A SET OF GENERALIZED NUMBERS SHOWING BEURLING'S THEOREM TO BE SHARP

BY

HAROLD G. DIAMOND[1]

Beurling [1] proved that the prime number theorem holds for generalized (henceforth g-) numbers if $N(x)$, the number of g-integers not exceeding x, satisfies $N(x) = cx + O(x \log^{-\gamma} x)$ for c a positive number and γ a number greater than $\frac{3}{2}$. Further, he showed that this result is sharp by giving an example of a "prime measure" and associated "integer measure" for which $\gamma = \frac{3}{2}$ but for which the prime number theorem is false. However, the measures of Beurling's example are continuous and thus differ from the usual (atomic) counting measures of prime number theory.

We shall give an example of g-primes and g-integers for which the prime number theorem fails but $N(x) = cx + O\{x(\log x)^{-3/2}\}$. Our construction is based on Beurling's example and a method of approximating measures that we have used in [2].

Let $\pi(x)$ be the number of g-primes not exceeding x, and define $\Pi(x) = \sum_{n=1}^{\infty} n^{-1}\pi(x^{1/n})$. Set $li(x) = k + \int_2^x (\log t)^{-1} dt$ (k a constant) and define $\tau(x)$ by

$$\tau(x) = \int_1^x \{1 - \cos(\log t)\}(\log t)^{-1} \, dt \quad \text{for } x \geq 1$$

and $\tau(x) = 0$ for $x < 1$.

PROPOSITION. *Let p_r, the r^{th} g-prime, be defined by $p_r = \tau^{-1}(r)$. Then*

$$N(x) = cx + O\{x(\log x)^{-3/2}\},$$

where $c = \exp\{\int_1^\infty t^{-1}(d\Pi - d\tau)(t)\}$, and $\pi(x)/li(x)$ does not have a limit as $x \to \infty$.

Sketch of proof. The number c is finite and positive because $\Pi(t) - \tau(t) = O(t^{1/2})$. Since $\pi(x) = [\tau(x)]$ and

$$\tau(x) = li(x) - \frac{x}{2\log x}\{\sin(\log x) + \cos(\log x)\} + O(x \log^{-2} x),$$

$\pi(x)/li(x)$ has no limit as $x \to \infty$. We now estimate $N(x)$, noting first that $dN = \exp d\Pi$ [1; p. 257], [2; §3.1]. The exponential is defined by its power series about the origin; the powers of a measure dA are defined by $dA^0 = \delta =$ point mass 1 at 1 and by $(dA)^n = (dA)^{n-1} * dA$ for $n \geq 1$, where $*$ denotes

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multiplicative convolution. Also define $dT = \exp d\tau$ and $C = \log c$. Now

$$dN = \exp d\Pi = c\, dT * \exp\{d\Pi - d\tau - C\delta\},$$

and by a technique of Dirichlet we have

$$\begin{aligned} N(x) &= c \int_{1-}^{\sqrt{x}+} T(x/u) \exp\{d\Pi - d\tau - C\delta\}(u) \\ &\quad + c \int_{1-}^{\sqrt{x}+} \left(\int_{\sqrt{x}+}^{x/u+} \exp\{d\Pi - d\tau - C\delta\} \right) dT(u). \end{aligned} \tag{1}$$

Our two principal tasks will be to show $\int_{1-}^{y+} \exp\{d\Pi - d\tau - C\delta\}$ suitably small and to obtain the first few terms of the asymptotic series for $T(y)$. We shall collect this information and return to (1).

LEMMA 1. *There exist numbers $\{c_j\}$ such that as $y \to \infty$,*

$$T(y) \sim y + 2 \operatorname{Re} \textstyle\sum_{j=1}^{\infty} c_j y^{1+i} (\log y)^{-j-1/2}.$$

Proof. We have

$$\int_{x=1}^{\infty} x^{-s}\, dT(x) = \int_{x=1}^{\infty} x^{-s} (\exp d\tau)(x) = \exp \int_{x=1}^{\infty} x^{-s}\, d\tau(x)$$
$$= \exp\left(\tfrac{1}{2} \log\{1 + (s-1)^{-2}\}\right).$$

The middle equality is a consequence of the fact that the map

$$dB(u) \to u^{\alpha}\, dB(u)$$

is an algebra homomorphism for any fixed α. Following [1; pp. 288–290], we set $\varphi(s) = \{1 + (s-1)^{-2}\}^{1/2}$ and for $x > 1$ express T in terms of φ by the Mellin inversion formula:

$$T(x) = \lim_{X \to \infty} \frac{1}{2\pi i} \int_{\sigma - iX}^{\sigma + iX} \varphi(s) x^s s^{-1}\, ds \quad \text{with } \operatorname{Re} s = \sigma > 1.$$

Deform the path of integration to yield

$$T(x) = \int_{l_\varepsilon(1)} + \int_{l_\varepsilon(1+i)} + \int_{l_\varepsilon(1-i)},$$

where $\int_{l_\varepsilon(1+ia)}$ denotes integration along the contour of points at distance ε from the ray from $1 + ia$ to $-\infty + ia$, taken in the positive sense with respect to the point $1 + ia$. As in [1], $\int_{l_\varepsilon(1)} = x + \varphi(0)$, and we now apply Beurling's analysis to the two remaining integrals. Set

$$s^{-1}\varphi(s) = \textstyle\sum_{n=0}^{\infty} A_n (s - 1 - i)^{n+1/2}$$

for $|s - 1 - i| < 1$. As $x \to \infty$ we have

$$\int_{l_\varepsilon(1+i)} \sim \sum_{n=0}^{\infty} A_n \frac{1}{2\pi i} \int_{l_\varepsilon(1+i)} (s - 1 - i)^{n+1/2} x^s\, ds$$
$$\sim \textstyle\sum A_n x^{1+i} (\log x)^{-n-3/2} / \Gamma(-n - \tfrac{1}{2}).$$

Since $s^{-1}\varphi(s)$ is real for s real, a conjugate expression holds about $s = 1 - i$, and we have

$$\int_{l_\varepsilon(1-i)} \sim \sum \bar{A}_n x^{1-i}(\log x)^{-n-3/2}/\Gamma(-n - \tfrac{1}{2}).$$

Setting $c_{j+1} = A_j/\Gamma(-j - \frac{1}{2})$, we have the desired expansion.

LEMMA 2.

$$\int_1^x u^{-1}\, dT(u) = O(\log x) \quad \textit{and} \quad \int_1^x u^{-1}\, dN(u) = O(\log x).$$

Proof. The first estimate is easy since $T(u) = O(u)$. For the second, write

$$\begin{aligned}\int_1^x u^{-1}\, dN(u) &= \int_1^x u^{-1} \exp\{d\pi + (d\Pi - d\pi)\}(u)\\ &= \int_1^x u^{-1}\{\exp d\pi * \exp(d\Pi - d\pi)\}(u)\\ &= \int_1^x (u^{-1}\exp d\pi) * (u^{-1}\exp\{d\Pi - d\pi\}).\end{aligned}$$

Now since $\exp\{d\Pi - d\pi\} \geq 0$,

$$\int_1^x u^{-1}\, dN(u) \leq \left(\int_1^x u^{-1}\exp d\pi\right)\left(\int_1^\infty u^{-1}\exp\{d\Pi - d\pi\}\right).$$

The last integral equals $\exp\{\int_1^\infty u^{-1}\{d\Pi - d\pi\}(u)\}$, and this is a finite number since $\Pi(y) - \pi(y) = O(y^{1/2})$.

We estimate $\int_1^x u^{-1}\exp d\pi$ in terms of $\int_1^x u^{-1}\exp d\tau$ by noting the following two simple facts: If $A(x)$ and $B(x)$ are increasing functions on $[1, \infty)$ and $A(x) \geq B(x)$ for all $x \geq 1$, then

$$\int_1^x (dA)^n \geq \int_1^x (dB)^n, \tag{2a}$$

$$\int_1^x u^{-1}\, dA(u) \geq \int_1^x u^{-1}\, dB(u). \tag{2b}$$

Now we have

$$\begin{aligned}\int_1^x u^{-1}\{\exp d\pi\}(u) &= \sum_0^\infty \int_1^x u^{-1}(d\pi)^n/n!\\ &\leq \sum_0^\infty \int_1^x u^{-1}(d\tau)^n/n! = \int_1^x u^{-1}\, dT(u) = O(\log x),\end{aligned}$$

and the proof of the lemma is complete.

Let C satisfy the hypothesis of the proposition, and define ν_j for $j \geq 0$

and $x \geq 1$ by

$$\nu_j(x) = \int_{1-}^{x+} u^{-1}(d\Pi - d\tau - C\delta)^j(u)$$

and set $\nu(x) = \nu_1(x)$. We remark that $\nu_0(x) \equiv 1$, $x \geq 1$, and $d\nu_j = (d\nu)^j$.

LEMMA 3. *There exist constants A_0 and A_1 such that for all $x \geq 1$ and all positive integers n,*

$$(3) \qquad | \nu_n(x) | \leq nA_0(2 \log \log ex + A_1)^{n-1} x^{-1/2n}.$$

Proof. By partial integration we see that $\nu(x) = O(x^{-1/2})$, and thus there exists A_0 such that (3) holds with $n = 1$. Also, for $x \geq 1$,

$$\int_{1-}^{x+} | d\nu | = 2 \int_1^x u^{-1} \, d\tau(u) + \nu(x) + C + | C | \leq 2 \log \log ex + B_0,$$

B_0 some constant. We now proceed by induction on n again using Dirichlet's device. Assume the truth of the n^{th} case $(n \geq 1)$ and set $y = x^{1/(n+1)}$ and $z = x/y$.

$$\begin{aligned} \nu_{n+1}(x) &= \iint_{st \leq x} (d\nu)^n(s) \, d\nu(t) \\ &= \int_{1-}^{y+} \nu_n(x/t) \, d\nu(t) + \int_{1-}^{z+} \nu(x/t)(d\nu)^n(t) - \nu(y)\nu_n(z). \end{aligned}$$

$$\begin{aligned} \left| \int_{1-}^{y+} \right| &\leq nA_0(2 \log \log ex + A_1)^{n-1} z^{-1/2n} \int_{1-}^{y+} | d\nu | \\ &\leq nA_0(2 \log \log ex + A_1)^{n-1}(2 \log \log ex + B_0) x^{-1/(2n+2)}. \end{aligned}$$

$$\begin{aligned} \left| \int_{1-}^{z+} \right| &\leq A_0(x/z)^{-1/2} \left(\int_{1-}^{x+} | d\nu | \right)^n \\ &\leq A_0(2 \log \log ex + B_0)^n x^{-1/(2n+2)}. \end{aligned}$$

$$| \nu(y)\nu_n(z) | \leq A_0 n A_0(2 \log \log ex + A_1)^{n-1} x^{-1/(2n+2)}.$$

Thus (3) is true with $n + 1$ in place of n if we take $A_1 = A_0 + B_0$, and the induction is complete.

LEMMA 4. *Let $f(y) = \int_{1-}^{y+} u^{-1}(\exp\{d\Pi - d\tau - C\delta\} - \delta)$. Then for $y > 1$,*

$$f(y) + 1 = \int_{1-}^{y+} \exp(d\nu),$$

and for each (fixed) positive number k, $f(y) = O(\log^{-k} y)$ as $y \to \infty$.

Proof. Since the map $dB(u) \to u^{-1} \, dB(u)$ is an algebra homomorphism,

$$f(y) = \int_{1-}^{y+} \exp(d\nu) - 1 = \textstyle\sum_{n=1}^{\infty} \nu_n(y)/n!,$$

and we use the estimates of the last lemma. Break the sum at

$$N = [(8 + k)(\log \log ey + A_1)]:$$

$$\sum_1^N \leq A_0 y^{-1/(2N)} \exp \{2 \log \log ey + A_1\} = O(\log^{-k} y).$$

$$\begin{aligned}\sum_{N+1}^{\infty} &\leq A_0 \sum_N^{\infty} \frac{(2 \log \log ey + A_1)^n}{n!} \\ &\leq 2\, A_0 \frac{(2 \log \log ey + A_1)^N}{N!} \\ &\leq 2A_0 \left(\frac{e}{8 + k}\right)^N \leq 2A_0\, e^{-N} = O(\log^{-k} y).\end{aligned}$$

Proof of the proposition. We show the second term in (1) to be small. We have the formula

$$\int_{1-}^{y+} \exp \{d\Pi - d\tau - C\delta\} = \int_{1-}^{y+} \{\delta + u\, df(u)\},$$

and integration by parts and use of the last lemma yield

$$\int_{1-}^{y+} u\, df(u) = O(y \log^{-3} y).$$

Also, $dT \geq 0$ and thus

$$\left| \int_{1-}^{\sqrt{x}+} \left\{ \int_{\sqrt{x}+}^{x/u+} \exp\,(d\Pi - d\tau - C\delta) \right\} dT(u) \right| \leq K \int_1^{\sqrt{x}} \frac{x}{u} \left(\log \frac{x}{u}\right)^{-3} dT(u) = O(x \log^{-2} x).$$

To estimate the first term in (1), set

$$T(x/u) = \frac{x}{u} + 2 \operatorname{Re} \sum_{j=1}^{2} c_j \left(\frac{x}{u}\right)^{1+i} \left(\log \frac{x}{u}\right)^{-j-1/2} + O\left\{\frac{x}{u}\left(\log \frac{x}{u}\right)^{-7/2}\right\}$$

and treat separately the three resulting integrals I + II + III.

$$\begin{aligned}\mathrm{I} &= \int_{1-}^{\sqrt{x}+} xu^{-1} \exp \{d\Pi - d\tau - C\delta\}(u) = x(f(\sqrt{x}) + 1) \\ &= x + O(x \log^{-2} x).\end{aligned}$$

$$\mathrm{II} = \int_{1-}^{\sqrt{x}+} 2 \operatorname{Re} \sum c_j \left(\frac{x}{u}\right)^{1+i} \left(\log \frac{x}{u}\right)^{-j-1/2} \exp \{d\Pi - d\tau - C\delta\}(u)$$

and may be expressed as a linear combination of four integrals

$$J_{a,b} = \int_{1-}^{\sqrt{x}+} \left(\frac{x}{u}\right)^{1+ia} \left(\log \frac{x}{u}\right)^{-b} \exp \{d\Pi - d\tau - C\delta\}(u),$$

with $a = 1, -1; b = \frac{3}{2}, \frac{5}{2}$. By the definition of f, we may write

$$J_{a,b} = x^{1+ia} \int_{1-}^{\sqrt{x}+} u^{-ia} \left(\log \frac{x}{u}\right)^{-b} df(u) + x^{1+ia} \log^{-b} x.$$

Integrating by parts and noting that $\int_1^\infty u^{-1}|f(u)|\,du < \infty$, we see that $J_{a,b} = O(x \log^{-b} x)$ and thus II $= O\{x(\log x)^{-3/2}\}$. Finally,

$$
\begin{aligned}
|\,\mathrm{III}\,| &\le K \int_{1-}^{\sqrt{x}+} \frac{x}{u}\left(\log \frac{x}{u}\right)^{-7/2} |\exp\{d\Pi - d\tau - C\delta\}(u)| \\
&\le K' x \log^{-7/2} x \int_1^{\sqrt{x}} u^{-1} \exp(d\Pi + d\tau) \\
&\le K' x \log^{-7/2} x \left(\int_1^x u^{-1}\,dN\right)\left(\int_1^x u^{-1}\,dT\right) = O\{x(\log x)^{-3/2}\},
\end{aligned}
$$

and we conclude that $N(x) = cx + O\{x(\log x)^{-3/2}\}$.

Bibliography

1. A. Beurling, *Analyse de la loi asymptotique de la distribution des nombres premiers généralisés*. I, Acta Math., vol. 68 (1937), pp. 255–291.
2. H. Diamond, *Asymptotic distribution of Beurling's generalized integers*, Illinois J. Math., vol. 14 (1970), pp. 12–28.

University of Illinois
Urbana, Illinois