-invariant. This proves (2.12g).

Now \( D_{j+1}, \ldots, D_t, \{C\} \) is an augmented Fitting subchain of \( A_{j+1}, \ldots, A_t, \{B\} \) by Proposition 2.11. Proposition 7.5, (7.6), and (7.8) imply that \( P \) centralizes each \( D_i \). By (7.10), \( P \) centralizes each \( C_i \). Since \( H \) normalizes each \( F_i \) (by Proposition 7.5) and \( E_i \) (by Proposition 7.3), it follows easily from (7.6) and (7.8) that it normalizes each \( D_i \). Since \( H \) normalizes each \( B_i \) and \( P \), it normalizes each \( C_i \) (by (7.10)). From (7.1a) and (4.19c) we see that \( j + 1 \leq 6 \). Therefore \( D_6, D_7, \ldots, D_t, \{C\} \) satisfy the conditions of Theorem 2.13. This completes the proof of that theorem.

8. Thompson’s conjecture

We first prove Thompson’s conjecture in the special case of solvable groups \( G \) whose Carter subgroups have a normal complement. We use the following lemma, which was mentioned to me by R. Carter:

LEMMA 8.1. Let \( G \) be a finite solvable group whose Carter subgroups have a normal complement \( K \). Let \( H \) be a Carter subgroup of \( G \) and \( L \) be an \( H \)-invariant subgroup of \( L \). Then \( H \) normalizes some Sylow system of \( L \).

Proof. Since \( H \) is a Carter subgroup of \( G \) and \( H \leq HL \), it is a Carter subgroup of \( HL \) (see Lemma VI, 7.9 and Theorem VI, 12.2 of [4]). So there is a system normalizer \( N \) of \( HL \) contained in \( H \) (see Theorem VI, 12.8 of [4]). Let \( C/D \) be any chief section of \( H \). Then \( CL/CL \simeq C/D \) is a chief section of \( HL \), since \( L \) is a complement to \( H \) in \( HL \). Clearly the nilpotence of \( H \) makes \( CL/CL \) a central chief section of \( HL \). So it is covered by \( N \) (see Theorem 11.10 of Chapter VI of [4]). Hence \( N \) covers every chief section \( C/D \) of \( H \). Therefore \( N = H \). If \( \{S\} \) is a Sylow system of \( H \) normalized by \( N = H \), then \( L \cap S \) is a Sylow system of \( L \) normalized by \( H \). So the lemma is true.

To construct \( H \)-invariant Fitting chains we use

LEMMA 8.2. Let \( K \) be a finite solvable group and \( H \) be a group acting on \( K \) and
leaving fixed some Sylow system of $K$. Then there exist sections $A_i = C_i/D_i$ of $K$, for $i = 1, \cdots, h = h(K)$, satisfying:

- $(8.3a)$ $A_i \in \mathcal{A}$, for $i = 1, \cdots, h$.
- $(8.3b)$ $A_i$ is $H$-invariant, for $i = 1, \cdots, h$.
- $(8.3c)$ $p(A_i) \neq p(A_{i+1})$, for $i = 1, \cdots, h - 1$.
- $(8.3d)$ $C_i$ normalizes $A_i$, for $1 \leq i \leq j \leq h$.
- $(8.3e)$ $D_i = \text{Ker}(C_i \text{ on } A_{i+1})$, for $i = 1, \cdots, h - 1$.
- $(8.3f)$ $D_h = \{1\}$.
- $(8.3g)$ $(H \cdot \prod_{i<h} C_i \text{ on } \bar{A}_h)$ is irreducible.
- $(8.3h)$ $[\Phi(A_{i+1}), C_i] = \{1\}$, for $i = 1, \cdots, h - 1$.
- $(8.3i)$ $C_h \leq F(K)$.

Proof. We use induction on $h$. If $h = 0$, there is nothing to prove. If $h = 1$, let $C_1$ be any minimal $H$-invariant subgroup of $K$ and $D_1 = \{1\}$. The relevant conditions $(8.3)$ are immediately verified in this case.

Now assume that $h > 1$ and that the lemma is true for all smaller values of $h$. Since $F(K)$ is a characteristic subgroup of $K$, the group $H$ acts on $K^* = K/F(K)$. The images in $K^*$ of the groups forming an $H$-invariant Sylow system of $K$ are obviously the members of an $H$-invariant Sylow system of $K^*$. For each $x = a, b, \cdots, i$, let $(8.3x)^*$ be $(8.3x)$ with $K, A_j, C_j, D_j, h$ replaced by $K^*, A_j^*, C_j^*, D_j^*, h - 1$, respectively, for all indices $j$. Clearly $h(K^*) = h - 1$. So induction gives us sections $A_i^* = C_i^*/D_i^*$ of $K^*$, for $i = 1, \cdots, h - 1$, satisfying $(8.3)^*$.

Let $S$ be a $p(A_{h-1}^*)$-Sylow subgroup of $K$ belonging to a Sylow system fixed by $H$. Then $S \cap F_2(K)$ is an $H$-invariant $p(A_{h-1}^*)$-Sylow subgroup of $F_2(K)$, and $N = N_K(S \cap F_2(K))$ is an $H$-invariant subgroup of $K$. Considering $S \cap F_2(K)$ as a $p(A_{h-1}^*)$-Sylow subgroup of the normal subgroup $(S \cap F_2(K))F(K)$ of $K$, we see by the Frattini argument that $NF(K) = K$. We denote by the natural epimorphism of $N$ onto $K/F(K) = K^*$.

For each $i = 1, \cdots, h - 2$, we define $C_i$ and $D_i$ to be the inverse images under $\varphi$ of $C_i^*, D_i^*$, respectively. Since $\varphi$ defines an $H$-isomorphism of $N/N \cap F(K)$ onto $K^*$, we see from $(8.3)^*$ that those parts of condition $(8.3)$ involving only those $C_i, D_i$ and $A_i = C_i/D_i$ with $i \leq h - 2$ are all satisfied.

The image $\varphi(S \cap F_2(K))$ is the $p(A_{h-1}^*)$-Sylow subgroup of $F(K^*) = F_2(K)/F(K)$. From $(8.3f, i)^*$ we see that $C_{h-1}^* \leq \varphi(S \cap F_2(K))$. Let $C_{h-1}$ be the inverse image in $S \cap F_2(K)$ of $C_{h-1}^*$ under $\varphi$. Since $N$ is the normalizer of $S \cap F_2(K)$, it follows from $(8.3b, d)^*$ that $H \cdot \prod_{i<h-1} C_i$ normalizes $C_{h-1}$.

Because $A_{h-1}^* \in \mathcal{A}$, we have $A_{h-1}^* \neq \{1\}$ (by $(1.4a)$). It follows that $C_{h-1} > S \cap F_2(K) \cap \text{Ker} \varphi = S \cap F(K)$.

So there must exist some prime $p \neq p(A_{h-1}^*)$ such that $C_{h-1}$ does not centralize the $p$-Sylow subgroup $T$ of $F(K)$. Then $H \cdot \prod_{i<h} C_i$ normalizes $T$ and $[T, C_{h-1}] \neq \{1\}$. The Hall-Higman theory (see Theorem III, 13.5 of [4]) gives us an $H \cdot \prod_{i<h} C_i$-invariant special subgroup $C_h$ of $T$ such that $(H \cdot \prod_{i<h} C_i$
on \( \tilde{C}_h \) is irreducible, \((C_{h-1} \text{ on } C_h)\) is non-trivial, and \([\Phi(C_h), C_{h-1}] = \{1\}\). If \( p \) is odd, we even have \(\exp(C_h) = p\). Hence \(C_h \in \mathfrak{a}\).

Now define \(D_{h-1}\) by \(D_{h-1} = \text{Ker}(C_{h-1} \text{ on } C_h)\). Since \(F(K)\) is nilpotent and
\[ p \neq p(A_{h-1}^*) \]
the subgroup \(S \cap F(K)\) centralizes \(T\). So \(D_{h-1} \geq S \cap F(K)\). But \(S \cap F(K)\) is the kernel of the natural epimorphism of \(C_{h-1}\) onto \(A_{h-1}^*\) (by (8.3f)\( ^* \)). The image \(E\) of \(D_{h-1}\) in \(A_{h-1}^*\) is evidently \(H \cdot \prod_{i < h-1} C_i^*\)-invariant and not equal to \(A_{h-1}^*\). By (8.3g)\( ^* \) we must have \(D_{h-1} \leq \Phi(A_{h-1}^*)\). Defining \(A_{h-1}\) to be \(C_{h-1}/D_{h-1}\), we see that \(\phi\) induces a natural isomorphism of \(\tilde{A}_{h-1}\) on \(A_{h-1}^*\).

Condition (8.3a) for \(i = h - 1\) comes from \(1 < A_{h-1} \cong A_{h-1}^*/E\) and (1.5). Condition (8.3b) for \(i = h - 1\) comes from the construction of \(C_{h-1}\) and \(D_{h-1}\).

Condition (8.3c) for \(i = h - 2\) comes from (8.3c)\( ^* \), since \(p(A_{h-2}) = p(A_{h-2}^*)\) and \(p(A_{h-1}) = p(A_{h-1}^*)\). Condition (8.3d) for \(j = h - 1\) comes from the construction of \(C_{h-1}, D_{h-1}\). Condition (8.3e) for \(i = h - 2\) comes from (8.3e)\( ^* \), since \(p(A_{h-2}) \neq p(A_{h-1})\) implies
\[
\text{Ker}(A_{h-2} \text{ on } A_{h-1}) = \text{Ker}(A_{h-2} \text{ on } A_{h-1}^*) = \{1\},
\]
and \(\phi\) induces an isomorphism of \(\text{Ker}(A_{h-2} \text{ on } A_{h-1})\) onto \(\text{Ker}(A_{h-2}^* \text{ on } A_{h-1}^*)\).

Finally, condition (8.3h) for \(i = h - 2\) comes from (8.3h)\( ^* \).

Set \(D_h = \{1\}\) and \(A_h = C_h/D_h\). The constructions of \(A_i\) and \(D_{h-i}\) give those conditions (8.3) involving \(A_h, C_h\) or \(D_h\) with no difficulty and complete the inductive proof of the lemma.

Now we can prove the special case of Thompson’s conjecture.

**Theorem 8.4.** Let \(G\) be a finite solvable group whose Carter subgroups have normal complement \(K\). If \(H\) is a Carter subgroup of \(G\), then \(h(K) \leq 5(2^{2^{\alpha}} - 1)\).

**Proof.** By Lemma 8.1, \(H\) normalizes a Sylow system of \(K\). So Lemma 8.2 gives us a chain \(A_1, \ldots, A_h, h = h(K)\), of sections of \(K\) satisfying (8.3). By (8.3d, e), \(A_i\) normalizes \(A_{i+1}\), for \(i = 1, \ldots, h - 1\). We claim that \(A_1, \ldots, A_h\) with these actions is a Fitting chain, i.e., that it satisfies (2.2). Indeed, property (2.2a) comes from (8.3a), property (2.2b) from (8.3c), property (2.2c) from (8.3h) and property (2.2d) from (8.3e). Since \(C_i\) normalizes both \(A_{i+1}\) and \(A_{i+2}\) (by (8.3d)), the action \((A_{i+1} \text{ on } A_{i+2})\) is \(C_i\)-invariant and therefore \(A_i\)-invariant. So (2.2e) holds, and \(A_1, \ldots, A_h\) is an \(H\)-invariant Fitting chain (by (8.3)).

Let \(i = 1, 2, \ldots, t - 2\) be a relevant index. By Lemma 8.1, \(H\) leaves invariant some \(p(A_i)\)-Sylow subgroup \(B_i\) of \(C_i\). Let \(q_i\) be the natural epimorphism of \(B_i\) onto \(A_i = C_i/D_i\). By (8.3d), \(B_i\) normalizes
\[
A_{i+1}, A_{i+2}, A_{i+3}, \ldots, A_h.
\]
This gives as a natural action of \(B_i\) on \(A_{i+2}\). Clearly \(B_i\) satisfies (2.10a,b).
Since $B_1 A_{i+2}$ acts on $A_{i+3}$, condition (2.10c) is also satisfied if $i \leq h - 3$. The $H$-invariance of $B_i$ implies that $A_1, \ldots, A_h, \{B_i\}$ is an $H$-invariant augmented Fitting chain.

Because $H$ is a Carter subgroup of $G$ and $H \cap K = \{1\}$, it centralizes no nontrivial section of $K$. Furthermore, $H$ is nilpotent. So Theorem 2.14 tells us that $h \leq 5(2^{l(H)} - 1)$, which is this theorem.

At last we have

**Theorem 8.5.** Let $H$ be a Carter subgroup of a finite solvable group $G$. Then $h(G) \leq 10(2^{l(H)} - 1) - 4l(H)$.

**Proof.** By induction on $l = l(H)$. If $l(H) = 0$, then $H = \{1\}$ and $G = \{1\}$. So $h(G) = 0 = 10(2^0 - 1) - 4\cdot0$, and the theorem is true in this case.

Now assume that $l > 0$, and that the theorem is true for all smaller values of $l(H)$.

Fix a Carter subgroup $H$ of $G$. The Fitting series satisfies

$$\{1\} = F_0(G) < F_1(G) < \cdots < F_h(G) = G,$$

where $h = h(G)$. So there exists an integer $k \geq 0$ such that

$$(8.6a) \quad F_k(G) \cap H = \{1\}.$$  
$$\quad (8.6b) \quad F_{k+1}(G) \cap H \neq \{1\}.$$  

Let $G_1 = G/F_{k+1}(G)$. The image $H_1$ of $H$ in $G_1$ is a Carter subgroup of $G_1$ (see Lemma VI, 12.3 of [4]). By (8.6b) we have $l(H_1) < l(H)$. So induction gives

$$h(G_1) = h - k - 1 \leq 10(2^{l(H_1)} - 1) - 4l(H_1)$$  
$$\quad \quad \leq 10(2^{l-1} - 1) - 4(l - 1).$$  

(8.7)

The subgroup $G_2 = H \cdot F_k(G)$ contains the Carter subgroup $H$ of $G$. So $H$ is a Carter subgroup for $G_2$ (see Lemma VI, 7.9 and Theorem VI, 12.2 of [4]). Clearly (8.6a) says that $F_k(G)$ is a normal complement to $H$ in $G_2$. From Theorem 8.4 we conclude that $h(F_k(G)) = k \leq 5(2^i - 1)$. Adding this to (8.7) we get

$$h = 1 + k + (h - k - 1)$$  
$$\leq 1 + 5(2^i - 1) + 10(2^{l-1} - 1) - 4(l - 1) = 10(2^i - 1) - 4l.$$

So the theorem is true.

**References**


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